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Uniqueness of the infinite connected component for the vacant set of random interlacements on amenable transient graphs*

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Abstract

We prove the uniqueness of the infinite connected component for the vacant set of random interlacements on general vertex-transitive amenable transient graphs. Our approach is based on connectedness of random interlacements and differs from the one used by Teixera [10] to prove the uniqueness of the infinite connected component for the vacant set of random interlacements on $\mathbb{Z}^d$.

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1 Introduction

The model of random interlacements was introduced by Sznitman in [8] on lattices $\mathbb{Z}^d$ ($d \geq 3$) and defined on arbitrary transient graphs by Teixeira in [9]. The vacant set of random interlacements has been an important example of percolation model with strong, algebraically decaying correlations, see e.g. [3, 4, 5] for comprehensive literature review. In this paper we study the vacant set of random interlacements on infinite vertex-transitive amenable transient graphs.

Let $G = (V, E)$ be a locally finite graph. G is *vertex-transitive* if for any $x, y \in V$ there exists a graph automorphism of G, which maps x to y. G is *amenable* if the vertex isoperimetric constant $\kappa_V(G)$, defined by

$$\kappa_V(G) = \inf\left\{\frac{|\partial A|}{|A|} \,:\, |A| < \infty\right\},$$

is equal to 0, where $\partial A = \{x \in A \,:\, (x,y) \in E \text{ for some } y \notin A\}$ is the (inner) vertex boundary of A. G is *transient* if a simple random walk on G is transient.

Random interlacements on G is the range of a Poisson cloud of doubly-infinite random walks on G, whose density is controlled by a parameter $u > 0$. We postpone its precise

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definition to Section 2 and just mention here that the law of random interlacements $\mathcal{I}^u$—as a random subset of V—is uniquely characterized by the relations

$$\mathbb{P}[\mathcal{I}^u \cap K = \emptyset] = e^{-u\mathrm{cap}(K)}, \quad \text{for every finite } K \subset V, \tag{1.1}$$

where $\mathrm{cap}(K)$ stands for the random walk capacity of K. The *vacant set of random interlacements at level* u is defined as the complement of $\mathcal{I}^u$:

$$\mathcal{V}^u = V \setminus \mathcal{I}^u.$$

Our main result concerns the number of infinite connected components of $\mathcal{V}^u$.

Theorem 1.1. *Let G be a vertex-transitive amenable transient graph. For any $u > 0$, the number of infinite connected components in $\mathcal{V}^u$ is either a.s. equal to 0 or a.s. equal to 1.*

The main challenge to prove uniqueness for the vacant set of random interlacements is lack of the so-called *finite energy property*. It is not clear how to overcome this difficulty using only the description of random interlacements by (1.1). In the previous works [8, 10], the characterization of random interlacements as the range of a Poisson cloud of random walks (see (2.7)) and careful rerouting of the random walks enabled to overcome the obstruction, but the method was very sensitive to geometry of the ambient graph. In particular, the result of Theorem 1.1 was obtained by Teixeira in [10] in the case of $\mathbb{Z}^d$ ($d \geq 3$), but his proof uses the structure of $\mathbb{Z}^d$ in a crucial way and cannot be extended to general graphs. To explain the novelty of our approach, let us briefly discuss the common strategy for the proof of uniqueness.

Let N be the number of infinite connected components in $\mathcal{V}^u$. By ergodicity (see [11] and Proposition 2.1), N is constant almost surely. If $N = k$ a.s. for some $2 \leq k < \infty$, then there is a sufficiently large finite set K, which intersects all k infinite components with positive probability. If one can change such configurations locally at “finite cost” to merge all the infinite components into one, then one obtains $N = 1$ with positive probability and arrives at a contradiction. If $N = \infty$ a.s., the Burton-Keane argument [2] leads to a contradiction by showing that there is a positive density of so-called trifurcations—locations where an infinite component locally splits into at least 3 infinite components. The existence of trifurcations is also proved by a local modification argument on the paths that intersect a large finite set K. In previous works [8, 10], the behavior of the random walk paths that visit K was modified *inside* K in order to assemble a desired configuration in K, which is very sensitive to the local geometry of the ambient graph. Our method is based on the *connectedness* of $\mathcal{I}^u$ (see [11] and Proposition 2.1); more precisely, we reroute the random walks from the Poisson cloud that ever visit K locally between their first and last visit to K through the random interlacement $\mathcal{I}^u$ *outside* of K, see the proofs of Proposition 3.3 and Lemma 3.5. In [7], we use similar local modification in the proof of uniqueness of the infinite connected component for the vacant set of Brownian interlacements.

