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A new three-dimensional extension of Bougerol's identity in law

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Abstract

Bougerol's identity in law is one of the most infamous in the study of Brownian motion. To that end, in this paper, we present a further extension of its study through a proof in the third dimension. Second-dimension proofs have been extensively explored, as have those of the third dimension. However, here we will extrapolate a previously two-dimensional proof into the third dimension through the use of a three-Brownian joint distribution.

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1 Introduction

The content of this paper demonstrates the extension of a previous two-dimensional proof of Bougerol's identity law given by Vakeroudis [4] into the third dimension. To illustrate Bougerol's identity in law consider the following exponential functional with an associated Brownian motion $(B_s, s \geq 0)$ such that

$$A_t = \int_0^t ds e^{2B_s}, \quad t \geq 0. \tag{1.1}$$

Bougerol's identity in law [2], states that, for every fixed t:

$$\sinh(B_t) \overset{(law)}{=} \beta_{A_t}, \tag{1.2}$$

where β is an independent, real Brownian motion. Before providing a new extension of Bougerol's identity in law in the third dimension, we will first gain a background by examining the one-dimensional and two-dimensional proofs analogous to our three-dimensional extension.

Consider the following quantity:

$$X_u^{(1)} = e^{-B_u} \int_0^u d\xi_v^{(1)} e^{B_v}. \tag{1.3}$$

$\xi_v^{(1)}$ and B_u are real, independent Brownian motions with $v \geq 0$ and $u \geq 0$, respectively. A proof of Bougerol's identity in law requires the study of two laws related to the above

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distribution conveyed in terms of processes and fixed u's, respectively. Our first burden of proof is at the level of processes and is the most commonly represented form of Bougerol's identity in law [2]. In one dimension it is described as follows:

$$X_u^{(1)} \overset{(law)}{=} \sinh(B_u), \quad d < B^{(1)} >_v = \tanh(B_t^{(1)}), \quad (B_t^{(1)}, t \geq 0). \tag{1.4}$$

The second relates to fixed u behavior defined by

$$X_u^{(1)} \overset{(law)}{=} \beta^{(1)}_{\int_0^u dve^{2B_v}}. \tag{1.5}$$

To prove the first statement we must prove the equality of the infinitesimal generators of the distribution at the level of processes under the assumption that Bougerol's identity in law holds. To prove the second we equate the fixed u density of the process level statement with that of the fixed u statement to confirm Bougerol's identity.

$$\sinh(B_u^{(1)}) \overset{(law)}{=} \beta^{(1)}_{\int_0^u dve^{2B_v}}, \tag{1.6}$$

and derive a density function of the form

$$p_u(x) = E\left[\frac{1}{\sqrt{2\pi}} \frac{\exp\left(\frac{-x^2}{2\int_0^u dve^{2B_v}}\right)}{\sqrt{\int_0^u dve^{2B_v}}}\right]. \tag{1.7}$$

The latter should represent the one-dimensional density of both $\sinh(B_u^{(1)})$ and $\beta^{(1)}_{\int_0^u dve^{2B_v}}$.

A similar method was outlined directly by Vakeroudis [4] in two dimensions with an extension of the method above. The changes are in the initial distribution and both conditions. The distribution is given by

$$\left(X_u^{(1)}, X_u^{(2)}\right) \overset{(law)}{=} \left(e^{-B_u}\int_0^u d\xi_v^{(1)} e^{B_v}, e^{-2B_u}\int_0^u d\xi_v^{(2)} e^{2B_v}\right), \tag{1.8}$$

where we now have two dependent Brownian motions of the form

$$d < B^{(1)}, B^{(2)} >_v = \tanh(B_v^{(1)}) \tanh(2B_v^{(2)}) dv. \tag{1.9}$$

In two dimensions, Bougerol's identity in law at the level of processes [4] is then expressed as

$$\left(X_u^{(1)}, X_u^{(2)}\right) \overset{(law)}{=} \left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)})\right). \tag{1.10}$$

