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The scaling limit of the weakly self-avoiding walk on a high-dimensional torus*

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Abstract

We prove that the scaling limit of the weakly self-avoiding walk on a d-dimensional discrete torus is Brownian motion on the continuum torus if the length of the rescaled walk is $o(V^{1/2})$ where V is the volume (number of points) of the torus and if $d > 4$. We also prove that the diffusion constant of the resulting torus Brownian motion is the same as the diffusion constant of the scaling limit of the usual weakly self-avoiding walk on $\mathbb{Z}^d$. This provides further manifestation of the fact that the weakly self-avoiding walk model on the torus does not feel that it is on the torus *up until* it reaches about $V^{1/2}$ steps, which we believe is sharp.

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1 Introduction and results

1.1 Introduction

A self-avoiding walk (SAW) on any graph is a path on this graph that does not visit any vertex more than once. Historically and still today, most of the research on this topic focuses on the case where the graph is a lattice and especially the Euclidean lattice $\mathbb{Z}^d$ where the dimension d plays a key role in the behaviour of this model. In dimension $d > 4$ the model is now very well understood following the introduction of the *lace-expansion* by Brydges and Spencer in 1985 [5] in the context of the closely related *weakly self-avoiding walk* (WSAW). Hara and Slade developed the lace-expansion to prove that the usual self-avoiding walk model has *mean-field* behaviour in dimension $d \geq 5$ in the early 90's [9, 11, 10] which means in short that the SAW behaves closely like the simple random walk, sharing for example the same critical exponents. It was shown in this context [11, 22] that the scaling limit of the SAW on $\mathbb{Z}^d$ is Brownian motion with some diffusion constant $D > 1$. The scaling limit is also believed to be Brownian motion at the critical dimension $d_c = 4$ but with extra $\log$ corrections to the rescaling factors [6]. Relatively recent rigorous results in that direction have been obtained through renormalization group arguments for the WSAW in [1, 2]. In lower dimensions the picture is quite different. In two dimension, the scaling limit of the SAW is conjecturally the so-called Schramm-Loewner Evolution with parameter $\kappa = 8/3$. And in fact this is

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known to be the case under the widely open assumption that the scaling limit exists and is conformally invariant following the breakthrough work of Lawler, Schramm and Werner [15]. In three dimensions there seems to be no good candidate as of today for the scaling limit and the model more generally remains poorly understood even from the physics standpoint.

Here we study the WSAW model where strictly SAW are assigned weight 1 and other walks get penalized for intersecting and have a smaller but non-zero weight. A more formal definition is given after. It is widely acknowledged even though conjectural that the asymptotic behaviour of the WSAW and of the SAW are the same in all dimensions and that both models lie in the same *universality class*. We are interested in the model on a d-dimensional torus (a box with periodic boundary conditions) of sidelength $r \geq 3$ where $d > 4$. It was pointed out and partly proven in [18] that the WSAW on the torus behaves the same as its $\mathbb{Z}^d$ counterpart provided that its length is less than $r^{d/2}$ and that this should be sharp. In this article we show that the scaling limit of the WSAW on the torus is Brownian motion on the continuum torus if the walk that we rescale has length much smaller than $r^{d/2}$ and much larger than r^2, which is possible only if $d > 4$. Our main tool to prove the above is the lace expansion. Since its introduction in [5], the lace expansion technology has been simplified and explained by several authors [16, 23, 4] (to which we refer for further background) as well as applied to a broad range of models (percolation, lattice trees and lattice animals, Ising, $|\varphi|^4$, etc. [7, 8, 19, 20]) above their critical dimensions.

1.2 Notation

We write $f \sim g$ to mean $\lim f/g = 1$, and use the usual Landau notations o and O. We also write for non-negative f and g that $f \ll g$ if $f/g = o(1)$ and $f \gg g$ if $g \ll f$ and that $f \asymp g$ if both $f/g = O(1)$ and $g/f = O(1)$. For any vector $u \in \mathbb{R}^d$, $|u|$ denotes the Euclidean norm of u. In what follows we repeatedly make use of the mention "$\beta > 0$ sufficiently small" which means that the variable β is strictly between 0 and some $\beta_0(d) > 0$ where β_0 is allowed to depend on d only, and is not allowed to depend on n or r. Constants in general are permitted to depend on d.

1.3 The weakly self-avoiding walk

An n-step walk on a graph $G = (V, E)$ is a function $\omega : \{0, 1, \ldots, n\} \to V$ with $\{\omega(i-1), \omega(i)\} \in E$ for $1 \leq i \leq n$. For an n-step walk ω on G, for $0 \leq s < t \leq n$ and for $\beta \in [0, 1]$, we define

$$U_{st}(\omega) = \begin{cases} -1 & (\omega(s) = \omega(t)) \\ 0 & (\omega(s) \neq \omega(t)) \end{cases}, \qquad K[0, n] = \prod_{0 \leq s < t \leq n} (1 + \beta U_{st}(\omega)) \tag{1.1}$$

where we keep the dependency on ω and β implicit to simplify the notation.

We let $\mathcal{W}_n$ denote the set of n-step walks starting at 0 on $\mathbb{Z}^d$ with usual nearest neighbour edges. Given $\beta \in [0, 1]$, we define the *partition function* $c_n = c_n(\beta)$ by

$$c_n = \sum_{\omega \in \mathcal{W}_n} K[0, n]. \tag{1.2}$$

The product in (1.1) discounts ω by a factor $1 - \beta$ for each pair s, t with an intersection $\omega(s) = \omega(t)$. When $\beta = 0$, c_n is simply the number of n-step walks and is thus equal to $(2d)^n$. For $\beta = 1$, c_n is the number of n-step strictly self-avoiding walks. The case $\beta \in (0, 1)$ is the *weakly self-avoiding walk*. In particular, it is proved in [13] that for $d > 4$ with $\beta > 0$ sufficiently small

$$c_n = A\mu^n(1 + O(\beta n^{-(d-4)/2})), \tag{1.3}$$

with $A = 1 + O(\beta)$ and where $\mu = \mu(\beta, \mathbb{Z}^d)$ is the WSAW *connective constant of* $\mathbb{Z}^d$.