An immediate application of Theorem 1.1 is the continuity of the percolation function $\theta(u)$—the probability that the connected component of a given vertex in $\mathcal{V}^u$ is infinite—in the supercritical phase of the vacant set of random interlacements. The proof follows a by now standard argument of van den Berg and Keane [1] (see e.g. [10, Corollary 1.2] for an application to the vacant set of random interlacements on $\mathbb{Z}^d$).

The rest of the paper is organized as follows. Section 2 contains definition of random interlacements point process as a Poisson point process on the space of doubly infinite paths (see Section 2.3) as well as some useful sampling procedure for the paths that visit a finite set (see (2.5) and (2.6)). The random interlacements at level u is defined in (2.7) as the range of all paths from the interlacement point process. We prove Theorem 1.1 in Section 3. A key ingredient for the proof is Proposition 3.1, in which we strengthen the result of Teixeira and Tykesson (see Proposition 2.1) about connectedness of random interlacements.

2 Random interlacements

In this section we define random interlacements on G precisely. For further details and proofs we refer the reader to [9].

Let W_+ be the space of nearest neighbor paths $w : \mathbb{N}_0 \to V$ on G such that $w(n) \to \infty$ for $n \to \infty$[1]. We denote by X_n, $n \geq 0$, the canonical process on W_+ (i.e. $X_n(w) = w(n)$) and by $\mathcal{W}_+$ the sigma-algebra on W_+ generated by the canonical process. Since G is transient, the law of the simple random walk on G started from $x \in V$, denoted by P_x, is a probability measure on $(\mathsf{W}_+, \mathcal{W}_+)$.

We write $p_n(x, x') = \mathsf{P}_x[X_n = x']$ for the n-step transition probability of the random walk and $g(x, x') = \sum_{n=0}^{\infty} p_n(x, x')$ for the respective Green function.

For a finite set K in V, we define the equilibrium measure of K by

$$e_K(x) = \mathsf{P}_x\big[X_n \notin K \text{ for all } n \geq 1\big].$$

Note that e_K is supported on ∂K. The total mass of e_K is called the capacity of K and is denoted by $\mathrm{cap}(K)$. Let $L_K = \sup\{n \geq 0 \,:\, X_n \in K\}$ be the last visit time in K. Then the joint law of L_K and X_{L_K} (under P_x) is given by

$$\mathsf{P}_x[L_K = n, X_{L_K} = y] = p_n(x, y) e_K(y). \tag{2.1}$$

For $x \in \partial K$, we denote by P_x^K the law of the simple random walk starting from x and conditioned on staying outside of K for all $n \geq 1$. We denote by $\mathsf{P}_{x,y}^n$ the random walk bridge measure in time $n \geq 0$ from x to y. Then

$$\begin{gathered}\text{under } \mathsf{P}_x, \text{ conditionally on } L_K = n \text{ and } X_{L_K} = y \in \partial K, \text{ the processes} \\ (X_s)_{0 \leq s \leq n} \text{ and } (X_{L_K+s})_{s \geq 0} \text{ are independent and have laws } \mathsf{P}_{x,y}^n \text{ resp. } \mathsf{P}_y^K.\end{gathered} \tag{2.2}$$

