Strong solutions to McKean–Vlasov SDEs with coefficients of Nemytskii-type[*][†]

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Abstract

We study a large class of McKean–Vlasov SDEs with drift and diffusion coefficient depending on the density of the solution's time marginal laws in a Nemytskii-type of way. A McKean–Vlasov SDE of this kind arises from the study of the associated nonlinear FPKE, for which is known that there exists a bounded Sobolev-regular Schwartz-distributional solution u. Via the superposition principle, it is already known that there exists a weak solution to the McKean–Vlasov SDE with time marginal densities u. We show that there exists a strong solution the McKean–Vlasov SDE, which is unique among weak solutions with time marginal densities u. The main tool is a restricted Yamada–Watanabe theorem for SDEs, which is obtained by an observation in the proof of the classical Yamada–Watanabe theorem.

1 Introduction

In this paper we will consider the following McKean–Vlasov stochastic differential equation (abbreviated by McKean–Vlasov SDE or MVSDE) in $\mathbb{R}^d$, $d \in \mathbb{N}$, with coefficients of Nemytskii-type, which in our case is of the form

$$dX(t) = E(X(t))b\left(\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)dt + \sqrt{2a\left(\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)}\mathbb{1}_{d \times d}\, dW(t),$$

$$X(0) = \xi, \tag{MVSDE.PME}$$

where $t \in [0,T]$, $T \in (0,\infty)$, $\mathbb{1}_{d \times d}$ is the d-dimensional unit matrix, $(W_t)_{t \in [0,T]}$ is a standard d-dimensional $(\mathscr{F}_t)$-Brownian motion and ξ an $\mathscr{F}_0$-measurable function on some stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \in [0,T]})$, i.e. a complete, filtered probability space, where $(\mathscr{F}_t)_{t \in [0,T]}$ is a normal filtration, and $\mathcal{L}_{X(t)} := \mathbb{P} \circ (X(t))^{-1}$, $t \in [0,T]$. Here, we assume that $E : \mathbb{R}^d \to \mathbb{R}^d$, $b : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$ are functions with $a(r) := \beta(r)/r$, $r \in \mathbb{R} \setminus \{0\}$, $a(0) := \beta'(0)$, such that the following assumptions hold:

(i) $\beta \in C^1(\mathbb{R})$, $\beta(0) = 0$;

(ii) There exists $\gamma_0 > 0$ such that for all $r_1, r_2 \in \mathbb{R}$, $\gamma_0|r_1 - r_2|^2 \leq (\beta(r_1) - \beta(r_2))(r_1 - r_2)$;

(iii) $E \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$, $\mathrm{div}E \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, $\mathrm{div}E \geq 0$;

(iv) $b \in C^1(\mathbb{R}) \cap C_b(\mathbb{R})$, $b \geq 0$.

Here, we would like to point out that conditions (i) and (ii) imply

$$a \geq \gamma_0 > 0, \tag{1.1}$$

which, in turn, means that the diffusion matrix of (MVSDE.PME) is assumed to be non-degenerate. There is a vast literature on the solvability of McKean–Vlasov SDEs under various assumptions on the coefficients. In 1966, McKean [23] initiated the study of diffusion processes related to certain non-linear PDEs arising from, for example, statistical mechanics (as in our case, see [2]). This work was closely followed up by important results such as [13, 31, 30] investigating the weak and/or strong solvability of McKean–Vlasov SDEs; for more references see [3]. For recent results consult, in particular, [24, 29, 29, 16, 17, 18], and also [9] and the references therein. In all of these papers, the authors assume the continuity of the coefficients in the measure-component with respect to the weak topology, some Wasserstein distance, or total variation norm. In [18], the authors consider also the coefficients' continuity with respect to a norm, which is stronger than the sum of a Wasserstein distance and total variation norm. However, the coefficients in (MVSDE.PME) do not bear any such continuity property in their measure-component.

Equation (MVSDE.PME) arises from the study of a nonlinear Fokker–Planck–Kolmogorov equation (in short: FPKE), which in this case is a porous medium equation perturbed by a nonlinear transport term of the following type

$$\partial_t u + \mathrm{div}(Eb(u)u) - \Delta\beta(u) = 0 \ \text{ on } [0, T] \times \mathbb{R}^d \ \text{ with } \ u|_{t=0} = u_0. \tag{PME}$$

This equation is to be understood in the Schwartz-distributional sense. We will say that a curve of $L^1(\mathbb{R}^d)$-functions $u = (u_t)_{t \in [0,T]}$ is a Schwartz-distributional solution to (PME) if $[0, T] \mapsto u_t(x)dx$ is narrowly continuous and

$$\int_{\mathbb{R}^d} \varphi(x)u_t(x)dx = \int_{\mathbb{R}^d} \varphi(x)u_0(x)dx + \int_0^t \int_{\mathbb{R}^d} E^i(x)b(u_s(x))\partial_i\varphi(x)u_s(x)dxds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \beta(u_s(x))\Delta\varphi(x)dxds \ \ \forall t \in [0, T], \tag{1.2}$$

for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ (using Einstein summation convention), where $E(x) = (E^i(x))_{i=1}^d$. In the case that u_t is even a probability density for all $t \in [0, T]$, then $u = (u_t)_{t \in [0,T]}$ is simply called a *probability solution* to (PME).

