Electron. Commun. Probab. **28** (2023), article no. 22, 1–12.
https://doi.org/10.1214/23-ECP529
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Non-universality in clustered ballistic annihilation*

Matthew Junge† Arturo Ortiz San Miguel‡ Lily Reeves§
Cynthia Rivera Sánchez¶

Abstract

In ballistic annihilation, infinitely many particles with randomly assigned velocities move across the real line and mutually annihilate upon contact. We introduce a variant with superimposed clusters of stationary particles, and provide a simple formula for the critical initial cluster density in terms of the mean and variance of the cluster size.

Keywords: statistical physics; interacting particle system; phase transition.
MSC2020 subject classifications: 60K35.
Submitted to ECP on April 11, 2023, final version accepted on May 14, 2023.

1 Introduction

Ballistic annihilation (BA) is a stochastic spatial system in which particles are placed throughout the real line with independent and identically distributed spacings and proceed to move at independently sampled velocities. Collisions result in mutual annihilation. Interest in annihilating dynamics with ballistic particle trajectories arose as an extremal case of diffusion-limited annihilating systems being studied by physicists and mathematicians in the late 20th century [23, 5, 6].

Droz et. al in [11] analyzed the symmetric three-velocity setting with velocities sampled from $-1, 0, 1$. Velocity 0 particles, which we will refer to as *blockades*, occur with probability p. Velocity $+1$ and -1 particles, which we will call *right* and *left arrows*, respectively, each occur with probability $(1-p)/2$. Let $\theta(p)$ be the probability that a given blockade is never annihilated. By ergodicity, the limiting proportion of surviving blockades converges to $p\theta(p)$. So,

$$p_c = \inf\{p\colon \theta(p) > 0\} \tag{1.1}$$

represents the critical initial blockade density for species survival.

Three-velocity BA has multiple collision types: arrow–blockade and arrow–arrow. The rates at which these occur are not obvious, and thus neither is the value of p_c. Another challenge is that BA exhibits long-range dependence. This makes it difficult to

*Junge, Ortiz San Miguel, and Rivera Sánchez were partially supported by NSF grant DMS-2115936. Part of this research was completed during the 2022 Baruch College Discrete Math REU partially supported by NSF grant DMS-2051026.

†Baruch College. E-mail: Matthew.Junge@baruch.cuny.edu
‡Brown University. E-mail: arturo_ortiz@brown.edu
§Cornell University. E-mail: zw477@cornell.edu
¶University of Puerto Rico Río Piedras. E-mail: cynthia.rivera15@upr.edu

extrapolate from finite systems and to account for multiple velocities. BA is also sensitive to perturbation. Changing an arrow to a blockade may increase the lifespans of other arrows. Thus, it is not obvious how to rigorously confirm the intuition that θ and related quantities are monotone in p. This is problematic. For example, it is not a priori obvious that the definition of p_c at (1.1) is equal to $\sup\{p\colon \theta(p) = 0\}$.

Droz et. al [11] and, later in more detail, Krapivsky et. al [19] worked out the phase-behavior of three-velocity BA and concluded that $p_c = 1/4$. A rough intuition for why (given in the introduction of [11]) is that arrow–arrow collisions should occur, on average, at twice the rate of arrow–blockade collisions. Under this assumption, on average, six arrows are removed for every two blockades. This suggests that the critical density starts with a 1:3 ratio of blockades to arrows, hence p_c should be $1/4$. This is difficult to make rigorous and ignores any spatial effects. The derivations in both works, though more sophisticated than this heuristic, were not completely rigorous. Despite some progress towards upper bounds on p_c [12, 22, 8], showing that $p_c = 1/4$ remained an open problem. Even proving the much weaker statement that $p_c > 0$ was a problem widely advertised by Sidoravicius in the mid 2010s. A breakthrough from Haslegrave, Sidoravicius, and Tournier introduced an exactly solvable approach that proved that $p_c = 1/4$ [14]. In the same work, the authors also worked out finer details such as tail survival probabilities and the “skyline” of collision types.

Many of the findings in [14] are *universal* in the sense that the results hold for any continuous law of particle spacings. For example, $p_c = 1/4$ so long as triple collisions almost surely do not occur [8, 14]. Note that [14] also proved results concerning universality when a random arrow survives a triple collision. Additional universality properties with respect to particle spacings were observed in the followup work by Haslegrave and Tournier [15] as well as Cruzado-Padro, Junge, and Reeves [20]. Broutin and Marckert discovered that a closely related bullet process with finitely many particles has a universal law governing the number of surviving particles that does not depend on velocity or spacing laws [7].

A canonical form of universality is invariance with respect to the average particle density. It is physically and mathematically natural to allow for clusters of superimposed particles, as is standard in other diffusion-limited annihilating systems [6]. To test the robustness of BA dynamics to the initial particle density, we introduce a variant of BA with random clusters of multiple blockades. We prove that the analogue of the critical value (1.1) depends on more than simply the average initial density of particles. Thus, three-velocity BA lacks this type of universality. To our knowledge, this is a new discovery that was not previously conjectured.

1.1 Notation

We let $(x_n)_{n\in\mathbb{Z}}$ be an ordered sequence of starting locations for particles. To standardize placements, set $x_0 = 0$ and assume that $x_n - x_{n-1}$ are sampled independently according to a continuous distribution with support contained in $(0,\infty)$. Let X be a nonnegative integer-valued random variable with probability distribution $\mu = (\mu_k)_{k\geq 0}$, and let $f(t) = \mathbf{E}[t^X] = \sum_{k=0}^{\infty} \mu_k t^k$ be the probability generating function. In an abuse of notation, we will write $\mathbf{E}[\mu]$ and $\operatorname{var}(\mu)$ for the mean and variance of X. Adopt the convention that $\operatorname{var}(\mu) = \infty$ whenever $\mathbf{E}[\mu] = \infty$. Take $(X_n)_{n\in\mathbb{Z}}$ to be independent and μ-distributed. Each site x_n either independently starts with a *cluster* of X_n-blockades with probability $p \in [0,1]$, or otherwise contains a single arrow whose velocity is sampled uniformly from ± 1. We will sometimes refer to the starting number of blockades in a cluster as the *size* and write k*-cluster* to refer to a cluster of size k. Blockades are stationary. Left and right arrows move with velocities -1 and $+1$, respectively.