There is a natural probability measure on $\mathcal{W}_n$ denoted $\mathbb{P}_{\beta,n}$ that comes with the WSAW model. If we let $\mathbb{E}_{\beta,n}$ be its corresponding expectation, then it is defined by

$$\mathbb{E}_{\beta,n} f = \frac{1}{c_n} \sum_{\omega \in \mathcal{W}_n} f(\omega) K[0,n] \tag{1.4}$$

for any bounded function $f\colon \mathcal{W}_n \to \mathbb{R}$. We write $\omega \sim \beta$-WSAW$_n$ to mean that ω has law $\mathbb{P}_{\beta,n}$. With this notation and when $d > 4$, it was proven in [5, 13] for small enough $\beta > 0$ and in [11] for $\beta = 1$, that the *mean-square displacement* defined by the left-hand side of (1.5) satisfies

$$\mathbb{E}_{n,\beta} |\omega(n)|^2 = Dn(1 + O(n^{-\varepsilon})) \tag{1.5}$$

for any $\varepsilon \in (0, 1/2)$ and where $D = D(\beta) > 1$ is called the *diffusion constant* of the WSAW. It is also known (see [7, Theorem 1.6] and [16, Theorem 6.1.8]) that the scaling limit of the strictly self-avoiding walk is Brownian motion for $d > 4$. In this case the (usual) diffusion constant of the limiting Brownian motion is equal to D.

We are mostly interested in walks on a discrete torus $\mathbb{T}_r^d = (\mathbb{Z}/r\mathbb{Z})^d$ for an integer $r \geq 4$ which will be taken to be large and even. The edge set is defined to be the set of pairs of vertices x, y such that x and y differ exactly by 1 mod r in one of the d directions. The *volume* of the torus is $V = r^d$. We let $\mathcal{W}_n^{\mathbb{T}}$ denote the set of n-steps torus walks starting at 0. In the same way as on $\mathbb{Z}^d$, we define torus quantities

$$c_n^{\mathbb{T}} = \sum_{\omega \in \mathcal{W}_n^{\mathbb{T}}} K[0,n], \qquad \mathbb{E}_{\beta,n,r}^{\mathbb{T}} g = \frac{1}{c_n^{\mathbb{T}}} \sum_{\omega \in \mathcal{W}_n^{\mathbb{T}}} g(\omega) K[0,n], \tag{1.6}$$

for any $g\colon \mathcal{W}_n^{\mathbb{T}} \to \mathbb{R}$. We write $\omega \sim \beta$-WSAW$_{n,r}^{\mathbb{T}}$ to mean that ω has law $\mathbb{P}_{\beta,n,r}^{\mathbb{T}}$.

1.4 Main result

We introduce some notations before rescaling the walk and stating our main result. Let $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, $T > 0$ and finite and define $\mathcal{C}_0^0([0,T], \mathbb{T}^d)$ be the space of continuous functions from $[0,T]$ to $\mathbb{T}^d$. For a torus walk $\omega \in \mathcal{W}_n^{\mathbb{T}}$ with $n \ll V^{1/2}$ and $n/r^2 \uparrow T$, we define the mapping $x^{(n)}$ from $\mathcal{W}_n^{\mathbb{T}}$ to $\mathcal{C}_0^0([0,T], \mathbb{T}^d)$ which is the proper rescaling of the torus walk by

$$x_{k/r^2}^{(n)}(\omega) = \frac{\omega(k)}{r} \text{ for } k \in \{0, 1, \cdots, n\}, \tag{1.7}$$

where we see $\frac{\omega(k)}{r}$ as an element of $\mathbb{T}^d$ rather than $r^{-1}\mathbb{T}_r^d$ by a canonical embedding. We then let $x_t^{(n)}(\omega)$ interpolate linearly between the above values up to $t = n/r^2$ and let $x_t = x_{n/r^2}$ for $t \in [n/r^2, T]$. Finally we define all of the above to the case $n/r^2 \uparrow \infty$ (i.e. $T = \infty$) by working instead on $\mathcal{C}_0^0([0,\infty), \mathbb{T}^d)$, the space of continuous functions from $[0,\infty)$ to $\mathbb{T}^d$ and by letting the latter interpolation satisfy instead $x_t = x_{n/r^2}$ for $t \in [n/r^2, \infty)$.

We define the stopped process $B^T = (B_t)_{0 \leq t \leq T}$ where B is an $\mathbb{R}^d$-Brownian motion defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying $\mathbb{E}\,|B_1|^2 = D$. So the diffusion constant of all the Brownian motions appearing in what follows is taken equal to D which is defined in (1.5). In the case $T = \infty$, B^T is simply B. Finally, we denote by B^T mod 1 the process with values in $\mathbb{T}^d$ obtained by taking all d coordinates of B modulo 1.

We defer the precise definition of convergence in law and continuity to Section 2.1. Our main result is the following:

Theorem 1.1. *Let $d > 4$ and $\beta > 0$ sufficiently small. If $\omega \sim \beta$-WSAW$_{n,r}^{\mathbb{T}}$ with $n = n_r$ satisfying $n \ll V^{1/2}$ and $n/r^2 \uparrow T$ where $T \in (0, \infty]$ then $X^{(n)} = x^{(n)}(\omega)$ converges*

in law to a Brownian motion on the torus with diffusion constant D and length T. In other words, for any continuous and bounded functional f on $\mathcal{C}_0^0([0,T],\mathbb{T}^d)$ (resp. on $\mathcal{C}_0^0([0,\infty),\mathbb{T}^d)$ for $T=\infty$) we have

$$\mathbb{E}^{\mathbb{T}}_{\beta,n,r} f(X^{(n)}) \to \mathbb{E} f(B^T \bmod 1) \text{ as } r \to \infty. \tag{1.8}$$

Theorem 1.1 follows from the classical fact that the scaling limit of the strictly self-avoiding walk is Brownian motion with diffusion constant D and from the following more recent result proven in [18] which itself carries more technicality than the proof of Theorem 1.1.

Theorem 1.2. *For $d>4$ and $C_0>0$, if $\beta>0$ is sufficiently small and if $n \le C_0 V^{1/2}$, then*

$$c_n^{\mathbb{T}} = A\mu^n \left[1 + O(\beta)\Big(\frac{1}{n^{(d-4)/2}} + \frac{n^2}{V}\Big)\right], \tag{1.9}$$

where A is the same constant as in (1.3) *and the error term depends on C_0 but not on n, V, β. In particular $c_n \sim c_n^{\mathbb{T}}$ for $n = o(V^{1/2})$ going to infinity.*

Note that although the statement of Theorem 1.2 holds for n up to and of order $V^{1/2}$, we state our scaling limit theorem only for walks of length $n = o(V^{1/2})$. This is a manifestation of the fact that Theorem 1.2 essentially captures the correct behaviour of $c_n^{\mathbb{T}}$ only for $n = o(V^{1/2})$ as we explain further in Section 1.5 and there is currently no hope of adapting our proof to obtain a scaling limit for walks of length $V^{1/2}$ (this is apparent in the proof of Lemma 2.4).