In [2] and [3], an approach was developed in order to solve general McKean–Vlasov SDEs by first solving the associated nonlinear FPKE. This approach is based on the superposition principle developed by Trevisan [32], which in turn relies on the fundamental result of Figalli [12]; for a very recent generalisation of the latter two publications see [7]. Note also the very recent superposition principle, which relates solutions to non-local FPKEs with solutions to SDEs with jumps [27]. Of course, solving first the McKean–Vlasov SDE, Itô's formula yields that the time marginal laws of the solution process solve the associated nonlinear FPKE. In this sense, solving the McKean–Vlasov equation is essentially equivalent to solving the associated nonlinear FPKE. In the special case when the nonlinear FPKE is of the type (PME), in [4], Barbu and Röckner solved this

equation under various assumptions on β, E, b and lifted the solution to a weak solution to (MVSDE.PME), whose time marginal laws coincide with the constructed probability solution u to (PME), provided u_0 is a bounded probability density.

The aim of this paper is to show that the constructed weak solution provided by [4] is a functional of the driving Brownian motion, i.e. a strong solution to (MVSDE.PME) under our assumptions. Our method relies on a proper modification of the Yamada–Watanabe theorem for SDEs. This modification makes it possible to prove the existence of a strong solution to an SDE provided weak existence and pathwise uniqueness holds in a certain subclass of weak solutions. We will provide the theorem in Section 2, as there seems to be uncertainty about this result; in [10], the authors stated, 'Note also that pathwise uniqueness is proved only for particular solutions [...], so we cannot use directly the result of Yamada and Watanabe to deduce strong existence [for the SDE under investigation].' ([10, p. 1502]). For the proof we refer to the author's thesis [15]. This modification of the Yamada–Watanabe theorem can be applied to (MVSDE.PME) by fixing the solution u to (PME) provided by [4] in the coefficients of (MVSDE.PME). This transfers the problem of strong existence for McKean–Vlasov SDEs to a problem for SDEs. Moreover, (MVSDE.PME) has already been studied in terms of weak existence and restricted pathwise uniqueness by Jabir and Bossy in [8] in the case $E \equiv b \equiv 0$ and under assumptions which strictly imply ours. They did not prove the existence of a strong solution in their case.

This paper is structured as follows. First, we will introduce some frequently used notation in this paper. Afterward, in Section 2, we will present an abstract modification of the famous Yamada–Watanabe theorem for SDEs based on [22, Appendix E], which, in particular, enables us to conclude strong existence provided one has proved weak existence and pathwise uniqueness for some subclass of weak solutions. This theorem will be the main tool to deduce the existence of a strong solution to (MVSDE.PME). In Section 3, we will apply the Yamada–Watanabe theorem for SDEs to general McKean–Vlasov SDEs by fixing the time marginal laws of a given curve of probability measures in the coefficients' measure component. Within the last section, Section 4, we will state the main result and its proof. This section is divided into four subsections. In Subsection 4.1, we will state the main result and the steps on how to prove it. In Subsection 4.2, we will discuss the existence and regularity of a probability solution u to (PME) under the conditions (i)–(iv). In Subsection 4.3, we will conclude the existence of a weak solution to (MVSDE.PME) with time marginal law densities u. In Subsection 4.4 we show that pathwise uniqueness holds among weak solutions to (MVSDE.PME) with time marginal law densities u. This subsection is divided into two further subsections. In Subsection 4.4.1, we will recall a pathwise uniqueness result for SDEs with bounded Sobolev-regular coefficients. In Subsection 4.4.2 we will apply the pathwise uniqueness result for SDEs from Subsection 4.4.1 to (MVSDE.PME). Here we will add condition (v) to the previous assumptions (i)–(iv).

Notation. For a topological space $(\mathbf{T}, \tau)$, $\mathcal{B}(\mathbf{T})$ shall denote the Borel σ-algebra on $(\mathbf{T}, \tau)$. Let $n \geq 1$. On $\mathbb{R}^n$, $B_R(x)$ denotes the open ball with center $x \in \mathbb{R}^n$ and radius $R > 0$, and $|\cdot|_{\mathbb{R}^n}$ the usual Hilbert–Schmidt norm. If there is no risk for confusion, we just write $|\cdot| = |\cdot|_{\mathbb{R}^n}$. Let (M, d) be a metric space. Then $\mathcal{P}(M)$ denotes the set of all Borel probability measures on (M, d). We will consider $\mathcal{P}(M)$ as a topological space with respect to the topology of weak convergence of probability measures. A curve of probability measures $(\nu_t)_{t \in [0,T]} \subset \mathcal{P}(M)$ is called narrowly continuous if $[0, T] \ni t \mapsto \int \varphi(x) \nu_t(dx)$ is continuous for all $\varphi \in C_b(M)$. By $\mathcal{P}_0(\mathbb{R}^n)$ we will denote the set of all probability densities with respect to Lebesgue measure, i.e. $\mathcal{P}_0(\mathbb{R}^n) = \left\{ \rho \in L^1(\mathbb{R}^n) \; : \; \rho \geq 0 \text{ a.e.}, \int_{\mathbb{R}^n} \rho(x) dx = 1 \right\}$. Let $(S, \mathscr{S}, \eta)$ be a measure space and E be a Banach space. For $1 \leq q \leq \infty$, $L^q(S; E)$ symbolises the usual Bochner space on S with values in E. If $S = \mathbb{R}^n$ and $E = \mathbb{R}$, we just

write $L^p(\mathbb{R}^n; \mathbb{R}) = L^p(\mathbb{R}^n)$. Moreover, $\mathrm{div}, \Delta, \nabla$ symbolise the divergence, Laplacian and gradient with respect to the spatial variable and are taken in the Schwartz-distributional sense. Further, the transpose of the distributional Jacobian matrix is also denoted by the gradient symbol.

2 A modification of the Yamada–Watanabe theorem for SDEs

Since it will be the core of the technique of this paper, we will start by presenting a restricted version of the famous Yamada–Watanabe theorem for SDEs.