Define μ*-clustered ballistic annihilation* to have the just-described starting configuration at time 0. As time evolves, particles move at their assigned velocities. When two arrows collide, both vanish from the system. When an arrow collides with a cluster containing $k \geq 1$ remaining blockades, the arrow vanishes and one blockade is removed from the cluster (so $k-1$ blockades remain). A more formal construction of BA that easily generalizes to include clusters can be found in [14].

We denote the events that a cluster starts at x_n by $\dot{\bullet}_n$, or that a left or right arrow starts at x_n by $\overleftarrow{\bullet}_n$ and $\vec{\bullet}_n$, respectively. When x_n contains a cluster, we denote the starting size with a superscript $\dot{\bullet}_n^{X_n}$. We will frequently refer to $\dot{\bullet}, \overleftarrow{\bullet}, \vec{\bullet}$ as particles. Accordingly, collision events and visits to a location $u \in \mathbb{R}$ are specified by

$$\begin{aligned}
\vec{\bullet}_m \longleftrightarrow \overleftarrow{\bullet}_n &= \{\vec{\bullet}_m \text{ and } \overleftarrow{\bullet}_n \text{ mutually annihilate}\}\\
\vec{\bullet}_m \longleftrightarrow \overleftarrow{\bullet} &= \{\vec{\bullet}_m \text{ mutually annihilates with an arrow}\}\\
\dot{\bullet}_n \leftarrow \overleftarrow{\bullet}_m &= \{\overleftarrow{\bullet}_m \text{ mutually annihilates with a blockade at } x_n\}\\
\dot{\bullet}_n^k \leftarrow \overleftarrow{\bullet}_m &= \{\overleftarrow{\bullet}_m \text{ mutually annihilates with a blockade at } x_n,\ X_n = k\}\\
\dot{\bullet} \leftarrow \overleftarrow{\bullet}_n &= \{\overleftarrow{\bullet}_n \text{ mutually annihilates with a blockade}\}\\
u \leftarrow \overleftarrow{\bullet} &= \{u \text{ is visited by a } \overleftarrow{\bullet}\}\\
u \overset{j}{\leftarrow} \overleftarrow{\bullet}_m &= \{\overleftarrow{\bullet}_m \text{ is the } j\text{th } \overleftarrow{\bullet} \text{ to arrive to } u\}.
\end{aligned}$$

The events $\vec{\bullet}_m \to \dot{\bullet}_n$, $\vec{\bullet}_m \to \dot{\bullet}$, $\vec{\bullet}_m \to \dot{\bullet}^k$, $\vec{\bullet} \to u$, and $\vec{\bullet} \overset{j}{\to} u$ are defined similarly. We denote complements of collision events with $\not\longleftrightarrow, \not\leftarrow$, and $\not\to$. Note that when an arrow hits a cluster we count that as visiting the site, so $\{\dot{\bullet}_k \leftarrow \overleftarrow{\bullet}\} \subseteq \{x_k \leftarrow \overleftarrow{\bullet}\}$.

It is often helpful to restrict to a system which only includes particles started in an interval $I \subseteq \mathbb{R}$. We notate this restriction by including I as a subscript on the event, for example, $(\dot{\bullet}_m \leftarrow \overleftarrow{\bullet}_n)_{[x_m, x_n]}$ is the event that $\bullet_m$ is a blockade that annihilates with a left arrow started at x_n in the process restricted to only the particles in $[x_m, x_n]$. Unless indicated otherwise, the default is that events are one-sided i.e., restricted to $(0,\infty)$. So, $\mathbf{P}(0 \leftarrow \overleftarrow{\bullet}) = \mathbf{P}((0 \leftarrow \overleftarrow{\bullet})_{(0,\infty)})$.

We now define the generalization of θ from the previous section for μ-clustered BA:

$$\theta = \theta(p,\mu) := \mathbf{P}((\vec{\bullet} \not\to 0)_{(-\infty,0)} \wedge (0 \not\leftarrow \overleftarrow{\bullet})_{(0,\infty)}).$$

It is convenient to instead work with the one-sided complement

$$q = q(p,\mu) := \mathbf{P}(0 \leftarrow \overleftarrow{\bullet}),$$

so that $\theta = (1-q)^2$. Define the critical value

$$p_c = p_c(\mu) := \inf\{p \colon \theta(p,\mu) > 0\}.$$

1.2 Results

Our main result is a simple formula for p_c that depends on both the mean and variance of μ. We also provide an implicit formula for q.

Theorem 1.1. *For μ-clustered BA it holds that*

$$p_c = \frac{1}{(\mathbf{E}[\mu]+1)^2 + \operatorname{var}(\mu)}. \tag{1.2}$$

Moreover, q is continuous, strictly decreasing on $[p_c, 1]$, and solves

$$\frac{(1-q)^2}{(1-q^2)\, q^2 f'(q) - 2qf(q) + q^2 + 1} = p \tag{1.3}$$

with $f(q) = \sum_{k=0}^{\infty} \mu_k q^k$ the probability generating function of μ.

A surprising consequence of Theorem 1.1 is that there is no phase transition whenever μ has infinite variance.

Corollary 1.2. *If* $\mathrm{var}(\mu) = \infty$*, then* $p_c = 0$*.*

Another corollary is that the value $p_c = 1/4$ in BA from [14] is maximal among all systems with $\mathbf{E}[\mu] = 1$.