Since we are considering the WSAW rather than the usual (strictly) SAW, the results in [16] do not apply directly but as we discuss below Lemma 2.7 it is elementary to adapt them to the easier case of the WSAW.

In fact we shall only prove the case $T=\infty$ which is the most interesting and from which the proof can be easily adapted to cover the easier case $T>0$ and finite. We exclude the case $T=0$ because then the rescaling in (1.7) is degenerate in the sense that the torus is properly scaled whereas the rescaled walk shrinks to 0 as is noted in Remark 2.5.

1.5 Discussion of the results and related problems

Different regimes of $c_n^{\mathbb{T}}$. We refer to the regime of values of n for which the limiting behaviour (such as the scaling limit) of the torus WSAW is the same as on $\mathbb{Z}^d$ as the subcritical regime or dilute phase. On the torus, it corresponds conjecturally to walks having length $o(V^{1/2})$. On the other hand it would be interesting to determine what the scaling limit of the walk is when its length is of order $V^{1/2}$ (critical regime) or much larger than $V^{1/2}$ (supercritical regime or dense phase). The above conjectural distinction of regimes: $n \ll V^{1/2}$, $n \asymp V^{1/2}$ and $n \gg V^{1/2}$ has been first identified for the simpler and solvable case of the SAW on the complete graph in [24] (see also [18, 25] for progress on the torus and the hypercube). In order to establish a version of Theorem 1.1 for the critical regime we would need at least to understand better how $c_n^{\mathbb{T}}$ behaves when $n \asymp V^{1/2}$. In this regime we do not expect $c_n^{\mathbb{T}}$ to be equivalent to c_n when $n \to \infty$ (although $c_n \asymp c_n^{\mathbb{T}}$ should hold). This prevents the proof contained in the present article to apply directly in this case and is a manifestation of the fact that new ideas are required to understand the critical regime as was already noted in [18, Remark 5.3]. A related open problem is thus to obtain good control of $c_n^{\mathbb{T}}$ for $n \asymp V^{1/2}$ or of $\chi^{\mathbb{T}}(z) := \sum_{n\ge 0} c_n^{\mathbb{T}} z^n$ for $z \in [z_c - \sigma V^{-1/2}, z_c + \sigma V^{-1/2}]$ where $\sigma > 0$ is fixed but arbitrary, which is the conjectural *critical window*.

Role of boundary conditions. Another interesting question lies in the determination of the role of boundary conditions for self-avoiding walk models on finite graphs. For

free boundary condition, i.e. for the model on a large box without identifying opposite sides, it seems reasonable to expect instead a distinction of regimes of the type $n \ll r^2$, $n \asymp r^2$ and $n \gg r^2$ for the behaviour of the corresponding partition function of n-step walks c_n^{F}. Putting aside the scaling limit, it would again be of considerable interest to first elucidate the behaviour of c_n^{F}, but this is not currently known, even for the dilute phase. We note that it is the role of the coming [17] to analyse and clarify the role of boundary conditions for a version of the $|\varphi|^4$ model for $d \geq 4$, the latter being closely related to the WSAW model. We refer to [18, Section 1.7] and [17, Section 1.5 and Section 1.6] for further discussion and precise conjectures.

Related models. It is natural to ask whether Theorem 1.1 holds in the case of the strictly self-avoiding walk ($\beta = 1$) for $d > 4$, in particular since the convergence to Brownian motion is known in this case. We believe the answer is "Yes" in general and moreover that the proof in the case where $d > 4$ is large enough could be obtained with reasonable efforts: In order to apply our proof to SAW, one needs a version of Theorem 1.2 for SAW where the $O(\beta)$-term is likely to be replaced by $O((2d)^{-1})$ for large enough $d > 4$, the so-called "small parameter" in the lace-expansion literature. Then, the proof of Theorem 1.2 is based on diagrammatic estimates which should still hold in the case of the SAW rather than WSAW provided that one can establish a *near-critical decay* estimate for the SAW two-point function as in [25, Theorem 1.1]. This last estimate is used repeatedly in [18] but is for now known only for WSAW and is thus the main hindrance in pushing the analysis further.

Finally, we can also consider the related long-range self-avoiding walk which is shown to converge in [12] to an α-stable process in general. Considering a torus counterpart of this problem would also be of interest and remains open.

2 Scaling limit on the torus

2.1 Set-up

For any $r \geq 3$, we recall that $\mathbb{T}_r^d$ is the d-dimensional discrete r-torus, that is $\mathbb{T}_r^d = (\mathbb{Z}/r\mathbb{Z})^d$. We let $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ be the continuum torus. We will often see the discrete r-torus and the continuum torus as $[-r/2, r/2) \cap \mathbb{Z}^d$ with addition modulo r and as $[-1/2, 1/2)$ with addition modulo 1 respectively. For $x \in \mathbb{T}^d$ we denote by $(x)_1$ (for $x \in \mathbb{T}_r^d$ we denote by $(x)_r$ respectively) the unique representative in the equivalence class of x that is inside $[-\frac{1}{2}, \frac{1}{2})$ (inside $[-\frac{r}{2}, \frac{r}{2})$ respectively). This is useful to canonically embed $\mathbb{T}_r^d$ into $\mathbb{Z}^d$ and $\mathbb{T}^d$ in $\mathbb{R}^d$. In the rest of the article we will mention convergence in law on several metric spaces that we now define. We let

- $\mathcal{C}^0(\mathbb{R}^d) = (\mathcal{C}_0^0([0,\infty), \mathbb{R}^d), \|\cdot\|_\infty)$ be the metric space of continuous functions x from $[0,\infty)$ to $\mathbb{R}^d$ with $x(0) = 0$ endowed with the topology of uniform convergence. By abuse of notation we shall also denote by $\mathcal{C}^0(\mathbb{R}^d)$ the corresponding measurable space with its associated Borel σ-algebra.
- $\mathcal{C}^0(\mathbb{T}^d) = (\mathcal{C}_0^0([0,\infty), \mathbb{T}^d), \|\cdot\|_\infty)$ is defined similarly with $\|x\|_\infty = \sup_{t\geq 0} |(x(t))_1|$ and the corresponding topology and Borel σ-algebra.

We denote by $\Rightarrow$ the convergence in law (see [3, Chapter 1]) of probability measures on any of the above metric spaces without distinction since it shall be clear from context where the convergence holds.