2.1 Compatible measures on doubly-infinite paths

Let W be the space of doubly-infinite nearest neighbor paths $w : \mathbb{Z} \to V$ tending to infinity at positive and negative infinite times. We denote by X_n, $n \in \mathbb{Z}$, the canonical process on W (i.e. $X_n(w) = w(n)$) and by $\mathcal{W}$ the sigma-algebra on W generated by the canonical process. We denote the canonical time shift on W by θ_n, $n \in \mathbb{Z}$. For a finite set $K \subset V$, we define the first entrance time of $w \in \mathsf{W}$ in K as $H_K(w) = \inf\{n \in \mathbb{Z} \,:\, X_n(w) \in K\}$, and write

$$\mathsf{W}_K = \{w \in \mathsf{W}\,;\, H_K(w) < \infty\}, \quad \mathsf{W}_K^0 = \{w \in \mathsf{W} \,:\, H_K(w) = 0\}$$

for the sets of paths that ever visit K, resp., visit K for the first time at time 0.

Consider the following measure on W_K^0:

$$Q_K\big[(X_{-n})_{n \geq 0} \in A,\, X_0 = x,\, (X'_n)_{n' \geq 0} \in A'\big] = \mathsf{P}_x^K[A]\, e_K(x)\, \mathsf{P}_x[A'], \quad A, A' \in \mathcal{W}_+. \tag{2.3}$$

The measures Q_K are compatible, in the sense that $Q_K = \theta_{H_K} \circ \big(\mathbb{1}_{\mathsf{W}_K} Q_{K'}\big)$, for any finite sets K and K' with $K \subseteq K'$, see e.g. the proof of [9, Theorem 2.1]. Note that Q_K is a finite measure with $Q_K[\mathsf{W}_K^0] = \mathrm{cap}(K)$.

For $K \subseteq V$ and $w \in \mathsf{W}_K$, we define the last visit time of w in K by

$$L_K(w) = \sup\{n \in \mathbb{Z} \,:\, X_n(w) \in K\}.$$

[1] $w(n) \to \infty$ if for any finite set $K \subset V$ there exists n_K such that $w(n) \notin K$ for all $n \geq n_K$.

By (2.1), (2.2) and (2.3), for any finite K and $A, A' \in \mathcal{W}_+$,

$$Q_K\big[(X_{-s})_{s\geq 0} \in A,\, (X_s)_{0\leq s\leq L_K} \in \cdot,\, (X_{L_K+s})_{s\geq 0} \in A'\big] \\ = \sum_{x,x'\in\partial K} e_K(x)\, e_K(x')\, g(x,x')\, \mathsf{P}^K_x[A] \Big(\sum_{n=0}^{\infty} \frac{p_n(x,x')}{g(x,x')} \mathsf{P}^n_{x,x'}[\cdot]\Big) \mathsf{P}^K_{x'}[A'], \quad (2.4)$$

where $(X_s)_{0\leq s\leq L_K}$ is viewed as a random element on the space W_{fin} of nearest neighbor paths of finite duration. The identity (2.4) states that under Q_K, the pieces of the random path before the first entrance time in K, after the last visit time in K, and between those times are conditionally independent, given the locations of the first and last visits of the path in K.

2.2 Random interlacement measure

We now define a suitable sigma-finite measure on doubly-infinite paths, whose restriction to every W_K is Q_K.

Two paths w and w' in W are called equivalent, if $w' = \theta_n(w)$ for some $n \in \mathbb{Z}$. The quotient set of W modulo this equivalence relation is denoted by W^*. The canonical projection $\pi^* : \mathsf{W} \to \mathsf{W}^*$ induces the sigma-algebra $\mathcal{W}^* = \{A \subseteq \mathsf{W}^* \,:\, (\pi^*)^{-1}(A) \in \mathcal{W}\}$ on W^*. For a finite set K in V, we denote by W^*_K the image of W_K under π^*. Note that π^* maps bijectively W^0_K onto W^*_K.

By [9, Theorem 2.1], there exists a unique sigma-finite measure ν on $(\mathsf{W}^*, \mathcal{W}^*)$, whose restriction to any W^*_K coincides with Q_K, more precisely,

$$\mathbb{1}_{\mathsf{W}^*_K}\, \nu = \pi^* \circ Q_K, \quad \text{for any finite } K \subset V.$$

Note that $\nu[\mathsf{W}^*_K] = Q_K[\mathsf{W}_K] = \mathrm{cap}(K)$.