Corollary 1.3. *If* $\mu_1 = 1$*, then* $p_c = 1/4$*. For all other* μ *with* $\mathbf{E}[\mu] = 1$*, we have* $p_c < 1/4$*.*

Lastly, Benitez, Junge, Lyu, Redman, and Reeves studied a coalescing version of ballistic annihilation in which particles sometimes survive collisions [4]. The primary interest was determining the analogue of p_c for these systems. However, they were unable to analyze the case in which blockades survive each collision with some fixed probability (see [4, Remark 5]). This is equivalent to μ-clustered ballistic annihilation with μ a geometric distribution. Thus, Theorem 1.1 gives the value of p_c in this unsolved case.

Corollary 1.4. *Let* $\beta \in (0,1)$ *and* μ *be a geometric distribution with parameter* β*, i.e.,* $\mu_k = (1-\beta)^{k-1}\beta$ *for* $k \geq 1$*. For* μ*-clustered ballistic annihilation it holds that*

$$p_c = \frac{\beta^2}{\beta^2 + \beta + 2}$$

and, by solving (1.3) *for* q*, we have for* $p \geq p_c$

$$q(p) = \frac{\sqrt{p^2\beta - p^2 - p\beta + 2p} - p\beta + \beta - 1}{p\beta^2 - p\beta + p - \beta^2 + 2\beta - 1}.$$

1.3 Discussion

There is no robust general theory that tells us whether or not a given interacting particle system will have a universal phase transition. On $\mathbb{Z}^d$, branching processes, diffusion-limited-annihilating systems, and activated random walk are processes known to have phase transitions that do not depend on the initial particle density [1, 9, 21]. The frog model and directed parking processes on d-ary trees have phase transitions that depend on more than the average density [10, 3, 17]. And, the number of visits to a distinguished site varies monotonically with the concentration of the initial particle placements for these processes on general families of graphs [16, 2].

It is a priori unclear whether or not p_c depends on more than $\mathbf{E}[\mu]$. On one hand, BA has dynamics similar to the systems considered in [2]. So, it is reasonable to expect some sensitivity to the initial density of particles. On the other hand, the mean-field heuristic presented in [11] and further clarified in [19, Section (b)] suggests that p_c might be universal. The explanation in [19] assumes that arrow–arrow collisions, on average, occur at twice the rate of blockade–left arrow collisions. This is “based on the expectation that the relative number of annihilation events is proportional to the relative velocities of the collision partners.” If this “expectation”, which seems to only depend on the relative velocities of particle types, still holds in μ-clustered BA, then the same heuristic would predict universality.

Theorem 1.1 settles the question. Put concisely, the more volatile μ becomes, the more space for arrow-arrow collisions, which enhances blockade survival. In a loose sense, our theorem says that the order particles are placed plays a role in determining p_c. A more detailed heuristic for why the variance plays a role in the formula for p_c in Theorem 1.1 comes from considering the extreme case in which $\mu_0 = (k-1)/k$ and $\mu_k = 1/k$ with k a large integer. As $\mathrm{var}(\mu) = k-1$ and $\mathbf{E}[\mu] = 1$, Theorem 1.1 implies that $p_c = 1/(k+3)$. To see intuitively why this is the correct order, suppose that 0 contains a k-cluster. Let x_N be the next site to the right of 0 that contains a k-cluster. We have

N is a geometric random variable with parameter p/k. Thus, we expect on the order of $(1-p)k/p$ arrows in $(0, x_N)$ along with some 0-clusters. The amount of arrows that reach the boundary of $(0, x_N)$ should be comparable to the magnitude of the discrepancy between left and right arrows started in $(0, x_N)$. By the central limit theorem, the discrepancy is on the order of $\sqrt{k/p}$, and so this order of left arrows from $(0, x_N)$ will reach 0 [13]. For these arrows to eliminate a significant portion of the k-cluster at 0, we would need $k \approx \sqrt{k/p}$, equivalently, $p \approx 1/k \sim 1/\operatorname{var}(\mu)$.

1.4 Proof overview

Our proof has three main parts. Section 2 is devoted to proving the recursive equation for q in Proposition 2.1. This is inspired by what was done in [14], but instead uses a version of the mass transport principle first observed in [18] and refined in [4]. The basic idea is to partition the event associated to q based on the velocity of $\bullet_1$.

An important probability for deriving this recursion is $s_k = \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \to \dot{\bullet}^k))$. In [14], it was observed that $s_1 = (1/2)pq^2$. Computing s_k for $k > 1$ in the proof of (2.2) is more involved. After applying the mass transport principle, this event partitions into various events in which $k+1$ left arrows arrive to 0 while satisfying non-symmetric spacing requirements. Remarkably, a broader symmetry than what was used in [14] (see (2.11) makes this case tractable and yields the simple formula $s_k = (1/2)p\mu_k k q^{k+1}$. In the proof of (2.3), we use similar methods to give a relatively simple formula for the companion probability $r_k = \mathbf{P}((0 \not\leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \to \dot{\bullet}^k))$. With these quantities in hand, it is straightforward to obtain (2.1). The second part is proving that q is continuous in p. The proof closely follows the argument that θ is continuous in asymmetric three-velocity ballistic annihilation from [18].

The last step, in Section 4, involves analyzing the recursion from Proposition 2.1. The recursion implies that $0 = (1-q)h(p,q)$ for an explicit function h. This tells us that either $q = 1$ or solves $h(p,q) = 0$. We prove that $h(u,1)$ has unique solution $u = p_c$ from Theorem 1.1. The goal is then to show that q, for any μ-clustered BA, continuously switches from being identically 1 for $p \le p_c$ to the unique curve determined by (1.3). A priori, it is not obvious how to prove that the roots of h are well-behaved and that q faithfully follows them. The continuity of q observed in Theorem 3.6 is crucial for ruling out the pathology that q jumps between 1 and solutions to $h = 0$. This more robust than past approaches [14, 4].