In order to compare walks in $\mathbb{T}_r^d$ and in $\mathbb{Z}^d$ we introduce *the lift $\hat{\ell}$ of a torus walk* which is to be thought of as the unwrapping to $\mathbb{Z}^d$ of a walk on $\mathbb{T}_r^d$. The occurence of the hat on top of the ℓ is to make explicit the distinction between the lift of walks and the lift of processes which is introduced in Proposition 2.2.

Definition 2.1 (The $\mathbb{Z}^d$-lift of torus walks)**.** *We define the* lift $\hat{\ell}$ *of a torus walk by the operator*

$$\begin{aligned} \hat{\ell}\colon \mathcal{W}_n^{\mathbb{T}} &\to \mathcal{W}_n \\ \omega &\mapsto \hat{\ell}[\omega] \end{aligned} \tag{2.1}$$

which to any $\omega \in \mathcal{W}_n^{\mathbb{T}}$ *associates the* $\mathbb{Z}^d$*-walk* $\hat{\ell}[\omega]$ *defined by* $\hat{\ell}[\omega](0) = 0$ *and*

$$\hat{\ell}[\omega](k) = \hat{\ell}[\omega](k-1) + (\omega(k) - \omega(k-1))_r \qquad (1 \le k \le n). \tag{2.2}$$

Due the nearest-neighbour constraint, when $r \ge 4$ *the map* $\hat{\ell}$ *is a bijection from* $\mathcal{W}_n^{\mathbb{T}}$ *onto* $\mathcal{W}_n$*.*

This bijection has the following consequence that $c_n^{\mathbb{T}}$ in (1.6) can be rewritten as a sum over $\mathbb{Z}^d$-walks where an intersection occurs if the walk visits two points that are equal modulo r (in every coordinate). This gives

$$c_n^{\mathbb{T}} = \sum_{\omega \in \mathcal{W}_n} K^{\mathbb{T}}[0,n], \qquad \text{where} \qquad K^{\mathbb{T}}[0,n] = \prod_{0 \le s < t \le n} (1 + U_{st}^{\mathbb{T}}(\omega)) \tag{2.3}$$

and with $U_{st}^{\mathbb{T}}$ defined by

$$U_{st}^{\mathbb{T}}(\omega) = \begin{cases} -1 & (\omega(s) = \omega(t) \mod r) \\ 0 & (\omega(s) \ne \omega(t) \mod r). \end{cases} \tag{2.4}$$

Similarly, we extend in Proposition 2.2 the definition of the lift to the setting of processes. This requires some additional yet elementary work which is why we make it a proposition. The lift of processes is useful because it enables us to partially convert the problem of the scaling limit on the torus to a similar problem on $\mathbb{Z}^d$ where the theory is already well developed.

Proposition 2.2 (The $\mathbb{R}^d$-lift of torus processes)**.** *There exists a linear operator* ℓ

$$\begin{aligned} \ell\colon \mathcal{C}^0(\mathbb{T}^d) &\to \mathcal{C}^0(\mathbb{R}^d) \\ x &\mapsto \ell[x] \end{aligned} \tag{2.5}$$

which we call the lift *of a process, which to any function* $x \in \mathcal{C}^0(\mathbb{T}^d)$ *associates* $y = \ell[x]$ *the unique continuous* $\mathbb{R}^d$ *function* $(y(t))_{t \ge 0}$ *starting at* 0 *such that* $x(t) = y(t) \mod 1$ *for all* $t \ge 0$*. Furthermore, the lift* ℓ *of a torus process is a linear homeomorphism of* $\mathcal{C}^0(\mathbb{T}^d)$ *onto its range* $\mathcal{C}^0(\mathbb{R}^d)$*.*

Proof. The proof consists in formalizing the obvious. By construction we shall show that the lift of a torus process is well defined and is unique. We recall that $(x)_1$ is the unique element in the equivalence class of x that is inside $[-\frac{1}{2}, \frac{1}{2})^d$. Let $x \in \mathcal{C}^0(\mathbb{T}^d)$, we construct $y \in \mathcal{C}^0(\mathbb{R}^d)$ as follows. Fix $y(0) = x(0) = 0$ and $t_0 = 0$. Consider the sequence (t_n) defined by

$$t_{n+1} = \inf\{t > t_n,\ |(x(t) - x(t_n))_1| = 1/8\} \tag{2.6}$$

and $t_{n+1} = \infty$ if the above set is empty. One sees from the continuity of x that $t_{n+1} > t_n$ and $t_n \to \infty$ (or $= \infty$ for some n). Then we set

$$y(t) = y(t_n) + (x(t) - x(t_n))_1 \qquad (\text{for } t \in (t_n, t_{n+1}] \text{ and } n \ge 0) \tag{2.7}$$

and define $\ell[x] = y$ which is in $\mathcal{C}^0(\mathbb{R}^d)$ by construction. The fact $x(t) = y(t) \mod 1$ is clearly satisfied and can be checked on $(t_n, t_{n+1}]$ by induction on n. To show the uniqueness suppose that $y_1, y_2 \in \mathcal{C}^0(\mathbb{R}^d)$ both satisfy $x(t) = y_i(t) \mod 1$ for all $t \ge 0$

and $i = 1, 2$. Then $y_1(t) - y_2(t) = 0 \mod 1$ so that $y_1(t) - y_2(t) = u(t)$ with $u(t) \in \mathbb{Z}^d$ and by continuity of both y_1 and y_2 and the fact that $y_1(0) = y_2(0) = 0$ this implies that $u = 0$. The fact that the range of ℓ is $\mathcal{C}^0(\mathbb{R}^d)$ follows from the fact that for any $y \in \mathcal{C}^0(\mathbb{R}^d)$, the canonical projection on the torus $y \mod 1$ is an element of $\mathcal{C}^0(\mathbb{T}^d)$ which satisfies $\ell[y \mod 1] = y$. This also settles the fact that ℓ^{-1} is just the canonical projection onto the torus. The continuity of ℓ^{-1} is clear from the fact that it is a projection. The linearity of ℓ then follows from the bijectivity of ℓ and the linearity of the canonical projection. Note that for any $\varepsilon > 0$ with $\varepsilon < 1/8$ we have that $\|x\|_\infty < \varepsilon$ implies that $\ell[x](t) = (x(t))_1$ for all $t \geq 0$ and thus that $\|\ell[x]\|_\infty < \varepsilon$ which proves the continuity of ℓ. □