The well-known Yamada–Watanabe theorem for SDEs (see, e.g. [22, Appendix E]; for the orginal work see [34]) provides a useful characterisation for the existence of a unique strong solution to an SDE; therefore, loosely speaking, it is necessary and sufficient to have a weak solution for *any* initial probability measure in combination with the pathwise uniqueness regarding *all* weak solutions.

Carefully checking the statements and the proofs in [22, Appendix E], it is possible to refine their definitions and results to a *restricted* Yamada–Watanabe theorem. This theorem is the result of a generalisation of the observation in [22, Remark E.0.16], which implies that, by the techniques employed in [22, Appendix E], a strong solution can be constructed from a weak solution with a fixed initial condition in the case that pathwise uniqueness is known for solutions with exactly this initial datum.

The restricted Yamada–Watanabe theorem will be of the following form. Let us fix a set P consisting of probability measures on the solution's path space, which have all the same initial time marginal laws. Assume that pathwise uniqueness holds among all weak solutions to an SDE whose laws lie in P and that there exists a weak solution (X, W) with $\mathcal{L}_X \in P$, where $\mathcal{L}_X$ denotes the distribution of X with respect to the underlying probability measure. Then, and only then, this is the case if there exists a unique strong solution to this SDE with law in P. For the precise statement see Theorem 2.5.

In the following, we will only present the necessary modification of the setup of [22, Appendix E] in which we slightly change the definitions of solutions and the corresponding statement of the main theorem. For the reader's convenience, we will stick to the finite dimensional setting. The adaption to the infinite dimensional case is essentially the same. The procedure from [22, Appendix E] leading to the presented restricted Yamada–Watanabe theorem can be rather easily adapted to our setup. Let us finally note that in this section we will consider SDEs on the time interval $[0, \infty)$. In fact, this section can analogously be stated for SDEs on a finite time interval $[0, T]$ without any difficulties. For the details and proofs consult [15], where also the infinite dimensional case is treated. For the finite dimensional case of [22, Appendix E], we refer to [26, Appendix E]. For the Yamada–Watanabe theorem in the mild solution framework, we refer to [25]. For the treatment of general stochastic models see [19, 20].

Let $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \geq 0})$ be a stochastic basis and $d_1 \in \mathbb{N}$. We will consider the Polish path spaces $(\mathbb{B}, \rho)$ and $(\mathbb{W}_0, \rho)$, where $\mathbb{B} := C([0, \infty); \mathbb{R}^d)$, $\mathbb{W}_0 := \{w \in C([0, \infty); \mathbb{R}^{d_1}) : w(0) = 0\}$, are respectively equipped with the metric $\rho(w_1, w_2) := \sum_{k=1}^{\infty} 2^{-k}(\max_{0 \leq t \leq k} | w_1(t) - w_2(t)| \wedge 1)$. The Borel σ-algebra of $\mathbb{B}$ and $\mathbb{W}_0$ are denoted by $\mathcal{B}(\mathbb{B})$ and $\mathcal{B}(\mathbb{W}_0)$, respectively. Furthermore, for $t \in [0, \infty)$, we define $\mathcal{B}_t(\mathbb{B}) := \sigma(\pi_s : 0 \leq s \leq t)$, where $\pi_s(w) := w(s), w \in \mathbb{B}$. $\mathcal{B}_t(\mathbb{W}_0)$ is defined analogously.

The equation under investigation is the following path-dependent stochastic differential equation

$$dX(t) = b(t, X)dt + \sigma(t, X)dW(t), \quad t \in [0, \infty), \qquad \text{(SDE.pd)}$$

where $b : [0, \infty) \times \mathbb{B} \to \mathbb{R}^d$ and $\sigma : [0, \infty) \times \mathbb{B} \to \mathbb{R}^{d \times d_1}$ are $\mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{B})/\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{B})/\mathcal{B}(\mathbb{R}^{d \times d_1})$-measurable, respectively, such that for each $t \in [0, \infty)$

$b(t, \cdot)$ is $\mathcal{B}_t(\mathbb{B})/\mathcal{B}(\mathbb{R}^d)$-measurable, and $\sigma(t, \cdot)$ is $\mathcal{B}_t(\mathbb{B})/\mathcal{B}(\mathbb{R}^{d \times d_1})$-measurable. Furthermore, W is a standard d_1-dimensional $(\mathscr{F}_t)$-Brownian motion and $\mathbb{P}_W$ denotes the distribution of W on $(\mathbb{W}_0, \mathcal{B}(\mathbb{W}_0))$. Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ and $P \subset \mathcal{P}(\mathbb{B})$. We will write $P = P_{\mu_0}$, if all the time marginal laws of the measures in P start with the same measure μ_0 at time $t = 0$.

Definition 2.1 (P_{μ_0}-weak solution)**.** *A pair (X, W) is called a P_{μ_0}-weak solution to* (SDE.pd)*, if $X = (X(t))_{t \geq 0}$ is an $(\mathscr{F}_t)$-adapted process with paths in $\mathbb{B}$, and W is a standard d_1-dimensional $(\mathscr{F}_t)$-Brownian motion on some stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \geq 0})$ such that the following holds:*

(i) $\mathbb{P}\left(\int_0^T |b(s, X)| + |\sigma(s, X)|^2 ds < \infty\right) = 1$*, for every $T \geq 0$,*

(ii) The following equation holds

$$X(t) = X(0) + \int_0^t b(s, X)ds + \int_0^t \sigma(s, X)dW(s), \text{ for every } t \geq 0 \; \mathbb{P}\text{-a.s.,} \qquad (2.1)$$

(iii) $\mathbb{P} \circ X^{-1} \in P_{\mu_0}$ *(in particular, $\mathbb{P} \circ X(0)^{-1} = \mu_0$).*

Remark 2.2. From the measurability assumptions on b and σ, it follows that, if X is as in Definition 2.1, then both processes $b(\cdot, X)$ and $\sigma(\cdot, X)$ are $(\mathscr{F}_t)$-adapted.