2 Recursion

The goal of this section is to prove the following recursive formula.

Proposition 2.1.

$$q = \frac{1-p}{2} + pqf(q) + s + q\left(\frac{1-p}{2} - s - r\right) \tag{2.1}$$

with

$$s := \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \to \dot{\bullet})) = \frac{pq^2}{2} f'(q) \tag{2.2}$$

$$r := \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \to \dot{\bullet})) = \frac{pq\big(q^2 f'(q) - qf'(q) - f(q) + 1\big)}{1-q}. \tag{2.3}$$

Proof of (2.1). We partition q in terms of the velocity of the first particle

$$q = \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \overleftarrow{\bullet}_1) + \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \dot{\bullet}_1) + \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \vec{\bullet}_1) \tag{2.4}$$

and will provide a formula for each summand. It is immediate that

$$\mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \overleftarrow{\bullet}_1) = \frac{1-p}{2}. \tag{2.5}$$

For the second summand, we further partition on the size of $\dot{\bullet}_1$ to write

$$\mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \dot{\bullet}) = \sum_{k=0}^{\infty} \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \dot{\bullet}_1 \wedge (X_1 = k)).$$

If $X_1 = k$, then $k+1$ left arrows must arrive at x_1 in order for 0 to be visited. This happens if and only if the jth left arrow to arrive reaches the starting location of the $(j-1)$th left arrow to arrive for $j = 2, \dots, k+1$. By similar reasoning as [14, Lemma 7], each of these arrivals is conditionally independent and has probability q. Thus, for $k \geq 0$ we have

$$\mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \dot{\bullet}_1 \wedge (X_1 = k)) = p \cdot \mu_k q^{k+1}.$$

Summing over k gives

$$\mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \dot{\bullet}_1) = pqf(q). \tag{2.6}$$

A similar argument as [22, Lemma 3.3] implies that all arrows are eventually annihilated. Since $\mathbf{P}(\vec{\bullet}_1 \to \dot{\bullet}) = s + r$, we may write

$$\begin{aligned}\mathbf{P}(\vec{\bullet}_1) = \frac{1-p}{2} &= \mathbf{P}(\vec{\bullet}_1 \to \dot{\bullet}) + \mathbf{P}(\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}) \\ &= s + r + \mathbf{P}(\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}). \end{aligned} \tag{2.7}$$

For 0 to be visited on the event $\{\vec{\bullet}_1\}$, the particle $\vec{\bullet}_1$ must first be annihilated. We partition on the collision type:

$$\begin{aligned}\mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge \vec{\bullet}_1) &= \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \to \dot{\bullet})) + \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet})) \\ &= s + q\mathbf{P}(\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}) \qquad (2.8) \\ &= s + q\left(\frac{1-p}{2} - s - r\right). \qquad (2.9)\end{aligned}$$

The equality at (2.8) follows from the definition of s and the fact that $\mathbf{P}(0 \leftarrow \overleftarrow{\bullet} \mid \vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}) = q$. This fact follows from the observation that conditional on $(\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}_j)$ for some $j > 1$, $(0 \leftarrow \overleftarrow{\bullet})$ occurs if and only if $(x_j \leftarrow \overleftarrow{\bullet})_{(x_j,\infty)}$, which has probability q. The move to (2.9) then uses (2.7). Combining (2.5), (2.6), and (2.9) in (2.4) gives (2.1). □

Next, we will prove the formulas for s and r at (2.2) and (2.3), respectively. These require the use of a Mass Transport Principle based on translation invariance.

Proposition 2.2 (Mass Transport Principle)**.** *Define a non-negative random variable $Z(m,n)$ for integers $m, n \in Z$ such that its distribution is diagonally invariant under translation, i.e., for any integer ℓ, $Z(m+\ell, n+\ell)$ has the same distribution as $Z(m,n)$. Then for each $m \in \mathbb{Z}$:*

$$\mathbf{E}\sum_{n \in Z} Z(m,n) = \mathbf{E}\sum_{n \in Z} Z(n,m).$$

Proof. Fubini's theorem and translation invariance give

$$\begin{aligned}\mathbf{E}\sum_{n \in Z} Z(m,n) &= \sum_{n \in Z} \mathbf{E}[Z(m,n)] \\ &= \sum_{n \in Z} \mathbf{E}[Z(2m-n,m)] = \sum_{n \in Z} \mathbf{E}[Z(n,m)] = \mathbf{E}\sum_{n \in Z} Z(n,m). \end{aligned}$$ □

Proof of (2.2). Let $s_k = \mathbf{P}((0 \leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \to \dot{\bullet}^k))$ so that $s = \sum_{k=0}^{\infty} s_k$. We will use the mass transport principle to relate the event associated to s_k to one that involves $k+1$ arrows arriving to the site containing a k-cluster. To this end, define

$$Z_k^j(a,b) = \sum_{c\in\mathbb{Z}} \Big[\mathbf{1}\{\dot{\bullet}_b^k \wedge (\vec{\bullet}_a \xrightarrow{j} x_b)_{[x_a,x_b)} \wedge (x_b \xleftarrow{k+1-j} \overleftarrow{\bullet}_c)_{(x_b,x_c]} \wedge (x_b - x_a < x_c - x_b)\}\Big]$$

for $a,b,j,k \in \mathbb{Z}$.