This above allows us to make the following definition. We let B be the usual Brownian motion on $\mathbb{R}^d$ and define $B^{\mathbb{T}}$, the *Brownian motion on the torus*, by

$$B^{\mathbb{T}} = \ell^{-1}(B) = B \mod 1. \tag{2.8}$$

2.2 Reduction of proof

In this section we reduce the proof of Theorem 1.1 to a convergence in law on $\mathbb{Z}^d$ by lifting the torus walks and their corresponding rescaled processes. We focus on the case $T = \infty$ which means that $n/r^2 \uparrow \infty$ because it is the most interesting one. The proof for an arbitrary finite and positive T is completely analogous and requires no additional effort and the case of $T = 0$ is settled in Remark 2.5. From now on B is a d-dimensional Brownian motion on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying $\mathbb{E}|B_1|^2 = D$ where we recall that D is the diffusion constant defined in (1.5). We define $B^{\mathbb{T}}$ as in (2.8). We want to show that whenever $\omega \sim \beta$-WSAW$^{\mathbb{T}}_{n,r}$ and $X^{(n)} = x^{(n)}(\omega)$ then for $r^2 \ll n \ll V^{1/2}$ $(X^{(n)}_t)_{t\geq 0} \Rightarrow (B^{\mathbb{T}}_t)_{t\geq 0}$. We will do so by studying instead the lift of torus walks which is equivalent from the following lemma.

Lemma 2.3 (Studying the lift of a process is enough)**.** *Let $P, P_1, P_2, \cdots$ be probability measures on $\mathcal{C}^0(\mathbb{T}^d)$. Then the following equivalence holds*

$$P_n \Rightarrow P \quad \textit{if and only if} \quad P_n \circ \ell^{-1} \Rightarrow P \circ \ell^{-1}. \tag{2.9}$$

Proof. This is an easy consequence of the *mapping theorem* (see [3, Theorem 2.7]) for convergence in law together with the fact that ℓ is a homeomorphism from $\mathcal{C}^0(\mathbb{T}^d)$ to $\mathcal{C}^0(\mathbb{R}^d)$ by Proposition 2.2. □

We define the corresponding $\mathbb{Z}^d$ β-WSAW renormalized in a similar way as X except that it is thus an $\mathbb{R}^d$-process. In detail, for ω a walk in $\mathcal{W}_n$ we define the mapping $y^{(n)}$ from $\mathcal{W}_n$ to $\mathcal{C}^0(\mathbb{R}^d)$ by

$$y^{(n)}_{k/r^2}(\omega) = \frac{\omega(k)}{r} \text{ for } k \in \{0, \cdots, n\} \tag{2.10}$$

and $y^{(n)}_t$ interpolates linearly between these values and also $y^{(n)}_t = y^{(n)}_{n/r^2}$ for $t \geq n/r^2$. From (1.7) and (2.10) we see that $y^{(n)}$ is nothing else than

$$y^{(n)} = \ell \circ x^{(n)} \circ \hat{\ell}^{-1}. \tag{2.11}$$

Lemma 2.4. *Let $\beta > 0$ be small enough, $n = o(V^{1/2})$, $\omega \sim \beta$-WSAW$_n$, $\omega^{\mathbb{T}} \sim \beta$-WSAW$^{\mathbb{T}}_{n,r}$ with $X^{(n)} = x^{(n)}(\omega^{\mathbb{T}})$, $Y^{(n)} = y^{(n)}(\omega)$ and suppose that $(Y^{(n)}_t)_{t\geq 0} \Rightarrow (B_t)_{t\geq 0}$. Then we have $(X^{(n)}_t)_{t\geq 0} \Rightarrow (B^{\mathbb{T}}_t)_{t\geq 0}$.*

Proof. We let $\beta > 0$ be small enough and X, Y as in the statement of Lemma 2.4. By assumption for any continuous bounded functional f on $\mathcal{C}^0(\mathbb{R}^d)$ we have that $\mathbb{E}_{\beta,n} f(Y^{(n)})$ converges to $\mathbb{E}f(B)$. We see from Lemma 2.3 that $(X_t^{(n)})_{t\geq 0} \Rightarrow (B_t^{\mathbb{T}})_{t\geq 0}$ if and only if $(\ell[X_\cdot^{(n)}](t))_{t\geq 0} \Rightarrow (B_t)_{t\geq 0}$. So it is enough to prove that the convergence in law holds for $Z^{(n)} = \ell[X^{(n)}]$ with limiting law that of a usual $\mathbb{R}^d$-Brownian motion with diffusion constant D. We use (1.1) and (2.3) to see that for any continuous and bounded functional f on $\mathcal{C}^0(\mathbb{R}^d)$

$$\begin{aligned}\mathbb{E}^{\mathbb{T}}_{\beta,n,r} f(Z^{(n)}) &= \frac{1}{c_n^{\mathbb{T}}} \sum_{\omega\in\mathcal{W}_n} f(\ell\circ x^{(n)}\circ \hat{\ell}^{-1}(\omega)) K^{\mathbb{T}}[0,n] \\ &= \frac{1}{c_n^{\mathbb{T}}} \sum_{\omega\in\mathcal{W}_n} f(y^{(n)}(\omega)) K^{\mathbb{T}}[0,n] \end{aligned} \tag{2.12}$$

where the last line follows from (2.11). By definition of $\mathbb{E}_{\beta,n} f(Y^{(n)})$ and letting $\mathcal{E}_n(f) := |\mathbb{E}^{\mathbb{T}}_{\beta,n,r} f(Z^{(n)}) - \frac{c_n}{c_n^{\mathbb{T}}} \mathbb{E}_{\beta,n} f(Y^{(n)})|$ we have

$$|\mathbb{E}^{\mathbb{T}}_{\beta,n,r} f(Z^{(n)}) - \mathbb{E}f(B)| \leq \mathcal{E}_n(f) + \left|1 - c_n/c_n^{\mathbb{T}}\right| \mathbb{E}_{\beta,n}|f(Y^{(n)})| + |\mathbb{E}_{\beta,n} f(Y^{(n)}) - \mathbb{E}f(B)|. \tag{2.13}$$

The third term in the right hand-side of (2.13) goes to zero by assumption. The second term goes to zero as n (and thus r) goes to infinity using the boundedness of f and the fact that $c_n^{\mathbb{T}} \sim c_n$ from Theorem 1.2 when $n = o(V^{1/2})$ and since $\beta > 0$ is taken small enough. To control $\mathcal{E}_n(f)$ we simply note that since $K^{\mathbb{T}}[0,n] \leq K[0,n]$ we have $\mathcal{E}_n(f) \leq \|f\|_\infty \frac{c_n - c_n^{\mathbb{T}}}{c_n^{\mathbb{T}}}$ which goes to zero as $r \to \infty$ by (1.3) and Theorem 1.2 since β is sufficiently small and $n \to \infty$ with $n = o(V^{1/2})$. □