Definition 2.3 (P_{μ_0}-weak uniqueness)**.** *We say that P_{μ_0}-weak uniqueness holds for* (SDE.pd)*, if any two P_{μ_0}-weak solutions (X, W), (X', W') on stochastic bases $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \geq 0})$ and $(\Omega', \mathscr{F}', \mathbb{P}'; (\mathscr{F}'_t)_{t \geq 0})$, respectively, have the same law on $\mathbb{B}$, i.e. $\mathbb{P} \circ X^{-1} = \mathbb{P}' \circ (X')^{-1}$.*

Definition 2.4 (P_{μ_0}-pathwise uniqueness)**.** *We say that P_{μ_0}-pathwise uniqueness holds for* (SDE.pd)*, if for any two P_{μ_0}-weak solutions (X, W), (Y, W) on a common stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \geq 0})$ with a common standard d_1-dimensional $(\mathscr{F}_t)$-Brownian motion W, $X(0) = Y(0)$ $\mathbb{P}$-a.s. implies $X(t) = Y(t)$ for all $t \geq 0$ $\mathbb{P}$-a.s.*

Let $\tilde{\mathcal{E}}_{\mu_0}$ to be the set of all maps $F_{\mu_0} : \mathbb{R}^d \times \mathbb{W}_0 \to \mathbb{B}$ which are $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{W}_0)}^{\mu_0 \otimes \mathbb{P}_W} / \mathcal{B}(\mathbb{B})$-measurable, where $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{W}_0)}^{\mu_0 \otimes \mathbb{P}_W}$ denotes the completion of $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{W}_0)$ with respect to the measure $\mu_0 \otimes \mathbb{P}_W$.

Definition 2.5 (P_{μ_0}-strong solution)**.** *The equation* (SDE.pd) *has a P_{μ_0}-strong solution if there exists $F_{\mu_0} \in \tilde{\mathcal{E}}_{\mu_0}$ such that, for μ_0-a.e. $x \in \mathbb{R}^d$, $F_{\mu_0}(x, \cdot)$ is $\overline{\mathcal{B}_t(\mathbb{W}_0)}^{\mathbb{P}_W}/\mathcal{B}_t(\mathbb{B})$-measurable for every $t \in [0, \infty)$, and for any standard d_1-dimensional $(\mathscr{F}_t)$-Brownian motion W on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \geq 0})$ and any $\mathscr{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable function $\xi : \Omega \to \mathbb{R}^d$ with $\mathbb{P} \circ \xi^{-1} = \mu_0$, one has that $(F_{\mu_0}(\xi, W), W)$ is a P_{μ_0}-weak solution to* (SDE.pd) *with $X(0) = \xi$ $\mathbb{P}$-a.s. Here, $\overline{\mathcal{B}_t(\mathbb{W}_0)}^{\mathbb{P}_W}$ denotes the completion with respect to $\mathbb{P}_W$ in $\mathcal{B}(\mathbb{W}_0)$.*

Definition 2.6 (unique P_{μ_0}-strong solution)**.** *The equation* (SDE.pd) *has a unique P_{μ_0}-strong solution, if there exists a function $F_{\mu_0} \in \tilde{\mathcal{E}}_{\mu_0}$ satisfying the adaptedness condition in Definition 2.5 and if the following two conditions are satisfied.*

1. *For every standard d_1-dimensional $(\mathscr{F}_t)$-Brownian motion W on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \geq 0})$ and any $\mathscr{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable $\xi : \Omega \to \mathbb{R}^d$ with $\mathbb{P} \circ \xi^{-1} = \mu_0$, $(F_{\mu_0}(\xi, W), W)$ is a P_{μ_0}-weak solution.*

2. *For any P_{μ_0}-weak solution (X, W) to* (SDE.pd) *we have $X = F_{\mu_0}(X(0), W)$ a.s.*

Remark 2.7. Let (X, W) be a P_{μ_0}-weak solution to (SDE.pd) on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \geq 0})$. Since $X(0)$ and W are independent, we have $\mathbb{P} \circ (X(0), W)^{-1} = \mu_0 \otimes \mathbb{P}_W$. In particular, P_{μ_0}-weak uniqueness holds for (SDE.pd) provided there exists a unique P_{μ_0}-strong solution to (SDE.pd).

Let us state the main theorem of this section.

Theorem 2.8 (restricted Yamada–Watanabe theorem)**.** *Let P_{μ_0}, b and σ be as above. Then the following statements regarding* (SDE.pd) *are equivalent:*

(i) *There exists a P_{μ_0}-weak solution and P_{μ_0}-pathwise uniqueness holds.*

(ii) *There exists a unique P_{μ_0}-strong solution.*

3 Application of the restricted Yamada–Watanabe theorem to (general) McKean–Vlasov SDEs

In this section, we will consider general McKean–Vlasov SDEs on $\mathbb{R}^d$, which are of the form

$$dX(t) = F(t, X(t), \mathcal{L}_{X(t)})dt + \sigma(t, X(t), \mathcal{L}_{X(t)})dW(t), \ \ t \in [0, T], \tag{MVSDE}$$

where $T \in (0, \infty)$ and $F : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d \times d}$ are $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d))/\mathcal{B}(\mathbb{R}^d)$- and $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d))/\mathcal{B}(\mathbb{R}^{d \times d})$-measurable, respectively. Let $(\mu_t)_{t \in [0, T]} \subset \mathcal{P}(\mathbb{R}^d)$ be a narrowly continuous curve of probability measures. In the following, we will use the notation $F^\mu(t, x) := F(t, x, \mu_t)$, $(t, x) \in [0, T] \times \mathbb{R}^d$. Note that F^μ and σ^μ are $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$- and $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^{d \times d})$-measurable, respectively. In the following, let us fix some notation for weak solutions to (MVSDE) and pathwise uniqueness for (MVSDE) among weak solutions with given time marginal laws. Therefore, we set

$$P^{(\mu_t)} := \{Q \in \mathcal{P}(C([0, T]; \mathbb{R}^d)) : Q \circ \pi_t^{-1} = \mu_t \ \forall t \in [0, T]\}.$$

We have the following definitions.