Observe that

$$s_k^j := \mathbf{P}((\vec{\bullet}_1 \xrightarrow{j} \dot{\bullet}^k) \wedge (0 \leftarrow \overleftarrow{\bullet})) = \mathbf{E}\sum_{b\in\mathbb{Z}} Z_k^j(1,b).$$

Define $\vec{D}_j$ to be the starting distance from x_1 of the jth particle to arrive to x_1 in the process restricted to particles in $(-\infty, x_1)$. We set $\vec{D}_j = \infty$ whenever fewer than j particles ever visit x_1. Define $\overleftarrow{D}_j$ similarly, but on (x_1,∞). By Proposition 2.2 and independence, s_k^j is equal to

$$\begin{aligned}\mathbf{E}\textstyle\sum_{a\in\mathbb{Z}} Z_k^j(a,1) &= \mathbf{P}(\dot{\bullet}_1^k)\mathbf{P}((\vec{\bullet} \xrightarrow{j} x_1)_{(-\infty,x_1)})\mathbf{P}((x_1 \xleftarrow{k+1-j} \overleftarrow{\bullet})_{(x_1,\infty)})\mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j}) \\ &= p\cdot \mu_k q^j q^{k+1-j}\mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j}). \end{aligned} \tag{2.10}$$

Since $s_k = \sum_{j=1}^k s_k^j$, (2.10) gives

$$s_k = p\cdot\mu_k q^{k+1}\sum_{j=1}^{k}\mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j}).$$

If k is even, then grouping summands gives

$$\sum_{j=1}^{k}\mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j}) = \sum_{j=1}^{k/2}\Big[\mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j}) + \mathbf{P}(\vec{D}_{k+1-j} < \overleftarrow{D}_j)\Big] = \frac{k}{2}. \tag{2.11}$$

We have $\mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j}) + \mathbf{P}(\vec{D}_{k+1-j} < \overleftarrow{D}_j) = 1$. This is because $\vec{D}_j$, $\overleftarrow{D}_{k+1-j}$, $\vec{D}_{k+1-j}$ and $\overleftarrow{D}_j$ are continuous and identically distributed random variables. Moreover, the $\vec{D}_j$ and $\overleftarrow{D}_{k+1-j}$ as well as $\vec{D}_{k+1-j}$ and $\overleftarrow{D}_j$ are pairwise independent. Using similar reasoning, if $k = 2m+1$ is odd, then we can write $\sum_{j=1}^k \mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j})$ as

$$\sum_{j=1}^{m}\Big[\mathbf{P}(\vec{D}_j < \overleftarrow{D}_{k+1-j}) + \mathbf{P}(\vec{D}_{k+1-j} < \overleftarrow{D}_j)\Big] + \mathbf{P}(\vec{D}_{m+1} < \overleftarrow{D}_{m+1}),$$

which equals $m + (1/2) = k/2$. Hence, $s_k = p\cdot\mu_k q^{k+1}(k/2)$. Summing gives

$$s = \sum_{k=1}^{\infty} s_k = \frac{pq^2}{2}\sum_{k=0}\mu_k k q^{k-1} = \frac{pq^2}{2} f'(q). \qquad \square$$

Proof of (2.3). Let $r_k = \mathbf{P}((0 \not\leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \to \dot{\bullet}^k))$ so that $r = \sum_{k=0}^{\infty} r_k$. As we did for the proof of (2.2), we apply the Mass Transport Principle with new indicators

$$W_k^{i,j}(a,b) = \textstyle\sum_{c\in\mathbb{Z}} \mathbf{1}\{\dot{\bullet}_b^k \wedge (\vec{\bullet}_a \xrightarrow{i} x_b)_{[x_a,x_b)} \wedge (x_b \xleftarrow{j} \overleftarrow{\bullet}_c)_{(x_b,x_c]} \wedge (x_c \not\leftarrow \overleftarrow{\bullet})_{(x_c,\infty)}\}$$

for $i,j,k,a,b \in \mathbb{Z}$. Let $(\vec{\bullet}_1 \xrightarrow{i} \dot{\bullet}^k \xleftarrow{j^*} \overleftarrow{\bullet})$ denote the event that $\vec{\bullet}_1$ is the ith right arrow to annihilate with a k-cluster and *exactly* j left arrows visit that same k-cluster. Observe that for $i+j \le k$ with $i \neq 0$, we have

$$r_k^{i,j} := \mathbf{P}((0 \not\leftarrow \overleftarrow{\bullet}) \wedge (\vec{\bullet}_1 \xrightarrow{i} \dot{\bullet}^k \xleftarrow{j^*} \overleftarrow{\bullet})) = \mathbf{E}\sum_{b\in\mathbb{Z}} W_k^{i,j}(1,b).$$

By Proposition 2.2 and independence,

$$\begin{aligned} r_k^{i,j} &= \mathbf{E} \sum_{a \in \mathbb{Z}} W_k^{i,j}(a,1) \\ &= \mathbf{P}(\dot{\bullet}_1^k)\mathbf{P}((\vec{\bullet} \xrightarrow{i} x_1)_{(-\infty,x_1)})\mathbf{P}((x_1 \xleftarrow{j} \overleftarrow{\bullet})_{(x_1,\infty)}) \times \mathbf{P}(\overleftarrow{D}_{j+1} = \infty \mid \overleftarrow{D}_j < \infty) \\ &= p \cdot \mu_k q^i q^j (1-q). \end{aligned}$$

We then have

$$r_k = \sum_{i=1}^{k} \sum_{j=0}^{k-i} r_k^{i,j} = p \cdot \mu_k \sum_{i=1}^{k} \sum_{j=0}^{k-i} q^{i+j}(1-q).$$

Applying the formula $\sum_{i=0}^m a^i = (1-a^{m+1})/(1-a)$ twice, gives

$$r_k = p \cdot \mu_k \frac{q\left(kq^{k+1} - kq^k - q^k + 1\right)}{1-q}.$$

Hence,

$$r = \sum_{k=1}^{\infty} r_k = \frac{pq\left(q^2 f'(q) - qf'(q) - f(q) + 1\right)}{1-q}. \qquad \square$$

3 Continuity

The goal of this section is to prove that q is continuous in p by proving that it is both upper and lower semi-continuous. We begin by recalling these definitions and stating a few classical facts. A function φ is *upper semi-continuous* (USC) at each $p_0 \in [0,1]$ if and only if $\limsup_{p \to p_0} \varphi(p) \le \varphi(p_0)$. It is *lower semi-continuous* (LSC) at each $p_0 \in [0,1]$ if and only if it holds that $\liminf_{p \to p_0} \varphi(p) \ge \varphi(p_0)$. Rather than working directly with the definition, we will apply the following properties. See [18] for proofs.