Its fixed u form is similarly extended from the one-dimensional version by

$$\left(X_u^{(1)}, X_u^{(2)}\right) \overset{(law)}{=} \left(\beta^{(1)}_{\int_0^u dve^{2B_v}}, \beta^{(2)}_{\int_0^u dve^{4B_v}}\right). \tag{1.11}$$

Again, we assume Bougerol's identity and prove the infinitesimal generators of the distribution and the process level are the same. Following the same line of proof as in the first dimension we then prove the second part by equating the densities of the fixed u and process level expressions given by the following:

$$\left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)})\right) \overset{(law)}{=} \left(\beta^{(1)}_{\int_0^u dve^{2B_v}}, \beta^{(2)}_{\int_0^u dve^{4B_v}}\right). \tag{1.12}$$

Vakeroudis [4] shows the densities to be equal and finds the probability density function to be an extended form of the first dimension represented as:

$$p_u(x,y) = E\left[\frac{1}{2\pi}\frac{\exp\left(\frac{-x^2}{2\int_0^u dve^{2B_v}}\right)}{\sqrt{\int_0^u dve^{2B_v}}}\frac{\exp\left(\frac{-y^2}{2\int_0^u dve^{2=4B_v}}\right)}{\sqrt{\int_0^u dve^{2=4B_v}}}\right]. \tag{1.13}$$

This provides us with an intuition of a method to extend Bougerol's identity in law to the third dimension in a new, more intuitive way, as outlined in Section 2.

2 The three-dimensional proof

Proofs of Bougerol's identity [2] in three dimensions are widely available through the less intuitive conventional methods of proof for the identity in law. We build upon Vakeroudis's [4] new two-dimensional extension in the third dimension, providing a more intuitive approach to foster higher dimensional expansion of this identity in law. Consider the following joint distribution:

$$\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)}\right) \overset{(law)}{=} \left(e^{-B_u}\int_0^u d\xi_v^{(1)}e^{B_v}, e^{-2B_u}\int_0^u d\xi_v^{(2)}e^{2B_v}, e^{-3B_u}\int_0^u d\xi_v^{(3)}e^{3B_v}\right), \tag{2.1}$$

where $\xi_v^{(1)}$, $\xi_v^{(2)}$, $\xi_v^{(3)}$, and B_u are independent Brownian motions such that

$$(\xi_v^{(1)}, v\geq 0), (\xi_v^{(2)}, v\geq 0), (\xi_v^{(3)}, v\geq 0), (B_u, u\geq 0). \tag{2.2}$$

The three-dimensional extension of Bougerol's identity in law can be obtained by the following:

Theorem 2.1 (Bougerol's Identity in Law in three dimensions)**.**
Consider three real dependent Brownian motions of the form:

$$(B_t^{(1)}, t\geq 0), (B_t^{(2)}, t\geq 0), (B_t^{(3)}, t\geq 0), \tag{2.3}$$

such that

$$d < B^{(1)}, B^{(2)}, B^{(3)} >_v = \tanh(B_v^{(1)})\tanh(2B_v^{(2)})\tanh(3B_v^{(3)})dv. \tag{2.4}$$

For the three-dimensional process $(X_u^{(1)}, X_u^{(2)}, X_u^{(3)})$ it holds that

$$\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)}\right) \overset{(law)}{=} \left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)}), \frac{1}{3}\sinh(B_u^{(3)})\right). \tag{2.5}$$

For the same process for a fixed $u\geq 0$:

$$\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)}\right) \overset{(law)}{=} \left(\beta^{(1)}_{\int_0^u dve^{2B_v}}, \beta^{(2)}_{\int_0^u dve^{4B_v}}, \beta^{(3)}_{\int_0^u dve^{6B_v}}\right). \tag{2.6}$$

Proof. We begin by proving the relation of the Theorem at the level of processes as given by the following proposition:

Proposition 2.2 (Process Level Equality)**.**
For the given three-dimensional process $(X_u^{(1)}, X_u^{(2)}, X_u^{(3)})$, recall the relation given by (2.5):

$$\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)}\right) \overset{(law)}{=} \left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)}), \frac{1}{3}\sinh(B_u^{(3)})\right).$$