Definition 3.1. *A tuple $(X, W) = (X(t), W(t))_{t \in [0, T]}$ consisting of two $(\mathscr{F}_t)$-adapted $\mathbb{R}^d$-valued stochastic processes on some given stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \in [0, T]})$ is called a $P^{(\mu_t)}$-weak solution to* (MVSDE) *if W is a standard d-dimensional $(\mathscr{F}_t)$-Brownian motion, and*

(i) $\mathbb{P}\left(\int_0^T |F(t, X(t), \mathcal{L}_{X(t)})| + |\sigma(t, X(t), \mathcal{L}_{X(t)})|^2 dt < \infty\right) = 1,$

(ii) *the following equality holds $\mathbb{P}$-a.s.:*

$$X(t) = X(0) + \int_0^t F(s, X(s), \mathcal{L}_{X(s)})ds + \int_0^t \sigma(s, X(s), \mathcal{L}_{X(s)})dW(s) \ \forall t \in [0, T],$$

(iii) $\mathbb{P} \circ (X(t))^{-1} = \mu_t$, *for all $t \in [0, T]$.*

Definition 3.2. *We say that $P^{(\mu_t)}$-pathwise uniqueness holds for* (MVSDE)*, if for any two $P^{(\mu_t)}$-weak solutions (X, W), (Y, W) on a common stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \in [0, T]})$ with a common standard d-dimensional $(\mathscr{F}_t)$-Brownian motion W, $X(0) = Y(0)$ $\mathbb{P}$-a.s. implies $X(t) = Y(t)$ for all $t \in [0, T]$ $\mathbb{P}$-a.s.*

Let us note that (X, W) is a $P^{(\mu_t)}$-weak solution to (MVSDE) if and only if it is a $P^{(\mu_t)}$-weak solution to the SDE(!)

$$dX(t) = F^\mu(t, X(t))dt + \sigma^\mu(t, X(t))dW(t), t \in [0, T], \tag{SDE$_\mu$}$$

and, obviously, $P^{(\mu_t)}$-pathwise uniqueness holds for (MVSDE) if and only if it holds for (SDE$_\mu$). Therefore, also the concept of a (unique) $P^{(\mu_t)}$-strong solution is the same for (MVSDE) and (SDE$_\mu$). This leads to the following application of the restricted Yamada–Watanabe theorem (Theorem 2.8) to (MVSDE).

Theorem 3.3. *Let $(\mu_t)_{t\in[0,T]} \subset \mathcal{P}_0(\mathbb{R}^d)$ be a narrowly continuous curve of probability measures. The following statements regarding* (MVSDE) *are equivalent.*

(i) There exists a $P^{(\mu_t)}$-weak solution and $P^{(\mu_t)}$-pathwise uniqueness holds.

(ii) There exists a unique $P^{(\mu_t)}$-strong solution.

Let us finally note that (MVSDE.PME) can be considered as an equation in the general form (MVSDE) as the following remark conveys.

Remark 3.4. The coefficients of (MVSDE.PME) fulfill the measurability conditions of the coefficients of (MVSDE) in the following sense. Consider the time-homogenous coefficients $F : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ defined via $F(x,\nu) := E(x)b(v_a(x))$, $\sigma(x,\nu) := \sqrt{2a\,(v_a(x))}\mathbb{1}_{d\times d}$, where $x \in \mathbb{R}^d, \nu \in \mathcal{P}(\mathbb{R}^d)$ and v_a is the following version of the dx-a.e. uniquely determined density of the absolutely continuous part of the probability measure ν with respect to the d-dimensional Lebesgue measure dx given by the Besicovitch derivation theorem

$$v_a(x) := \begin{cases} \lim_{r\to\infty} \frac{\nu(B_r(x))}{dx(B_r(0))}, & x \in E_\nu, \\ 0, & x \in E_\nu^{\complement}, \end{cases} \quad \text{for all } \nu \in \mathcal{P}(\mathbb{R}^d),$$

where $E_\nu := \left\{ x \in \mathbb{R}^d : \exists \lim_{r\to\infty} \frac{\nu(B_r(x))}{dx(B_r(0))} \in \mathbb{R} \right\} \in \mathcal{B}(\mathbb{R}^d)$ and $dx(E_\nu^{\complement}) = 0$ (cf. [1, Theorem 2.22]). Hence, it is easy to see that $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \ni (x,\nu) \mapsto v_a(x) \in [0,\infty)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d))/\mathcal{B}(\mathbb{R})$ measurable. In the following, we will always consider this version of the absolutely continuous part of a probability measure with respect to the d-dimensional Lebesgue measure.

Now, we are well equipped to translate Theorem 3.3 into action.

4 Strong solvability of (MVSDE.PME)

4.1 The procedure and the main result

Our overall goal is to apply Theorem 3.3, which will enable us to show that there exists a strong solution to (MVSDE.PME) (see Theorem 3.3). In order to achieve this, we will do the following steps.