2.3 Random interlacement point process

Consider the space of point measures

$$\Omega = \Big\{\omega = \sum_{i\geq 1} \delta_{(w^*_i, u_i)} \,:\, \omega\big(\mathsf{W}^*_K \times [0,u]\big) < \infty \text{ for all finite } K \subset V \text{ and } u > 0\Big\}$$

on $\mathsf{W}^* \times \mathbb{R}_+$, endowed with the sigma-algebra $\mathcal{A}$ generated by the evaluation maps

$$\omega \mapsto \omega(E), \quad E \in \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+),$$

and denote by $\mathbb{P}$ the Poisson point measure on $\mathsf{W}^* \times \mathbb{R}_+$ with intensity $\nu \otimes du$; the random point measure with law $\mathbb{P}$ is called the *random interlacement point process on G*.

The random variable

$$N_{K,u} = N_{K,u}(\omega) = \omega\big(\mathsf{W}^*_K \times [0,u]\big),$$

which counts the number of trajectories with labels $\leq u$ (in ω) that visit K, has Poisson distribution with parameter $u\mathrm{cap}(K)$.

For any finite $K \subset V$, given $N_{K,u} = n$, the n trajectories of the random interlacement point process that visit K and have labels $\leq u$ are independent random elements of W^*_K with the common distribution $\frac{1}{\mathrm{cap}(K)}\big(\pi^* \circ Q_K\big)$, whose labels are independent uniformly distributed on $[0,u]$. By (2.4), each of them can be sampled (independently) as follows:

- Sample the locations of the first entrance and last visit in K, (X_i, X'_i), from the distribution
$$\frac{1}{\mathrm{cap}(K)}\, g(x,x')\, e_K(x)\, e_K(x'), \quad x, x' \in \partial K; \quad (2.5)$$

- Given $X_i = x_i$ and $X_i' = x_i'$, sample independently random paths γ_i and γ_i' in W_+ and $\widetilde{\gamma}_i$ in W_{fin} respectively from the distributions

$$\mathsf{P}^K_{x_i}, \quad \mathsf{P}^K_{x_i'}, \quad \sum_{n=0}^{\infty} \frac{p_n(x,x')}{g(x,x')} \mathsf{P}^n_{x,x'}[\cdot]; \tag{2.6}$$

- Let w_i be the concatenation of the time reversal of γ_i, $\widetilde{\gamma}_i$ and γ_i', so that $w_i(n) = \gamma_i(-n)$ for $n \leq 0$. (Note that w_i is a random path in W^0_K with the law $\frac{1}{\mathrm{cap}(K)} Q_K$.)
- To get the desired random element of $\mathsf{W}^*_K \times [0,u]$, we project w_i onto W^* and assign it an independent label from the uniform distribution on $[0,u]$.

For any $u > 0$, the random point measure ι^u on W^*, defined by

$$\omega = \sum_{i \geq 1} \delta_{(w_i^*, u_i)} \ \mapsto \ \iota^u(\omega) = \sum_{i \geq 1:\, u_i \leq u} \delta_{w_i^*},$$

is called *random interlacement point process on G at level u*. Note that (under $\mathbb{P}$) ι^u is a Poisson point process on W^* with intensity $u\nu$.

2.4 Random interlacement

For any $u > 0$, the *random interlacement at level u* is defined as

$$\mathcal{I}^u(\omega) = \bigcup_{i \geq 1:\, u_i \leq u} \mathrm{range}(w_i^*), \quad \omega = \sum_{i \geq 1} \delta_{(w_i^*, u_i)} \in \Omega, \tag{2.7}$$

where $\mathrm{range}(w^*) = \bigcup_{n \in \mathbb{Z}} w(n)$ for $w^* \in \mathsf{W}^*$ and any $w \in \pi^{-1}(w^*)$. The complement of $\mathcal{I}^u$ is called the *vacant set (of random interlacement) at level u* and is denoted by $\mathcal{V}^u$.