Fact 3.1. *The following hold.*

(a) φ *is continuous if and only if* φ *is USC and LSC.*

(b) *If there exists a sequence of LSC functions* φ_n *with* $\varphi_n \uparrow \varphi$*, then* φ *is LSC.*

(c) *If* $\varphi(p) = \sup_n(\varphi_n(p))$ *with* φ_n *LSC, then* φ *is LSC.*

(d) *If* φ_1 *and* φ_2 *are LSC, then* $\max(\varphi_1, \varphi_2)$ *is LSC.*

(e) φ *is LSC if and only if* $-\varphi$ *is USC.*

(f) *If* ψ *is continuous and* φ *is LSC, then* $\psi \circ \varphi$ *is LSC. Similarly, if* φ *is USC, then* $\psi \circ \varphi$ *is USC.*

(g) *If* φ *and* ψ *are both LSC or USC, then so is* $\varphi + \psi$*.*

That q is LSC follows almost immediately from its definition.

Proposition 3.2. q *is LSC for* $p \in [0,1]$.

Proof. The events $Q_n = \{(0 \leftarrow \overleftarrow{\bullet})_{(0,x_n)}\}$ involve finitely many particles. After conditioning on the velocities of these particles and integrating over all possible spacings, $\mathbf{P}(Q_n)$ is a finite degree polynomial in p, and thus continuous. Moreover, $Q_n \subseteq Q_{n+1}$, thus the $\mathbf{P}(Q_n)$ are increasing in n. Since $q = \lim_{n\to\infty} \mathbf{P}(Q_n)$, it follows from Fact 3.1 (b) that q is LSC. $\square$

We next aim to prove that q is USC. This is more difficult and involves an indirect characterization of $\theta = (1-q)^2$ that takes a supremum over functionals of configurations with only finitely many particles. Let $\dot{N}(j,k)$ be the number of blockades that survive in ballistic annihilation restricted to the particles in $[x_j, x_k]$. Similarly, let $\overleftarrow{N}(j,k)$ and $\vec{N}(j,k)$ count the number of surviving left and right arrows. Define the random variables that track the difference between the number of surviving blockades and arrows in the process restricted to only the particles in $[x_j, x_k]$:

$$W(j,k) = \dot{N}(j,k) - \overleftarrow{N}(j,k) - \vec{N}(j,k).$$

Lemma 3.3. *$n^{-1}\mathbf{E}_p[W(1,n)]$ is continuous in p for all $n \geq 1$.*

Proof. The random variables $W(1,n)$ involve only finitely many particles. $\mathbf{E}_p[W(1,n)]$ is thus a finite degree polynomial in p and is continuous. □

Lemma 3.4. *$\theta = \max\left(0, \sup_{n\geq 1} n^{-1}\mathbf{E}_p[W(1,n)]\right)$ for all $p \in [0,1]$.*

Proof. The proof has four steps. Fortunately, it requires little modification from the blueprint developed in [18]. We explain the basic idea of each step and refer the reader to the appropriate reference.

Step 1. For all integers $j < k < \ell$ it holds that $W(j,\ell) \geq W(j,k) + W(k+1,\ell)$.

Proof. This superadditivity property is proven in [4, Lemma 15] for a more general variant of ballistic annihilation in which particles sometime survive collisions. The basic idea is that surviving arrows from the restrictions to $[x_j, x_k]$ and $[x_{k+1}, x_\ell]$ have a non-decreasing effect on $W(j,\ell)$. Surviving arrows either destroy other surviving arrows, which augments $W(j,\ell)$. Or, surviving arrows destroy blockades, which may cause a chain reaction, but, regardless, the effect is worst-case neutral on $W(j,\ell)$. The argument does not change if multiple blockades are present at a site. □

Step 2. $\lim_{k\to\infty} k^{-1}\overleftarrow{N}(1,k) = 0 = \lim_{k\to\infty} k^{-1}\vec{N}(1,k)$.

Proof. This is proven in [18, Proposition 12] for asymmetric ballistic annihilation. It is much simpler to deduce for symmetric systems. Using translation invariance of the velocity configuration, Birkhoff's Ergodic Theorem gives that the limits equal the probability an arrow is never annihilated. [22, Lemma 3.3] observes that this quantity must be zero, as otherwise, Birkhoff's Ergodic Theorem gives the contradiction that there is a positive densities of surviving left and right arrows. This reasoning still applies with the possibility of multiple blockades at a single site. □

Step 3. Let $N_{\mathbb{R}}(1,k)$ denote the number of blockades that survive in $[x_1, x_k]$ in ballistic annihilation with all particles in $\mathbb{R}$ present. If $\theta > 0$, then

$$\lim_{k\to\infty} k^{-1}\dot{N}(1,k) = \theta = \lim_{k\to\infty} k^{-1}N_{\mathbb{R}}(1,k).$$

Proof. This is proven in [18, Proposition 12]. It follows from the definition of θ and the strong law of large numbers that $\lim_{k\to\infty} k^{-1}N_{\mathbb{R}}(1,k) = \theta$. So, it suffices to prove that

$$\lim_{k\to\infty} k^{-1}[\dot{N}(1,k) - \dot{N}_{\mathbb{R}}(1,k)] = 0.$$

First, observe that blockade survival is a decreasing event as the interval of restriction is expanded. So, $\dot{N}(1,k) - \dot{N}_{\mathbb{R}}(1,k) \geq 0$. From there, the main idea is that at most a geometric random variable with parameter q, call it R_k, of the surviving blockades in ballistic annihilation restricted to $[x_1, x_k]$ are removed from right arrows entering at x_1, and the same for an independent and identically distributed geometric random variable of left arrows entering at x_k, call it L_k. So, $\dot{N}(1,k) - N_{\mathbb{R}}(1,k) \le L_k + R_k$ Since these random variables have exponential tails and constant parameter q, it is easy to infer from the Borel-Cantelli lemma that $\lim_{k\to\infty} k^{-1}[R_k + L_k] = 0$ almost surely. □

Step 4. Let $\theta_0 := \max\left(0, \sup_{k\ge 1} k^{-1}\mathbf{E}_p[W(1,k)]\right)$. It holds that $\theta = \theta_0$.