Remark 2.5. Following a similar proof as above, it is easy to check that if $\omega^{\mathbb{T}} \sim \beta$-WSAW$^{\mathbb{T}}_{n,r}$ with $n \ll r^2$ then any rescaling of $\omega^{\mathbb{T}}$ and $\mathbb{T}_r^d$ converges to 0 in probability in the sense that for every $\varepsilon > 0$

$$\lim_{\substack{n,r\to\infty \\ n\ll r^2}} \mathbb{P}^{\mathbb{T}}_{\beta,n,r}(r^{-1} \sup_{1\leq k\leq n} |\omega^{\mathbb{T}}(k)| > \varepsilon) = 0. \tag{2.14}$$

Indeed we have that for any $\varepsilon > 0$ and similarly as in the previous proof that

$$\begin{aligned}\mathbb{P}^{\mathbb{T}}_{\beta,n,r}(r^{-1} \sup_{1\leq k\leq n} |\omega^{\mathbb{T}}(k)| > \varepsilon) &\leq \mathbb{P}^{\mathbb{T}}_{\beta,n,r}(r^{-1} \sup_{1\leq k\leq n} |\ell[\omega^{\mathbb{T}}](k)| > \varepsilon) \\ &\leq \mathbb{P}_{\beta,n}(r^{-1} \sup_{1\leq k\leq n} |\omega(k)| > \varepsilon) \\ &= \mathbb{P}_{\beta,n}(n^{1/2} r^{-1} \sup_{1\leq k\leq n} n^{-1/2}|\omega(k)| > \varepsilon) \end{aligned} \tag{2.15}$$

and the proof follows from the hypothesis that $n \ll r^2$ and from the tightness of $(\sup_{1\leq k\leq n} n^{-1/2}|\omega(k)|)_n$ which is a consequence of the convergence in law of the $\mathbb{Z}^d$-WSAW to Brownian motion on $\mathbb{Z}^d$ when $d > 4$.

2.3 Proof of Theorem 1.1

In order to conclude the proof of the main theorem it is enough by Lemma 2.4 to show that $(Y_t^{(n)})_{t\geq 0} \Rightarrow (B_t)_{t\geq 0}$. We thus need to show that our rescaled (see (2.10)) $\mathbb{Z}^d$-WSAW converges to an infinite Brownian path. This latter fact is not known. Indeed, the original $\mathbb{Z}^d$-scaling limit result presented in [16, Theorem 6.1.8] is different. It is shown there that a length-n β-WSAW ω rescaled by $\sqrt{n}$ converges in distribution to $(B_t)_{0\leq t\leq 1}$. This last subsection thus gives a proof that $(Y_t^{(n)})_{t\geq 0} \Rightarrow (B_t)_{t\geq 0}$. To reduce the proof further

we use [14, Proposition 16.6] which states that the convergence in law of continuous random processes indexed by $\mathbb{R}^+$ and taking values in some general metric space holds if and only if the convergence holds for the restrictions to any compact subset of $\mathbb{R}^+$. In fact $\mathbb{R}^+$ could be replaced by much more general spaces but we do not need the full generality of the proposition here. In our case [14, Proposition 16.6] means that it is enough to prove the following proposition to conclude.

Proposition 2.6. *Let β be small enough. For $\omega \sim \beta$-WSAW$_n$ on $\mathbb{Z}^d$ and $Y^{(n)} = y^{(n)}(\omega)$ and for any $T \in (0,\infty)$ we have $(Y^{(n)})_{0\le t\le T} \Rightarrow (B_t)_{0\le t\le T}$, and the convergence in law holds in the space $\mathcal{C}_0^0([0,T],\mathbb{R}^d)$ instead of $\mathcal{C}_0^0([0,\infty),\mathbb{R}^d)$.*

We thus need to prove that for all $T \in (0,\infty)$, the subwalk starting from 0 of length r^2T of a length-n $\mathbb{Z}^d$-WSAW converges to Brownian motion once rescaled by r. As usual, the proof has two steps, we need to show that the finite-dimensional distributions (fdd) of $Y^{(n)}$ converge to those of a Brownian motion with diffusion constant D and then prove tightness of the sequence of processes. The proof requires a small generalization of [16, Chapter 6] that we explain in the next two subsections. Before doing so we introduce the following notation which we use in both parts of the proof. We note that for $\omega \in \mathcal{W}_n$

$$Y_t^{(n)}(\omega) = Y_{t_r}^{(n)}(\omega) + \varphi_{r,n,t}(\omega), \tag{2.16}$$

$$\varphi_{r,n,t}(\omega) = (r^2t - \lfloor r^2t\rfloor)(Y_{t_r+r^{-2}}^{(n)}(\omega) - Y_{t_r}^{(n)}(\omega)) \tag{2.17}$$

with $t_r = \lfloor r^2t\rfloor/r^2$ and where $\varphi_{r,n,t}(\omega)$ satisfies

$$|\varphi_{r,n,t}(\omega)| \le \frac{1}{r} \quad \text{uniformly in } \omega, t, T. \tag{2.18}$$

2.3.1 Finite-dimensional distributions

Note that we can equivalently determine the fdd of Y_t by replacing in the computations Y_t by Y_{t_r}. Indeed this follows from (2.16), (2.18) together with the discussion along [3, p.88–89]. Then, since the convergence of the fdd is implied by the convergence of the characteristic function, the statement of convergence of the fdd takes the form of Lemma 2.7. Also in the next lemma we let k_n be any increasing sequence going to infinity and satisfying $Tk_n \le n$ for n large enough. Here k_n plays the role of r^2 and is typically $\ll n$ for our purposes but the following lemma does not require this condition.

Lemma 2.7 (Finite-dimensional distributions of $Y_t^{(n)}(\omega)$)**.** *Let $d > 4$, $\beta > 0$ small enough and $T > 0$. For any increasing sequence $k = k_n$ such that $Tk_n \le n$ for all n large enough and $k_n \to \infty$, and for any integer $N \ge 1$, $0 = t_0 < t_1 < \cdots < t_N \le T$ and $u_1, \cdots, u_N \in \mathbb{R}^d$, the following holds*

$$\lim_{n\to\infty} \mathbb{E}_{\beta,n} \exp\Big[i\sum_{j=1}^N \frac{u_j}{\sqrt{Dk_n}} \cdot \Big(\omega(\lfloor t_jk_n\rfloor) - \omega(\lfloor t_{j-1}k_n\rfloor)\Big)\Big] = \exp\Big(-\frac{1}{2d}\sum_{j=1}^N |u_j|^2(t_j - t_{j-1})\Big). \tag{2.19}$$

The proof follows from an adaptation (and application) of [16, Theorem 6.6.2] that is given hereafter. While it is true that this theorem has been established for the SAW rather than the WSAW, it readily adapts to the easier case of the WSAW. Furthermore, the most technical part of the proof of the aforementioned result is the scaling limit of the endpoint [16, Theorem 6.6.1] which was proven in the original article [5] this time for the WSAW model and we shall therefore use [16, Theorem 6.6.2] in the case $d > 4$ and $\beta > 0$ small enough.