Any random interlacement $\mathcal{I}^u$ is a measurable map from $(\Omega, \mathcal{A})$ to $(\Sigma, \mathcal{F})$, where Σ is the set of all subsets of V and $\mathcal{F}$ is the sigma-algebra on Σ generated by the π-system $\big\{\{F \in \Sigma : F \cap K = \emptyset\},\, K \subset V \text{ finite}\big\}$. The law of $\mathcal{I}^u$ on $(\Omega, \mathcal{A}, \mathbb{P})$ is the probability measure Q^u on $(\Sigma, \mathcal{F})$ uniquely determined by the identities

$$Q^u\big[\{F \in \Sigma : F \cap K = \emptyset\}\big] = \mathbb{P}[\mathcal{I}^u \cap K = \emptyset] = e^{-u\,\mathrm{cap}(K)}, \quad K \subset V \text{ finite}.$$

Finally, we recall some results from [11].

Proposition 2.1. *Let G be a vertex-transitive amenable transient graph. Let $\mathrm{Aut}(G)$ be the group of automorphisms on G. The following statements hold for every $u > 0$.*

- *([11, (40)]) $\mathrm{Aut}(G)$ is a measure preserving ergodic flow on $(\Sigma, \mathcal{F}, Q^u)$.*
- *([11, Theorem 3.3]) $\mathcal{I}^u$ is connected almost surely.*

3 Proof of Theorem 1.1

Let $u > 0$ be fixed and let N be the number of infinite connected components in $\mathcal{V}^u$. By Proposition 2.1, N is constant almost surely. We prove that $N \in \{0,1\}$ a.s. by ruling out separately the two cases $N = k$ for some $2 \leq k < \infty$ (in Proposition 3.3) and $N = \infty$ a.s. (in Proposition 3.6). In Lemma 3.5, we prove that $N = \infty$ a.s. implies a positive density of trifurcations (see Definition 3.4). Our approach (in the proofs of Proposition 3.3 and Lemma 3.5) is based on a novel local modification procedure to reroute random walk paths that visit a finite set K through the random interlacements outside of K. A key auxiliary result about a connectedness of random interlacements outside any finite set K is given in Proposition 3.1. Some common notation that we use in the proofs are collected in Section 3.1.

3.1 Random interlacements outside finite set

In this section we introduce notation that we use in the proofs of Proposition 3.3 and Lemma 3.5 and prove a result about connectedness of random interlacement outside finite sets, which is key to justify local moditications in Proposition 3.3 and Lemma 3.5.

Let $u > 0$ and finite $K \subset V$ be fixed.

Let ι^u be a random interlacement point process at level u. We decompose ι^u into the point process ι' of trajectories that visit K and ι'' of trajectories that do not visit K. Note that ι' and ι'' are independent.

Recall the sampling procedure for interlacement trajectories that visit K from Section 2.4. Let N_K be the number of trajectories in ι'. On the event $\{N_K = n\}$,

- let (X_i, X_i'), $1 \le i \le n$, be the locations of the first and last visits to K of the interlacement trajectories ι'; and
- let $(\gamma_i, \widetilde{\gamma}_i, \gamma_i')$, $1 \le i \le n$, be the three fragments of the interlacement trajectories ι', respectively, the time reversal of the part before the first entrance in K, the part between the first entrance and the last visit in K, and the part after the last visit in K.

Given $N_K = n$, (X_i, X_i'), $1 \le i \le n$, are i.i.d. with distribution (2.5), and given their locations on ∂K, $(\gamma_i, \widetilde{\gamma}_i, \gamma_i')$, $1 \le i \le n$, are conditionally independent with law (2.6). We define

$$\mathcal{I}_{K,n} = \bigcup_{w^* \in \iota''} \text{range}(w^*) \cup \bigcup_{i=1}^{n} \text{range}(\gamma_i) \cup \bigcup_{i=1}^{n} \text{range}(\gamma_i')$$

and $\mathcal{V}_{K,n} = V \setminus \mathcal{I}_{K,n}$. (Note that, given $N_K = n$, $\mathcal{I}^u = \mathcal{I}_{K,n} \cup \bigcup_{i=1}^{n} \text{range}(\widetilde{\gamma}_i)$.)