Proof. The proof is similar to [18, Lemma 10]. First, we prove that $\theta \le \theta_0$. Combining Step 2, Step 3, and Fatou's lemma gives

$$\theta = \lim_{k\to\infty} k^{-1}W(1,k) = \mathbf{E}_p\left[\liminf_{k\to\infty} k^{-1}W(1,k)\right] \le \liminf_{k\to\infty} k^{-1}\mathbf{E}_p[W(1,k)] \le \theta_0.$$

Next, we show that $\theta \ge \theta_0$. This is immediate when $\theta_0 = 0$, so suppose that $\theta_0 > 0$. Then, there is an integer k with $\mathbf{E}_p[W(1,k)] > 0$. Letting $K_m = km$ for $m \ge 0$, we see that $S_n := \sum_{m=0}^{n-1} W(K_m+1, K_{m+1})$ is a random walk with positive drift. The law of large numbers gives that $S_n > 0$ for all $n \ge 1$ with positive probability. Step 1 implies that

$$W(1, K_n) \ge S_n \qquad \forall n \ge 1. \tag{3.1}$$

This is enough to deduce that 0 is never visited with positive probability, which gives $\theta > 0$. See the proof of [18, Lemma 10] for more details.

We will use this framework to prove that $\theta > \delta$ for arbitrary $\delta \in (0,1)$ with $\delta < \theta_0$. Let $k \ge 1$ be such that $k^{-1}\mathbf{E}_p[W(1,k)] > \delta$. Step 2 and Step 3 imply that $\theta = \lim_{n\to\infty} \frac{1}{n} W(1,n)$. Multiplying by n/n, applying (3.1) and then the strong law of large numbers gives

$$\theta = \liminf_{n\to\infty} \frac{n}{K_n}\frac{1}{n} W(1,K_n) \ge \liminf_{n\to\infty} \frac{n}{K_n}\frac{1}{n} S_n = k^{-1}\mathbf{E}_p[W(1,k)] > \delta$$

as desired. □

□

Proposition 3.5. *q is USC for $p \in [0,1]$.*

Proof. It follows that θ is LSC from Lemmas 3.3 and 3.4 along with Fact 3.1 (c) and (d). Since $\theta = (1-q)^2$, we have $q = 1 - \sqrt{\theta}$. Fact 3.1 (e) and (f) imply that $-\sqrt{\theta}$ is USC. Since 1 is USC, q can be expressed as the sum of two USC functions and by Fact 3.1 (g) is USC. □

Theorem 3.6. *q is continuous for $p \in [0,1]$.*

Proof. This follows immediately from Propositions 3.2 and 3.5 along with Fact 3.1 (a). □

4 Proof of Theorem 1.1

Proof of Theorem 1.1. Subtracting q from both sides of (2.1) in Proposition 2.1 gives $0 = g(p,q)$ with $g\colon [0,1]^2 \to \mathbb{R}$ defined as

$$g(u,v) := \frac{u\left(1-v^2\right)v^2 f'(v) + 2uvf(v) - uv^2 + u + v^2 - 2v + 1}{2(1-v)}. \tag{4.1}$$

Let $h(u,v) = g(u,v)/(1-v)$ so that Proposition 2.1 implies

$$0 = (1-q)h(p,q). \tag{4.2}$$

The goal is to show that (p,q) solves $1 - v = 0$ for $p \le p_c$ and transitions to solving $h(u,v) = 0$ for $p \ge p_c$.

Inspecting (4.1), we see that $h(u,v)$ is linear in u. Solving $h(u,v)=0$ yields

$$u = \frac{(1-v)^2}{(1-v^2)\,v^2 f'(v) - 2vf(v) + v^2 + 1} =: F(v).$$

Thus,

Fact 4.1. *If* $h(u,v)=0$*, then* $u=F(v)$*.*

Using L'Hospital's rule twice and basic generating function properties

$$\lim_{v\to 1} F(v) = \frac{1}{f(1)+3f'(1)+f''(1)} = \frac{1}{(1+\mathbf{E}[X])^2+\mathrm{var}(X)} =: p_*.$$

By Fact 4.1,

Fact 4.2. $(u,v)=(p_*,1)$ *is the unique solution to* $1-v=0=h(u,v)$*.*

Since q is continuous (Theorem 3.6) with $q(1)=0$, it follows from (4.2) and Fact 4.2 that $(p_*,1)$ is the only point at which (p,q) can continuously transition from solving $1-v=0$ to solving $h(u,v)=0$. So,

Fact 4.3. *If* $p \geq p_*$*, then* $h(p,q(p))=0$*.*

Combining Fact 4.1 and Fact 4.3 gives

Fact 4.4. $p=F(q(p))$ *for* $p\geq p_*$*.*

Fact 4.4 says that F is a left inverse of q on the domain $p\geq p_*$ i.e., if $q(p)=y$ for $p\geq p_*$, then $F(y)=p$. It is an elementary exercise in analysis that this and continuity of q imply that

Fact 4.5. q *is continuous and strictly decreasing for* $p\geq p_*$*.*

Fact 4.4 and Fact 4.5 (along with Theorem 3.6) imply (1.3) in Theorem 1.1.