In other terms, the infinitesimal generator of $(x_t^{(1)}, x_t^{(2)}, x_t^{(3)})$ *and* $(X_u^{(1)}, X_u^{(2)}, X_u^{(3)})$ *are the same such that:*

$$(x_t^{(1)}, x_t^{(2)}, x_t^{(3)}) = \left(\sinh(B_t^{(1)}), \frac{1}{2}\sinh(2B_t^{(2)}), \frac{1}{3}\sinh(B_t^{(3)}) \right). \tag{2.7}$$

Lemma 2.3. *The infinitesimal generator of* $\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)}\right)$ *is*

$$\begin{aligned}\frac{1}{2}\Bigg[&(1+x^2)\frac{\partial^2}{\partial x_1^2} + (1+4x_2^2)\frac{\partial^2}{\partial x_2^2} + (1+9x_3^2)\frac{\partial^2}{\partial x_3^2} + 4x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} \\ &+ 9x_1x_3\frac{\partial^2}{\partial x_1\partial x_3} + 36x_2x_3\frac{\partial^2}{\partial x_2\partial x_3}\Bigg] + \frac{x_1}{2}\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} + \frac{9x_3}{2}\frac{\partial}{\partial x_3}.\end{aligned} \tag{2.8}$$

Proof. We start by defining a general term for the elements of the three-dimensional process represented as

$$X_u^{(\alpha)} = e^{-\alpha B_u}\int_0^u d\xi_v^{(\alpha)} e^{\alpha B_v}, \quad \alpha = 1,2,3. \tag{2.9}$$

Next, we apply Itô's formula exactly like Vakeroudis [4].

$$\begin{aligned} X_u^{(\alpha)} &= \xi_u^{(\alpha)} + \int_0^u \Big(e^{-\alpha B_v}(-\alpha dB_v) + \frac{\alpha^2}{2}e^{-\alpha B_v}dv \Big) \times \Big(\int_0^v d\xi_h^{(\alpha)} e^{\alpha B_h} \Big) \\ &= \xi_u^{(\alpha)} + \int_0^u \Big(-\alpha dB_v X_v^{(\alpha)} + \frac{\alpha^2}{2} X_v^{(\alpha)} dv \Big) \\ &= \int_0^u d\eta_v^{(\alpha)} \sqrt{\Big(1+\alpha^2 \big(X_v^{(\alpha)}\big)^2\Big)} + \frac{\alpha^2}{2}\int_0^u X_v^{(\alpha)} dv. \end{aligned} \tag{2.10}$$

Using this general formula, we then substitute in 1, 2, and 3 for α.

$$\begin{aligned} X_u^{(1)} &= \int_0^u d\eta_v^{(1)} \sqrt{\Big(1+1^2\big(X_v^{(1)}\big)^2\Big)} + \frac{1^2}{2}\int_0^u X_v^{(1)} dv \\ &= \int_0^u d\eta_v^{(1)} \sqrt{\Big(1+\big(X_v^{(1)}\big)^2\Big)} + \frac{1}{2}\int_0^u X_v^{(1)} dv, \end{aligned} \tag{2.11}$$

$$\begin{aligned} X_u^{(2)} &= \int_0^u d\eta_v^{(2)} \sqrt{\Big(1+2^2\big(X_v^{(2)}\big)^2\Big)} + \frac{2^2}{2}\int_0^u X_v^{(2)} dv \\ &= \int_0^u d\eta_v^{(2)} \sqrt{\Big(1+4\big(X_v^{(2)}\big)^2\Big)} + 2\int_0^u X_v^{(2)} dv, \end{aligned} \tag{2.12}$$