Proof of Lemma 2.7. Let $d > 4$, $\beta > 0$, $T > 0$ and $k = k_n$ such that $Tk_n \le n$ and $k_n \to \infty$ and fix arbitrary $N \ge 1$, $0 = t_0 < t_1 < \cdots < t_N \le T$ and $u_1, \cdots, u_N \in \mathbb{R}^d$. For any vector

v we let $v^{(k)} := v/\sqrt{Dk}$ and introduce further

$$\Delta_j^{(k)}(\omega) = \omega(\lfloor t_j k \rfloor) - \omega(\lfloor t_{j-1} k \rfloor) \tag{2.20}$$

where the dependency on t is omitted for simplicity. Then (2.19) rewrites as

$$\lim_{n\to\infty} c_n^{-1} M_n(u,t) = \exp\Big(-\frac{1}{2d} \sum_{j=1}^N |u_j|^2 (t_j - t_{j-1}) \Big) \tag{2.21}$$

with

$$M_n(u,t) = \sum_{\omega \in \mathcal{W}_n} \exp\Big[i \sum_{j=1}^N u_j^{(n)} \cdot \Delta_j^{(k_n)}(\omega) \Big] K[0,n].$$

In order to prove (2.21) we use a "KJK expansion" to decouple the two parts of the walk before and after time $\lfloor k_n t_N \rfloor$. For this we appeal to Lemma 5.2.5 in [16] where it is shown that for any integer $m \in [0,n]$

$$K[0,n] = \sum_{I \ni m} K[0,I_1] J[I_1,I_2] K[I_2,n] \tag{2.22}$$

where the sum is over intervals I of the form $[I_1,I_2]$ where either $0 \le I_1 < m < I_2 \le b$ or $I_1 = I_2 = m$. We do not really need the definition of $J[s,t] = J[s,t](\omega)$ which can be found in [5, 16, 23] but rather only need to know that it is an interaction term (like $K[s,t]$) that only depends on ω between times s and t and satisfies

$$\sum_{n=0}^\infty n z_c^n \sum_{\omega \in \mathcal{W}_n} |J[0,n]| < \infty \tag{2.23}$$

where $z_c = \mu^{-1}$ as was originally proven in [5, Section 5]. We refer to [16, Theorem 6.2.9] for a more concise and systematic treatment of this fact and other related *diagrammatic estimates*. The input of (2.22) gives

$$M_n(u,t) = \sum_{I \ni \lfloor k_n t \rfloor} \sum_{\omega \in \mathcal{W}_n} \exp\Big[i \sum_{j=1}^N u_j^{(n)} \cdot \Delta_j^{(k_n)}(\omega) \Big] K[0,I_1] J[I_1,I_2] K[I_2,n]. \tag{2.24}$$

We split the walk ω into three pieces from times 0 to I_1, from I_1 to I_2 and from I_2 to n. We denote these walks ω_1, ω_2 and ω_3 respectively and see from the above decoupling that

$$M_n(u,t) = \sum_{I \ni \lfloor k_n t \rfloor} \sum_{\substack{\omega_1 \in \mathcal{W}_{I_1} \\ \omega_2 \in \mathcal{W}_{I_2 - I_1}}} \exp\Big[i \sum_{j=1}^N u_j^{(n)} \cdot \Delta_j^{(k_n)}(\omega_1 \circ \omega_2) \Big] K[0,I_1] J[0,I_2 - I_1] c_{n-I_2} \tag{2.25}$$

where $\omega_1 \circ \omega_2$ is the the walk obtained by concatenation of ω_1 and ω_2. We show that the main contribution to the above sum is given by those intervals I such that $|I| = I_2 - I_1 \le b_n$ where b_n ultimately needs to satisfy $b_n = o(k_n^{1/2})$ and $b_n \to \infty$. Consider first the case $|I| \le b_n$ (resp. $|I| > b_n$) and denote by $M^{\le}(u,n)$ (resp $M^{>}(u,n)$) the corresponding sum restricted to this range of $|I|$ such that $M_n(u,t) = M_n^{\le}(u,t) + M_n^{>}(u,t)$. In this case for n large enough we have $I_1 > k_n t_{N-1}$ and $I_1 \in [k_n t_N (1 - {k_n}^{-1/2}), k_n t_N]$ $I_1 \in [k_n t_N (1 - {k_n}^{-1/2}), k_n t_N]$ uniformly in $I \le b_n$ which implies that $\Delta_j^{(k_n)}(\omega_1 \circ \omega_2) = \Delta_j^{(k_n)}(\omega_1)$

for all $1 \le j \le N-1$. By adding and substracting $iu_N^{(n)} \cdot \omega_1(I_1)$ inside the exponential we see that [16, Theorem 6.6.2] applies (with k_n in place of n) for the sum over ω_1 and gives

$$\sum_{\omega_1 \in \mathcal{W}_{I_1}} \exp\Big[i \sum_{j=1}^{N-1} u_j^{(n)} \cdot \Delta_j^{(k_n)}(\omega_1) + iu_N^{(n)} \cdot (\omega_1(I_1) - \omega_1(\lfloor k_n t_{N-1} \rfloor))\Big] K[0, I_1]$$
$$= c_{I_1} \exp(-\frac{1}{2d} \sum_{j=1}^{N} |u_j|^2 (t_j - t_{j-1}))(1 + o(1)) \tag{2.26}$$

and the $o(1)$ goes to zero as $n \to \infty$ uniformly in $|I| \le b_n$. For the sum over ω_2 we note that $\exp(iu_N^{(n)} \cdot \omega_2(\lfloor k_n t_N \rfloor - I_1) = 1 + o(1)$ uniformly in $|I| \le b_n$ so that overall

$$M_n^{\le}(u,t) = (1 + o(1)) \exp(-\frac{1}{2d} \sum_{j=1}^{N} |u_j|^2 (t_j - t_{j-1})) \sum_{\substack{I \ni \lfloor k_n t \rfloor \\ |I| \le b_n}} c_{n-I_2} c_{I_1} \sum_{\omega_2 \in \mathcal{W}_{|I|}} J[0, I_2 - I_1]. \tag{2.27}$$