Proposition 3.1. *For any $u > 0$, finite $K \subset V$ and $n \in \mathbb{N}$, given $N_K = n$, $\mathcal{I}_{K,n}$ is connected almost surely.*

Proof. Let A be the event that $\mathcal{I}_{K,N_K}$ is connected. We prove that $\mathbb{P}[A] = 1$.

For $\omega = \sum_{i \ge 1} \delta_{(w_i^*, u_i)}$ and $0 \le u' < u''$, we denote by $\iota^{u',u''}$ the random interlacement point process with labels between u' and u'':

$$\iota^{u',u''} = \iota^{u',u''}(\omega) = \sum_{i \ge 1:\, u_i \in [u',u'']} \delta_{w_i^*}.$$

Note that $\iota^{u',u''}$ is independent from $\iota^{u'} (= \iota^{0,u'})$ and has the same distribution as $\iota^{u''-u'}$.

For $\varepsilon > 0$, we write the random interlacement point process ι^u as the sum of independent interlacement point processes $\iota^{u-\varepsilon} + \iota^{u-\varepsilon,u} =: \hat{\iota} + \check{\iota}$.

Let $\hat{N}_K$ be the number of trajectories of $\hat{\iota}$ that visit K. Let $\hat{X}_i, \hat{X}_i' \in \partial K$, $1 \le i \le \hat{N}_K$, be the locations of the first resp. last visit of these trajectories to K and let $\hat{\gamma}_i$ and $\hat{\gamma}_i'$ be the time reversed past of the trajectories before the first visit in K resp. the future of the trajectories after the last visit in K. By (2.6), given $\hat{N}_K$ and $\{(\hat{X}_i, \hat{X}_i'), 1 \le i \le \hat{N}_K\}$, the paths $\hat{\gamma}_i$ and $\hat{\gamma}_i'$ are all independent and distributed as simple random walks conditioned on never returning to K; furthermore, they are independent from $\check{\iota}$.

Since $\mathbb{P}[v \in \check{\iota}] = c(\varepsilon) > 0$, an independent simple random walk (with arbitrary starting point) hits the range of $\check{\iota}$ almost surely. Since simple random walk on G is transient, also an independent simple random walk conditioned on never returning to K hits the range of $\check{\iota}$ almost surely. Thus, each $\hat{\gamma}_i$ and $\hat{\gamma}_i'$ hit the range of $\check{\iota}$ almost surely. We denote this event by $\hat{A}$.

Let G be the event that none of the trajectories from $\check{\iota}$ visits K. Note that $\mathbb{P}[G] \xrightarrow[\varepsilon \to 0]{} 1$. Furthermore, since the trace of $\check{\iota}$ is connected almost surely, $\hat{A} \cap G \subseteq A$. Thus, for every $\varepsilon \in (0, u)$, $\mathbb{P}[A] \ge \mathbb{P}[\hat{A} \cap G] = \mathbb{P}[G]$, which gives $\mathbb{P}[A] = 1$. The proof is completed. □

Remark 3.2. An immediate corollary from Proposition 3.1 is that for every $u > 0$ and finite $K \subset V$, $\mathcal{I}^u \setminus K$ contains exactly one infinite connected component almost surely.

3.2 Number of infinite components is 0, 1 or ∞

In this section we rule out the case $N = k$ a.s. for some $2 \le k < \infty$.

Proposition 3.3. *$\mathbb{P}[N = k] = 0$ for all $2 \le k < \infty$.*

Proof. Assume on the contrary that $N = k$ a.s. for some $2 \le k < \infty$.

Fix a finite $K \subset V$ such that $\mathrm{int}(K) = K \setminus \partial K$ is connected and intersects all k infinite connected components of $\mathcal{V}^u$ with positive probability. Denote this event by E_K.

Fix n such that the probability of event $E_{K,n} = E_K \cap \{N_K = n\}$ is positive. On the event $E_{K,n}$, the infinite connected component of $\mathcal{V}_{K,n}$ is unique and contains $\mathrm{int}(K)$.