$$\begin{aligned} X_u^{(3)} &= \int_0^u d\eta_v^{(3)} \sqrt{\Big(1+3^2\big(X_v^{(3)}\big)^2\Big)} + \frac{3^2}{2}\int_0^u X_v^{(3)} dv \\ &= \int_0^u d\eta_v^{(3)} \sqrt{\Big(1+9\big(X_v^{(3)}\big)^2\Big)} + \frac{9}{2}\int_0^u X_v^{(3)} dv, \end{aligned} \tag{2.13}$$

where $\eta^{(1)}$, $\eta^{(2)}$, and $\eta^{(3)}$ denote three dependent Brownian motions with quadratic variation. We can use these equations to find the infinitesimal generator of $(X_u^{(1)}, X_u^{(2)}, X_u^{(3)})$ by an extension of the infinitesimal generator of geometric Brownian motion extended to three dimensions given by

$$\begin{aligned} Af(x_1,x_2,x_3) &= Af(x_1) + Af(x_2) + Af(x_3) \\ &\quad + (1)(4)x_1x_2 f_{x_1x_2} + (1)(9)x_1x_3 f_{x_1x_3} + (4)(9)x_2x_3 f_{x_2x_3}. \end{aligned} \tag{2.14}$$

Where $Af(x_\alpha)$ is given by:

$$Af(x_\alpha) = \frac{1}{2}(\vartheta_\alpha^2 x_\alpha^2)\frac{\partial^2}{\partial x_\alpha^2} + r_\alpha x_\alpha \frac{\partial}{\partial x_\alpha}, \tag{2.15}$$

where ϑ and r can be derived generally using the general form of $X_u^{(\alpha)}$ we derived in (2.11) since $Af(x_\alpha)$ is the generator of a geometric Brownian motion whose form is the stochastic differential equation

$$dX_u^{(\alpha)} = \vartheta_\alpha X_u^{(\alpha)} d\eta_u^{(\alpha)} + r_\alpha X_u^{(\alpha)} du. \tag{2.16}$$

We write now (2.11) in differential form and we have:

$$X_u^{(\alpha)} = \int_0^u d\eta_v^{(\alpha)} \sqrt{\left(1 + \alpha^2 \big(X_v^{(\alpha)}\big)^2\right)} + \frac{\alpha^2}{2}\int_0^u X_v^{(\alpha)} dv$$
$$dX_u^{(\alpha)} = \left(\sqrt{\left(1 + \alpha^2 \big(X_v^{(\alpha)}\big)^2\right)}\right) d\eta_v^{(\alpha)} + \frac{\alpha^2}{2} X_u^{(\alpha)} du.$$

In this form, it is clear that the generalized values of $\vartheta_\alpha x_\alpha$, r_α, and $Af(x_\alpha)$ are

$$\vartheta_\alpha x_\alpha = \sqrt{1 + \alpha^2 \big(X_u^{(\alpha)}\big)^2}, \quad r_\alpha = \frac{\alpha^2}{2}, \quad \alpha = 1, 2, 3; \tag{2.17}$$

$$Af(x_\alpha) = \frac{1}{2}(1 + \alpha^2 x_\alpha^2)\frac{\partial^2}{\partial x_\alpha^2} + \frac{x_\alpha \alpha^2}{2}\frac{\partial}{\partial x_\alpha}. \tag{2.18}$$

By plugging in $\alpha = 1, 2, 3$ we get a generator for each component of the three-dimensional joint distribution.

$$Af(x_1) = \frac{1}{2}(1 + x_1^2)\frac{\partial^2}{\partial x_1^2} + \frac{x_1}{2}\frac{\partial}{\partial x_1}, \tag{2.19}$$

$$Af(x_2) = \frac{1}{2}(1 + 4x_2^2)\frac{\partial^2}{\partial x_2^2} + 2x_2\frac{\partial}{\partial x_2}, \tag{2.20}$$

$$Af(x_3) = \frac{1}{2}(1 + 9x_3^2)\frac{\partial^2}{\partial x_3^2} + \frac{9x_3}{2}\frac{\partial}{\partial x_3}. \tag{2.21}$$

These can then be combined with (2.15) to give us the infinitesimal generator for our three-dimensional distribution, proving Lemma 2.3.