By using (2.22) again and dividing both sides by c_n this gives

$$c_n^{-1} M_n^{\le}(u,t) = \exp(-\frac{1}{2d} \sum_{j=1}^{N} |u_j|^2 (t_j - t_{j-1}))(1 + o(1))$$
$$- O(1) \sum_{\substack{I \ni \lfloor k_n t \rfloor \\ |I| > b_n}} \frac{c_{n-I_2} c_{I_1}}{c_n} \sum_{\omega_2 \in \mathcal{W}_{|I|}} J[0, |I|]. \tag{2.28}$$

Similarly for $M_n^{>}(u,t)$, but treating the exponential in the summand simply as a $O(1)$, we obtain directly by (2.22) that

$$c_n^{-1} M_n^{>}(u,t) = O(1) \sum_{\substack{I \ni \lfloor k_n t \rfloor \\ |I| > b_n}} \frac{c_{n-I_2} c_{I_1}}{c_n} \sum_{\omega_2 \in \mathcal{W}_{|I|}} J[0, |I|]. \tag{2.29}$$

By summing both contributions (2.28) and (2.29) we see that to conclude it is enough to show that $c_n^{-1} M_n^{>}(u,t) \to 0$ as $n \to \infty$. To see this we use the fact that $c_n = O(z_c^{-n})$ from (1.3) to obtain

$$c_n^{-1} M_n^{>}(u,t) = O(1) \sum_{\substack{I \ni \lfloor k_n t \rfloor \\ |I| > b_n}} z_c^{|I|} \sum_{\omega_2 \in \mathcal{W}_{|I|}} J[0, |I|]. \tag{2.30}$$

Finally we see that $|I|$ being fixed there are at most $|I| + 1$ ways to choose $I \ni \lfloor k_n t \rfloor$ so that

$$c_n^{-1} M_n^{>}(u,t) = O(1) \sum_{|I| = b_n + 1}^{\infty} |I| z_c^{|I|} \sum_{\omega_2 \in \mathcal{W}_{|I|}} J[0, |I|]. \tag{2.31}$$

The absolute bound (2.23) achieves the proof since $b_n \to \infty$ and the above sum is convergent. □

2.3.2 Tightness

We use the following criterion taken from [14, Corollary 16.9] and adapted to our purposes to obtain tightness of random continuous processes under some moment condition. The proof of (2.32) itself resembles that of [21] where the proof of tightness for the ordinarily rescaled SAW originates. We note that the criterion used here is slightly simpler than the one used in [21].

Lemma 2.8 (Tightness of $Y^{(n)}(\omega)$)**.** *Let $d > 4$, $\beta > 0$ small enough, $T > 0$ and $\omega \sim \beta$-WSAW$_n$ then the sequence $(Y_t^{(n)}(\omega))_{0\le t\le T}$ is tight if there exists $A > 0$ such that for all $0 \le s < t \le T$ and uniformly in n*

$$\mathbb{E}_{\beta,n}|Y_t^{(n)}(\omega) - Y_s^{(n)}(\omega)|^2 \le A|s-t|. \tag{2.32}$$

We now prove that (2.32) holds therefore completing the proof of Proposition 2.6 by combining Lemma 2.7 and Lemma 2.8. Let $0 \le s < t \le T$ and suppose first that $t_r = s_r$ where we recall that for any real u, we let $u_r = \lfloor r^2 u\rfloor/r^2$. Then we have by (2.16)–(2.18) that for any walk $\omega \in \mathcal{W}_n$

$$|Y_t^{(n)}(\omega) - Y_s^{(n)}(\omega)| = (r^2t - r^2s)|Y^{(n)}_{t_r+r^{-2}}(\omega) - Y^{(n)}_{t_r}(\omega)| \le r(t-s). \tag{2.33}$$

and since $t_r = t_s$ implies $r \le |t-s|^{-1/2}$ the result follows with $A = 1$ by squaring the above equation and taking the expectation on both sides.

Now we suppose that $t_r \ne s_r$, then by (2.16)–(2.18) again, we can write

$$|Y_t^{(n)}(\omega) - Y_s^{(n)}(\omega)| \le |Y^{(n)}_{t_r}(\omega) - Y^{(n)}_{s_r}(\omega)| + \frac{2}{r}. \tag{2.34}$$

Since $t_r \ne s_r$ implies $r \ge |t-s|^{-1/2}$ we further obtain using $(a+b)^2 \le 2(a^2+b^2)$ that

$$\mathbb{E}_{\beta,n}|Y_t^{(n)}(\omega) - Y_s^{(n)}(\omega)|^2 \le 2\mathbb{E}_{\beta,n}|Y^{(n)}_{t_r}(\omega) - Y^{(n)}_{s_r}(\omega)|^2 + 8|t-s|. \tag{2.35}$$

And thus to conclude with the proof of (2.32) we see that it is enough to show that

$$\mathbb{E}_{\beta,n}|Y^{(n)}_{t_r}(\omega) - Y^{(n)}_{s_r}(\omega)|^2 \le A|t-s|. \tag{2.36}$$

By definition the left-hand side of (2.36) is equal to

$$\frac{1}{r^2c_n}\sum_{\omega\in\mathcal{W}_n}|\omega(\lfloor r^2t\rfloor) - \omega(\lfloor r^2s\rfloor)|^2K[0,n]. \tag{2.37}$$

Using the fact that $K[0,n] \le K[0,\lfloor r^2s\rfloor]\,K[\lfloor r^2s\rfloor,\lfloor r^2t\rfloor]\,K[\lfloor r^2t\rfloor,n]$, decomposing ω into three subwalks corresponding to these time intervals and using that (A may change from line to line)

$$c_n^{-1} \le Ac^{-1}_{\lfloor r^2s\rfloor}c^{-1}_{\lfloor r^2t\rfloor-\lfloor r^2s\rfloor}c^{-1}_{n-\lfloor r^2t\rfloor} \tag{2.38}$$

from [16, Theorem 6.1.1(a)] we get that (2.37) satisfies

$$\frac{1}{r^2c_n}\sum_{\omega\in\mathcal{W}_n}|\omega(\lfloor r^2t\rfloor) - \omega(\lfloor r^2s\rfloor)|^2K[0,n] \le \frac{A}{r^2}\mathbb{E}_{\lfloor r^2t\rfloor-\lfloor r^2s\rfloor,\beta}|\omega(\lfloor r^2t\rfloor - \lfloor r^2s\rfloor)|^2. \tag{2.39}$$

which by the bound on the mean square displacement in (1.5) gets bounded further by $A|t-s|$ and thus achieves the proof.

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