1. We will use the recent result [4] (and the techniques of [5]) for (PME), in order to guarantee the existence of a probability solution u with sufficient Sobolev-regularity under the conditions (i)–(iv) (see Theorem 4.2).

2. We will apply the superposition principle procedure for McKean–Vlasov SDEs from [3, Section 2] in combination with the result of Step 1 in order to obtain a weak solution to (MVSDE.PME) with time marginal law densities u (see Theorem 4.3).

3. Afterwards, we will prove pathwise uniqueness for (MVSDE.PME) among weak solutions with time marginal law densities u via a pathwise uniqueness result for SDEs (see Theorem 4.4) and Step 1 in Theorem 4.6.

For the ease of notation, we set $P^{(u_t)} := P^{(\mu_t)}$, whenever $(\mu_t)_{t\in[0,T]}$ is a narrowly continuous curve of probability measures with $\mu_t = u_t(x)dx, u_t \in \mathcal{P}_0(\mathbb{R}^d)$, $t \in [0,T]$.

The steps will be carried out in the subsequent subsections. Combining the results of the steps with Theorem 3.3 yield the main result of this section and paper.

Theorem 4.1 (main result). *Let $d \neq 2$. Assume that conditions (i)–(iv) and* (v) *(see below) are fulfilled and that $u_0 \in \mathcal{P}_0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then,* (MVSDE.PME) *has a unique $P^{(u_t)}$-strong solution, where u is the constructed probability solution to* (PME) *provided by Theorem 4.2.*

The proof of Theorem 4.1 will be postponed to the end of this section.

4.2   Existence of a bounded Sobolev-regular probability solution u to (PME)

As described in the procedure in Section 4.1, the first step is to conclude the existence of a sufficiently regular solution to (PME) from [4] under the assumptions (i)–(iv). Combining [4, Theorem 2.2] with the techniques of [5], Barbu and Röckner showed that, under more general assumptions on the coefficients than we require in (i)–(iv), there exists a unique mild solution to (PME), interpreted as a Cauchy problem driven by an m-accretive operator. This solution is also an integrable and bounded Schwartz-distributional solution to (PME) if u_0 is integrable and bounded as well. If, in addition, β is non-degenerate, i.e. $\beta' \geq \gamma_0 > 0$, this *specific* solution can be proved to have certain desirable Sobolev-regularity under our conditions. Our approach relies on exactly this regularity when proving the pathwise uniqueness result for (MVSDE.PME) (cf. Theorem 4.6).

Theorem 4.2 (probability solution to (PME)). *Let $d \neq 2$ and $u_0 \in \mathcal{P}_0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Under the assumptions (i)–(iv), there exists a probability solution u to (PME) such that*

$$u \in L^2([0,T]; H^1(\mathbb{R}^d)) \cap L^\infty([0,T] \times \mathbb{R}^d). \tag{4.1}$$

Proof. By [4, Theorem 2.2], we know that there exists a Schwartz-distributional solution u to (PME) with $u \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T] \times \mathbb{R}^d)$, and with the property that $u_0 \in \mathcal{P}_0(\mathbb{R}^d)$ implies $u_t \in \mathcal{P}_0(\mathbb{R}^d)$, for all $t \in [0,T]$. Here, we should note that the authors require $\beta \in C^2(\mathbb{R}^d)$. However, due to (ii) and $\operatorname{div} E \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, this condition can be relaxed to $\beta \in C^1(\mathbb{R})$; this works analogous to [5, p. 20, proof of (2.6)]. Further, using the technique in [5], our assumptions imply that $u \in L^2([0,T]; H^1(\mathbb{R}^d))$. For an elaborate proof of these facts, see [15]. □

4.3   Existence of a $P^{(u_t)}$-weak solution to (MVSDE.PME)

The second step of the procedure in Section 4.1 is to show the existence of a weak solution to (MVSDE.PME). The following theorem is a variant of [4, Theorem 6.1 (a)] and is based on a superposition principle procedure for McKean–Vlasov SDEs as described in [3, Section 2], which generalises the procedure in [2, Section 2]).

Theorem 4.3 ($P^{(u_t)}$-weak solution). *Let $d \neq 2$ and $u_0 \in \mathcal{P}_0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Assume that conditions (i)–(iv) are fulfilled. Then, there exists a $P^{(u_t)}$-weak solution (X,W) to (MVSDE.PME), where u is the probability solution provided by Theorem 4.2.*

Proof. It is clear that conditions (i) and (iv) imply that b and a are continuous. By Theorem 4.2, we have that, in particular, $u \in L^\infty([0,T] \times \mathbb{R}^d)$. Hence, $a(u) \in L^\infty([0,T] \times \mathbb{R}^d)$, and, using (iii), $Eb(u) \in L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$. This yields

$$\int_0^T \int_{\mathbb{R}^d} \left[|a(u(t,x))| + |E(x)b(u(t,x))| \right] u(t,x) dx dt < \infty.$$

This enables us to use the superposition principle procedure for McKean–Vlasov SDEs in [3, Section 2] which provides us with a $P^{(u_t)}$-weak solution (X, W) to (MVSDE.PME). This finishes the proof. □

4.4 $P^{(u_t)}$-pathwise uniqueness for (MVSDE.PME)

The third step of the procedure in Section 4.1 is to show $P^{(u_t)}$-pathwise uniqueness for (MVSDE.PME), where u is the probability solution to (PME) provided by Theorem 4.2. As explained in the beginning of Section 4, showing $P^{(u_t)}$-pathwise uniqueness for (MVSDE.PME) is the same as showing $P^{(u_t)}$-pathwise uniqueness for (SDE$_\mu$), where $\mu = (u_t dx)_{t \in [0,T]}$. Since the coefficients $Eb(u)$ and $\sqrt{a(u)}$ are not continuous in the spacial variable, we will recall a pathwise uniqueness result for SDEs with time-dependent

Sobolev-coefficients from [28] in Subsection 4.4.1. In Subsection 4.4.2, we will then apply this result to show $P^{(u_t)}$-pathwise uniqueness for (MVSDE.PME).