$$Af(x_1, x_2, x_3) = \frac{1}{2}\Bigg[(1 + x^2)\frac{\partial^2}{\partial x_1^2} + (1 + 4x_2^2)\frac{\partial^2}{\partial x_2^2} + (1 + 9x_3^2)\frac{\partial^2}{\partial x_3^2} + 4x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} \tag{2.22}$$
$$+ 9x_1x_3\frac{\partial^2}{\partial x_1 \partial x_3} + 36x_2x_3\frac{\partial^2}{\partial x_2 \partial x_3}\Bigg] + \frac{x_1}{2}\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} + \frac{9x_3}{2}\frac{\partial}{\partial x_3}.$$

□

Lemma 2.4. *Thus, the infinitesimal generator of* $(x_t^{(1)}, x_t^{(2)}, x_t^{(3)})$ *is*

$$\frac{1}{2}\Bigg[(1 + x^2)\frac{\partial^2}{\partial x_1^2} + (1 + 4x_2^2)\frac{\partial^2}{\partial x_2^2} + (1 + 9x_3^2)\frac{\partial^2}{\partial x_3^2} + 4x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} \tag{2.23}$$
$$+ 9x_1x_3\frac{\partial^2}{\partial x_1 \partial x_3} + 36x_2x_3\frac{\partial^2}{\partial x_2 \partial x_3}\Bigg] + \frac{x_1}{2}\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} + \frac{9x_3}{2}\frac{\partial}{\partial x_3}.$$

Proof. Recall from (2.7) that

$$(x_t^{(1)}, x_t^{(2)}, x_t^{(3)}) = \left(\sinh(B_t^{(1)}), \frac{1}{2}\sinh(2B_t^{(2)}), \frac{1}{3}\sinh(B_t^{(3)}) \right). \tag{2.24}$$

We can now undertake an approach similar to Lemma 2.3 to deduce the infinitesimal generator of $(x_t^{(1)}, x_t^{(2)}, x_t^{(3)})$. Let's begin by constructing a general form given by

$$x_t^{(\alpha)} = \frac{1}{\alpha}\sinh(B_t^{(\alpha)}). \tag{2.25}$$

We once again apply Itô's formula to retrieve the following

$$\begin{aligned} x_t^{(\alpha)} &= \int_0^t \cosh(B_v^{(\alpha)}) dB_v^{\alpha} + \frac{\alpha^2}{2}\int_0^t \sinh(B_v^{(\alpha)}) dv \\ &= \int_0^t \sqrt{\left(1 + (x_v^{(\alpha)})^2\right)} dB_v^{(\alpha)} + \frac{\alpha^2}{2}\int_0^t x_v^{(\alpha)} dv. \end{aligned} \tag{2.26}$$

It is clear to see that $x_t^{(\alpha)}$'s general form is equivalent to that of $X_u^{(\alpha)}$ in (2.11). Further calculation would prove repetitive as deriving the infinitesimal generator would be identical to the process discovered in Lemma 2.3. So the infinitesimal generator of $(x_t^{(1)}, x_t^{(2)}, x_t^{(3)})$ is

$$\begin{aligned} Af(x_1, x_2, x_3) = \frac{1}{2}\Bigg[&(1 + x^2)\frac{\partial^2}{\partial x_1^2} + (1 + 4x_2^2)\frac{\partial^2}{\partial x_2^2} + (1 + 9x_3^2)\frac{\partial^2}{\partial x_3^2} \\ &+ 4x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} + 9x_1x_3\frac{\partial^2}{\partial x_1 \partial x_3} + 36x_2x_3\frac{\partial^2}{\partial x_2 \partial x_3}\Bigg] + \frac{x_1}{2}\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} + \frac{9x_3}{2}\frac{\partial}{\partial x_3}. \end{aligned} \tag{2.27}$$