It remains to prove that $p_c=p_*$ as claimed at (1.2). Suppose that $p>p_*$. Fact 4.3 implies that $h(p,q(p))=0$. Fact 4.2 ensures that $q(p_*)=1$. Fact 4.1 requires that $q(p)\neq 1$. Since $q(p)$ is a probability, we then have $q(p)<1$. So, $p_c \leq p_*$.

To see the reverse inequality, suppose that there exists $p_0<p_*$ with $q(p_0)=v<1$. Fact 4.5 and $q(1)=0$ imply that $q\colon [p_*,1]\to[0,1]$ is a continuous bijection. Thus, there is $p_1>p_*$ with $q(p_1)=v$. As $v<1$, (4.2) implies that $h(p_0,v)=0=h(p_1,v)$. This contradicts Fact 4.1, which requires that $p_0=p_1=F(v)$. So, $q=1$ for all $p\geq p_*$. Thus, $p_c\geq p_*$. □

References

[1] Krishna B Athreya and Peter E Ney, *Branching processes*, Courier Corporation, 2004. MR0373040

[2] Riti Bahl, Philip Barnet, Tobias Johnson, and Matthew Junge, *Diffusion-limited annihilating systems and the increasing convex order*, Electronic Journal of Probability **27** (2022), 1–19. MR4453592

[3] Riti Bahl, Philip Barnet, and Matthew Junge, *Parking on supercritical Galton-Watson trees*, ALEA **18** (2021). MR4332220

[4] Luis Benitez, Matthew Junge, Hanbaek Lyu, Maximus Redman, and Lily Reeves, *Three-velocity coalescing ballistic annihilation*, Electronic Journal of Probability. **28** (2023) 1–18. MR4578056

[5] Maury Bramson and Joel L Lebowitz, *Asymptotic behavior of densities in diffusion dominated two-particle reactions*, Physica A: Statistical Mechanics and its Applications **168** (1990), no. 1, 88–94. MR1077199

[6] Maury Bramson and Joel L Lebowitz, *Asymptotic behavior of densities for two-particle annihilating random walks*, Journal of statistical physics **62** (1991), no. 1, 297–372. MR1105266

[7] Nicolas Broutin and Jean-François Marckert, *The combinatorics of the colliding bullets*, Random Structures & Algorithms **56** (2020), no. 2, 401–431. MR4060351

[8] Debbie Burdinski, Shrey Gupta, and Matthew Junge, *The upper threshold in ballistic annihilation*, Latin American Journal of Probability and Mathematical Statistics **16** (2019), 1077. MR4017187

[9] M. Cabezas, L. T. Rolla, and V. Sidoravicius, *Recurrence and density decay for diffusion-limited annihilating systems*, Probability Theory and Related Fields **170** (2018), no. 3, 587–615. MR3773795

[10] Nicolas Curien and Olivier Hénard, *The phase transition for parking on Galton–Watson trees*, arXiv:1912.06012 (2019). MR4396227

[11] Michel Droz, Pierre-Antoine Rey, Laurent Frachebourg, and Jarosław Piasecki, *Ballistic-annihilation kinetics for a multivelocity one-dimensional ideal gas*, Physical Review **51** (1995), no. 6, 5541–5548 (eng).

[12] Brittany Dygert, Christoph Kinzel, Matthew Junge, Annie Raymond, Erik Slivken, Jennifer Zhu, et al., *The bullet problem with discrete speeds*, Electronic Communications in Probability **24** (2019). MR3962477

[13] Yves Elskens and Harry L Frisch, *Annihilation kinetics in the one-dimensional ideal gas*, Physical Review A **31** (1985), no. 6, 3812.

[14] John Haslegrave, Vladas Sidoravicius, and Laurent Tournier, *Three-speed ballistic annihilation: phase transition and universality*, Selecta Mathematica **27** (2021), no. 84. MR4304559

[15] John Haslegrave and Laurent Tournier, *Combinatorial universality in three-speed ballistic annihilation*, In and Out of Equilibrium 3: Celebrating Vladas Sidoravicius, vol. 77, Springer International Publishing, 2021, pp. 487–517. MR4237283

[16] Tobias Johnson and Matthew Junge, *Stochastic orders and the frog model*, Annales Henri Poincaré **54** (2018), no. 2, 1013–1030. MR3795075

[17] Tobias Johnson and Leonardo T Rolla, *Sensitivity of the frog model to initial conditions*, Electronic Communications in Probability **24** (2019), 1–9. MR3962479

[18] Matthew Junge and Hanbaek Lyu, *The phase structure of asymmetric ballistic annihilation*, The Annals of Applied Probability **32** (2022), no. 5, 3797–3816. MR4497858

[19] PL Krapivsky, S Redner, and F Leyvraz, *Ballistic annihilation kinetics: The case of discrete velocity distributions*, Physical Review E **51** (1995), no. 5, 3977.

[20] Darío Cruzado Padró, Matthew Junge, and Lily Reeves, *Arrivals are universal in coalescing ballistic annihilation*, arXiv:2209.09271 (2022).

[21] Leonardo T Rolla, Vladas Sidoravicius, and Olivier Zindy, *Universality and sharpness in activated random walks*, Annales Henri Poincaré, vol. 20, Springer, 2019, pp. 1823–1835. MR3956161

[22] Vladas Sidoravicius and Laurent Tournier, *Note on a one-dimensional system of annihilating particles*, Electronic Communications in Probability **22** (2017), 9 pp. MR3718709

[23] Doug Toussaint and Frank Wilczek, *Particle–antiparticle annihilation in diffusive motion*, The Journal of Chemical Physics **78** (1983), no. 5, 2642–2647.

Acknowledgments. We are grateful to L.P. Arguin and the anonymous referees for their helpful comments.