4.4.1 A pathwise uniqueness result for SDEs with bounded Sobolev-regular coefficients

There are a lot of pathwise uniqueness results for SDEs across the literature, but numerous classical as well as recent results on this topic require the diffusion coefficient to be continuous in the spacial variable. For example, in [35] and [21] (see also the classic result [33]), the authors consider a (singular) drift coefficient in $L^q([0, T]; L^p(\mathbb{R}^d))$, $d/p + 2/q < 1$, where $p, q \in (2, \infty)$, and an elliptic diffusion coefficient, whose distributional derivative is in $L^q([0, T]; L^p(\mathbb{R}^d))$, for the same choice of p and q as before. Additionally, the diffusion coefficient is considered to be uniformly continuous in the x-variable locally uniformly in time.

To the best of our knowledge, the best pathwise uniqueness results for SDEs, with no a priori continuity assumption on the diffusion term, can be obtained through [28] and [10]; these works require Sobolev-regularity of the coefficients in the x-variable, but no a priori continuity property. Here, we will just focus on the result obtained in [28], as it allows the coefficients to only have local Sobolev-regularity in the spatial variable.

In this subsection, we will provide the reader with a simple modification of a restricted pathwise uniqueness result for SDEs in the proof of [28, Theorem 1.1]. The strength of this result is that there is a trade-off between the regularity of the densities of the time marginal laws of a solution process and the regularity of the coefficients of the equation. Further, the estimate [11, Lemma A.3] and [11, Lemma A.2], involving the (local) Hardy-Littlewood maximal function, shows that sufficient Sobolev-regular coefficients of (SDE) (see below) satisfy (4.3), see Remark 4.5. Here, we also refer to [10], where the authors developed an interesting modification of this estimate (see [10, Lemma 3.2]), which turned out to be very useful when showing a restricted pathwise uniqueness result in the critical case when the drift coefficient of an SDE has $L^1([0, T]; W^{1,1}(\mathbb{R}^d))$-regularity (cf. [10, Theorem 1.1]).

Let us consider the following stochastic differential equation

$$dX(t) = \boldsymbol{F}(t, X(t))dt + \boldsymbol{\sigma}(t, X(t))dW(t), t \in [0, T], X(0) = \xi. \qquad \text{(SDE)}$$

where $\boldsymbol{F} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\boldsymbol{\sigma} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$- and $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^{d \times d})$-measurable functions, respectively; the initial condition ξ and the d-dimensional Brownian motion W are considered to be analogous to those introduced in the beginning of this work. We have the following theorem.

Theorem 4.4 (restricted pathwise uniqueness for (SDE)). *Let $\boldsymbol{F}, \boldsymbol{\sigma} \in L^\infty([0, T] \times \mathbb{R}^d)$. Fix $p, q, p', q' \in [1, \infty]$, such that $1/p + 1/p' = 1/q + 1/q' = 1$. Let $(X, W), (Y, W)$ be two (usual) weak solutions to (SDE) up to time T on a common stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \in [0,T]})$ with $X(0) = Y(0)$ $\mathbb{P}$-a.s., such that*

$$d\mathcal{L}_{X(\cdot)}/dx, d\mathcal{L}_{Y(\cdot)}/dx \in L^{q'}([0, T]; L^{p'}_{loc}(\mathbb{R}^d)). \qquad (4.2)$$

If for any radius $R > 0$, there exists a function $f_R \in L^q([0, T]; L^p(B_R(0))$, such that for almost all $(t, x, y) \in [0, T] \times B_R(0) \times B_R(0)$

$$2\langle x - y, \boldsymbol{F}(t, x) - \boldsymbol{F}(t, y)\rangle_{\mathbb{R}^d} + |\boldsymbol{\sigma}(t, x) - \boldsymbol{\sigma}(t, y)|^2 \le (f_R(t, x) + f_R(t, y)) \cdot |x - y|^2. \qquad (4.3)$$

Then, $\sup_{t \in [0,T]} |X(t) - Y(t)| = 0$ $\mathbb{P}$-a.s.

Proof. The proof is essentially contained in the proof of [28, Theorem 1.1]. However, since we allow different integrability in space and time for the time marginal law

densities in (4.2) and for f_R, we need to separately apply Hölder-estimates in the proof of [28, Theorem 1.1]. We omit the details here, since the adaption of the proof is straightforward. □

Remark 4.5. Let $q = p = 1$. Then, the proof of Theorem 4.4 allows to replace (4.2) by $d\mathcal{L}_{X(\cdot)}/dx, d\mathcal{L}_{Y(\cdot)}/dx \in L^\infty_{loc}([0,T] \times \mathbb{R}^d)$. Further, (4.3) is satisfied if, for some $\epsilon > 0$, $\boldsymbol{F} \in L^1([0,T]; W^{1,1+\epsilon}_{loc}(\mathbb{R}^d; \mathbb{R}^d))$, $\boldsymbol{\sigma} \in L^2([0,T]; W^{1,2}_{loc}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$. For details see [Remark 1.2,RO10].