By Proposition 3.1, $\mathcal{I}_{K,n}$ is connected almost surely. Thus, every X_i is connected to X'_i, for $1 \le i \le n$, by a path in $\mathcal{I}_{K,n} \cap K'$ for some large enough finite $K' \supseteq K$. Denote by $E_{K,n,K'}$ the event that $E_{K,n}$ occurs and every X_i is connected to X'_i, for $1 \le i \le n$, inside $\mathcal{I}_{K,n} \cap K'$. Then $E_{K,n,K'}$ is $\sigma(N_K, \mathcal{I}_{K,n}, \{(X_i, X'_i)\}_{1 \le i \le n})$-measurable and $\mathbb{P}[E_{K,n,K'}] > 0$ for all large enough finite $K' \subset V$. Fix such K'. There exists $I \subset K' \setminus \mathrm{int}(K)$ and $x_i, x'_i \in \partial K$, for $1 \le i \le n$, such that

$$\mathbb{P}\big[E_{K,n,K'}, \mathcal{I}_{K,n} \cap K' = I,\, (X_i, X'_i) = (x_i, x'_i) \text{ for all } 1 \le i \le n\big] > 0.$$

Note that every x_i is connected to x'_i inside of I.

Let G be the event that every bridge $\widetilde{\gamma}_i$ is a simple path in I from X_i to X'_i, for $1 \le i \le n$. Then there exists $c = c(n, K, K') > 0$ such that

$$\mathbb{P}\big[G \,|\, N_K = n,\, \mathcal{I}_{K,n} \cap K' = I,\, (X_i, X'_i) = (x_i, x'_i) \text{ for all } 1 \le i \le n\big] \ge c > 0.$$

Hence

$$\mathbb{P}\big[G,\, E_{K,n,K'}, \mathcal{I}_{K,n} \cap K' = I,\, (X_i, X'_i) = (x_i, x'_i) \text{ for all } 1 \le i \le n\big] > 0.$$

However, if this event occurs, then $\mathcal{V}^u = \mathcal{V}_{K,n}$, which implies that $\mathcal{V}^u$ contains a unique infinite connected component. Thus, we have shown that $\mathbb{P}[N = 1] > 0$, which contradicts the initial assumption. The proof is completed. □

3.3 Ruling out infinitely many infinite components

To rule out the possibility of infinitely many infinite connected components in $\mathcal{V}^u$ we follow the classical Burton-Keane argument with a bit more general notion of trifurcation. For $x \in V$, let $B(x,t)$ be the ball of radius t centered in x and denote by $\mathcal{C}_{x,t}$ the connected component of x in $\mathcal{V}^u \cap B(x,t)$.

Definition 3.4. *Let $t > 0$. We say that $x \in V$ is a t-*trifurcation *if there is an infinite connected component $\mathcal{C}$ of the vacant set $\mathcal{V}^u$, such that*

- *(a) $x \in \mathcal{C}$;*
- *(b) $\mathcal{C} \setminus \mathcal{C}_{x,t}$ contains at least 3 infinite connected components.*

Lemma 3.5. *Assume that $N = \infty$ a.s. Then there exists $t > 0$, such that (for any $x \in V$)*

$$\mathbb{P}\big[x \text{ is a } t\text{-trifurcation}\big] > 0.$$

Proof. The proof is very similar to the proof of Proposition 3.3.

Let $x \in V$. Fix a finite $K \subset V$ such that $\mathrm{int}(K) = K \setminus \partial K$ is connected, contains x, and intersects at least 3 infinite connected components of $\mathcal{V}^u$ with positive probability.

Denote this event by E_K. Fix n such that the probability of event $E_{K,n} = E_K \cap \{N_K = n\}$ is positive.