□

Returning to Proposition 2.2 we have now proven the equality of the infinitesimal generators between $(x_t^{(1)}, x_t^{(2)}, x_t^{(3)})$ *and* $(X_u^{(1)}, X_u^{(2)}, X_u^{(3)})$*, proving equality in law (2.5) in Theorem 2.1 We now continue on to the second portion of Theorem 2.1 for fixed* u*'s.*

Proposition 2.5 (Fixed u equality)**.** *Recall the content of (2.6) in Theorem 2.1 For the given three-dimensional process* $(X_u^{(1)}, X_u^{(2)}, X_u^{(3)})$*:*

$$\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)} \right) \stackrel{(law)}{=} \left(\beta^{(1)}_{\int_0^u dve^{2B_v}}, \beta^{(2)}_{\int_0^u dve^{4B_v}}, \beta^{(3)}_{\int_0^u dve^{6B_v}} \right).$$

Alternatively, the density function of the triples above should be equal.

Proof. First, recall the result of Proposition 2.2 gives us an easy density function to calculate via hyperbolic Brownian functionals inspired by Alili [1] and Hariya [3]:

$$\left(\sinh(B_t^{(1)}), \frac{1}{2}\sinh(2B_t^{(2)}), \frac{1}{3}\sinh(B_t^{(3)}) \right) \stackrel{(law)}{=} \left(\beta^{(1)}_{\int_0^u dve^{2B_v}}, \beta^{(2)}_{\int_0^u dve^{4B_v}}, \beta^{(3)}_{\int_0^u dve^{6B_v}} \right). \tag{2.28}$$

The general form of the expectation of the density in the one-dimensional case of our distribution is given by:

$$p_u(x) = E\left[\frac{1}{\sqrt{2\pi A_t}} \exp\left(\frac{-x^2}{2A_t} \right) \right]. \tag{2.29}$$

We are also aware that density functions are multiplicative, implying

$$p_u(x, y, z) = p_u(x)p_u(y)p_u(z). \tag{2.30}$$

Since Vakeroudis [4] gives us $p_u(x, y)$ we simply need to calculate the density, $p_u(z)$, and multiply as shown below.

$$p_u(z) = E\left[\frac{1}{\sqrt{2\pi\left(\int_0^u dve^{6B_v}\right)}}\exp\left(\frac{-z^2}{2\int_0^u dve^{6B_v}}\right)\right]. \tag{2.31}$$

Substituting in $p_u(x, y)$ and $p_z(x, y)$ we get the density of $\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)}\right)$:

$$p_u(x, y, z) = p_u(x, y)p_u(z)$$

$$\begin{aligned} p_u(x,y,z) &= E\left[\frac{1}{2\pi\sqrt{2\pi}}\frac{\exp\left(\frac{-x^2}{2\int_0^u dve^{2B_v}}\right)}{\sqrt{\int_0^u dve^{2B_v}}}\frac{\exp\left(\frac{-y^2}{2\int_0^u dve^{2=4B_v}}\right)}{\sqrt{\int_0^u dve^{4B_v}}}\right]E\left[\frac{\exp\left(\frac{-z^2}{2\int_0^u dve^{6B_v}}\right)}{\sqrt{2\pi\left(\int_0^u dve^{6B_v}\right)}}\right] \\ &= E\left[\frac{1}{2\pi\sqrt{2\pi}}\frac{\exp\left(\frac{-x^2}{2\int_0^u dve^{2B_v}}\right)}{\sqrt{\int_0^u dve^{2B_v}}}\frac{\exp\left(\frac{-y^2}{2\int_0^u dve^{2=4B_v}}\right)}{\sqrt{\int_0^u dve^{4B_v}}}\frac{\exp\left(\frac{-z^2}{2\int_0^u dve^{6B_v}}\right)}{\sqrt{\int_0^u dve^{6B_v}}}\right], \end{aligned} \tag{2.32}$$

where $p_u(x, y, z)$ is the density function of the triple $\left(X_u^{(1)}, X_u^{(2)}, X_u^{(3)}\right)$. This proves Proposition 2.5, wrapping up our extension of Bougerol's identity in the third dimension. □

□

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