4.4.2 Application to (MVSDE.PME)

In this subsection, we will apply the pathwise uniqueness result for SDEs from the previous section to (MVSDE.PME). We impose the following additional assumption on a and E, respectively.

$$a \text{ is locally Lipschitz continuous, } \nabla E \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^{d\times d}). \tag{v}$$

Note that if $\beta \in C^2(\mathbb{R}^d)$, then $a \in C^1(\mathbb{R}^d)$ with $a'(0) = \frac{1}{2}\beta''(0)$ and the first part of (v) is automatically satisfied. We have the following

Theorem 4.6 ($P^{(u_t)}$-pathwise uniqueness). *Let $d \neq 2$ and $u_0 \in \mathcal{P}_0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Assume that the conditions (i)–(iv) and (v) are fulfilled. Let $(X, W), (Y, W)$ be $P^{(u_t)}$-weak solutions to (MVSDE.PME) on the same filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\in[0,T]})$, with the same Brownian motion W, and $X(0) = Y(0)$ $\mathbb{P}$-a.s., where u is the probability solution provided by Theorem 4.2. Then, $\sup_{t\in[0,T]} |X(t) - Y(t)| = 0$ $\mathbb{P}$-a.s.*

Proof of Theorem 4.6. Let u denote the probability solution provided by Theorem 4.2. As explained before, we exactly need to show that $P^{(u_t)}$-pathwise uniqueness holds for (SDE) with coefficients $\boldsymbol{F} := Eb(u)$ and $\boldsymbol{\sigma} := \sqrt{2a(u)}\mathbb{1}_{d\times d}$. We will now check the conditions in Theorem 4.4. By Remark 4.5, these will be implied by the following conditions together with (4.1).

(a) $Eb(u) \in L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d), a(u) \in L^\infty([0,T] \times \mathbb{R}^d)$,

(b) $\nabla(\sqrt{a(u)}) \in L^2([0,T]; L^2(\mathbb{R}^d)), \nabla(Eb(u)) \in L^2([0,T]; L^2_{loc}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$.

Clearly, (a) is satisfied, since b and a are continuous and E and u are bounded. Let us now show condition (b). Due to (1.1) and (v), $\sqrt{a}$ is locally Lipschitz continuous. Let $L > 0$ be the Lipschitz constant of $\sqrt{a}$ on the interval $[- \|u\|_{L^\infty}, \|u\|_{L^\infty}]$. By (4.1), $\sqrt{a(u_t)}$ has a weak gradient satisfying $|\nabla \sqrt{a(u_t)}| \leq L|\nabla u_t|$ a.e., for almost all $t \in [0,T]$ (cf. [36, Theorem 2.1.11.]). Using (4.1), we see that condition (b) is fulfilled for a. Analogous to the proof of (b), condition (4.1) and $b \in C^1(\mathbb{R})$ imply that $\nabla(b(u)) \in L^2([0,T]; L^2(\mathbb{R}^d))$. Having in mind (iii), (iv) and (v), for each $R > 0$ we estimate using the usual product rule for Sobolev functions

$$\|\nabla(Eb(u))\|_{L^2([0,T]; L^2(B_R(0)))} \leq T^{1/2} \|b(u)\|_{L^\infty} \|\nabla E\|_{L^2(B_R(0))} + d \|E\|_{L^\infty} \|\nabla(b(u))\|_{L^2} < \infty,$$

This finishes the proof. □

The proof of the main result, Theorem 4.1, is straight forward.

Proof of Theorem 4.1. On the one hand, by Theorem 4.3, there exists a $P^{(u_t)}$-weak solution to (MVSDE.PME). On the other hand, Theorem 4.6 shows that $P^{(u_t)}$-pathwise uniqueness holds for (MVSDE.PME). Now, the assertion follows from Theorem 3.3. □

Remark 4.7. In fact, we can relax the integrability condition on E in (v) to $\nabla E \in L^1_{loc}(\mathbb{R}^d; \mathbb{R}^{d \times d})$. Indeed, one can use the stopping time technique of the proof of [28, Theorem 1.1] and the general technique of the proof of [10, Theorem 1.1 (ii)] to prove, in particular, a pathwise uniqueness result for (SDE) with coefficients $\boldsymbol{F} \in L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d) \cap L^1([0,T]; W^{1,1}_{loc}(\mathbb{R}^d; \mathbb{R}^d))$ and $\boldsymbol{\sigma} \in L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^{d \times d}) \cap L^2([0,T]; H^1_{loc}(\mathbb{R}^d; \mathbb{R}^{d \times d}))$ among weak solutions with bounded time marginal law densities. The details are carried out in the author's thesis [15].

Remark 4.8. With a similar technique as in this work, we are able to prove the existence of a strong solution to the degenerate McKean–Vlasov SDE associated to the classical porous medium equation in one dimension

$$\partial_t u = \partial_x^2(|u|^{m-1}u), \quad t \in [0,T], u|_{t=0} = u_0 \in \mathcal{P}_0(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

where $m > 3$. From [14, Theorem 1.2], it is known that there exists a unique entropy solution satisfying $u^{\frac{m-1}{2}} \in L^2([0,T]; H^{\frac{1}{2}}(\mathbb{R}))$. If u_0 is chosen as above, u is a bounded probability solution to this equation. As above, the existence of a $P^{(u_t)}$-weak solution is provided by the superposition principle procedure for MVSDEs. Employing the one dimensional pathwise uniqueness result [10, Theorem 1.2], we can show $P^{(u_t)}$-pathwise uniqueness to the corresponding MVSDE. Using a similar procedure to the one in Section 4.1, we obtain a unique $P^{(u_t)}$-strong solution. To the best of the author's knowledge, this result seems to be new. Note that in [6] the authors proved the existence of a unique strong solution in the case $d = 1$ under the stronger assumption that the initial distribution density is given by $u_0 \in \mathcal{P}_0(\mathbb{R}) \cap C^1_b(\mathbb{R})$ with the property that u_0 is always strictly positive.

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