Since the bridges $\widetilde{\gamma}_i$ have finite range, if $E_{K,n}$ occurs, then $\mathcal{V}_{K,n} \setminus K' = \mathcal{V}^u \setminus K'$ for all large enough finite $K' \supseteq K$; in particular, $\mathcal{V}_{K,n}$ contains an infinite component $\mathcal{C}'$, such that $\mathrm{int}(K) \subset \mathcal{C}'$ and $\mathcal{C}' \setminus K'$ contains at least 3 infinite connected components. Let $E_{K,n,K'}$ be the event that

(a) $N_K = n$;
(b) there is an infinite connected component $\mathcal{C}'$ in $\mathcal{V}_{K,n}$, such that $\mathrm{int}(K) \subset \mathcal{C}'$ and $\mathcal{C}' \setminus K'$ contains at least 3 infinite connected components;
(c) for all $1 \le i \le n$, X_i is connected to X_i' in $\mathcal{I}_{K,n} \cap K'$.

By Proposition 3.1, $\mathbb{P}[E_{K,n,K'}] > 0$ for all large enough finite $K' \supseteq K$. Now, exactly as in the proof of Proposition 3.3—by rerouting the bridges $\widetilde{\gamma}_i$ through $\mathcal{I}_{K,n}$—, we obtain that $\mathbb{P}[E_{K,n,K'}, \mathcal{I}_{K,n} = \mathcal{I}^u] > 0$. Thus, with positive probability, $\mathcal{V}^u$ contains an infinite connected component $\mathcal{C}$, such that $x \in \mathcal{C}$ and $\mathcal{C} \setminus K'$ contains at least 3 infinite connected components. Call this event $F_{x,n,K'}$. We claim that $F_{x,n,K'}$ implies that x is a t-trifurcation for all t large enough.

Assume that $F_{x,n,K'}$ occurs and let $\{\mathcal{C}_i\}_{i\in I}$ be all (≥ 3) the infinite connected components of $\mathcal{C} \setminus K'$. Choose t large enough, so that $\mathcal{C} \cap K' \subseteq \mathcal{C}_{x,t}$. Since $\mathcal{C}_i$'s are not connected in $\mathcal{C} \setminus K'$, they are also not connected in $\mathcal{C} \setminus \mathcal{C}_{x,t}$. Thus, $\mathcal{C} \setminus \mathcal{C}_{x,t}$ consists of at least 3 infinite connected components. Hence x is a t-trifurcation.

Finally, since $\mathbb{P}[F_{K,n,K'}] > 0$, x is a t-trifurcation with positive probability for some t. The proof is completed. □

Proposition 3.6. $\mathbb{P}[N = \infty] = 0$.

Proof. Assume on the contrary that $N = \infty$ a.s. and fix a t as in Lemma 3.5.

By arguing exactly as in the proof of [6, Theorem 2.4], we notice that for any finite set of t-trifurcations $\mathcal{T}$ of an infinite connected component $\mathcal{C}$ such that $\mathcal{C}_{x,t} \cap \mathcal{C}_{x',t} = \emptyset$ for all different $x, x' \in \mathcal{T}$, the set $\mathcal{C} \setminus \bigcup_{x\in\mathcal{T}} \mathcal{C}_{x,t}$ contains at least $|\mathcal{T}| + 2$ infinite connected components. Thus, an infinite component with j t-trifurcations in a finite set W, which are pairwise at distance at least $2t+1$ from each other, intersects $\partial W'$ in at least $j+2$ vertices, where $W' = \bigcup_{w\in W} B(w, t+1)$. Consequently, using the vertex-transitivity of G, the total number $\mathcal{T}(W)$ of t-trifurcations in W is at most $|B(w, 2t+1)|(|\partial W'| - 2)$, which is at most $|B(w, 2t+1)|\,|B(w, t+1)|\,|\partial W| =: c_1|\partial W|$. On the other hand, by Lemma 3.5 and the vertex-transitivity of G, $\mathbb{E}[\mathcal{T}(W)] = |W|\,\mathbb{P}[x \text{ is } t\text{-trifurcation}] = c_2|W|$ for some $c_2 > 0$. We conclude that for any finite $W \subset V$,

$$\frac{|\partial W|}{|W|} \geq \frac{c_2}{c_1} > 0,$$

thus also the vertex isoperimetric constant $\kappa_V(G)$ is positive. This contradicts with the assumption that G is amentable. The proof is completed. □

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