The Boué–Dupuis formula and the exponential hypercontractivity in the Gaussian space[*]

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Abstract

This paper concerns a variational representation formula for Wiener functionals. Let $B = \{B_t\}_{t \geq 0}$ be a standard d-dimensional Brownian motion. Boué and Dupuis (1998) showed that, for any bounded measurable functional $F(B)$ of B up to time 1, the expectation $\mathbb{E}\left[e^{F(B)}\right]$ admits a variational representation in terms of drifted Brownian motions. In this paper, with a slight modification of insightful reasoning by Lehec (2013) allowing also $F(B)$ to be a functional of B over the whole time interval, we prove that the Boué–Dupuis formula holds true provided that both $e^{F(B)}$ and $F(B)$ are integrable, relaxing conditions in earlier works. We also show that the formula implies the exponential hypercontractivity of the Ornstein–Uhlenbeck semigroup in $\mathbb{R}^d$, and hence, due to their equivalence, implies the logarithmic Sobolev inequality in the d-dimensional Gaussian space.

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1 Introduction

Given a positive integer d, let $B = \{B_t\}_{t \geq 0}$ be a standard d-dimensional Brownian motion. In [6], Boué and Dupuis established the following formula for any bounded measurable function F mapping $C([0,1]; \mathbb{R}^d)$ into $\mathbb{R}$:

$$\log \mathbb{E}\left[e^{F(B)}\right] = \sup_v \mathbb{E}\left[F\left(B + \int_0^{\cdot} v_t \, dt\right) - \frac{1}{2}\int_0^1 |v_t|^2 \, dt\right]. \tag{1.1}$$

Here the supremum runs over all progressively measurable processes v with respect to the augmented natural filtration of B such that $\int_0^1 |v_t|^2 \, dt$ is integrable. In [6], formula (1.1) was proven useful in deriving various large deviation asymptotics such as Laplace principles for small noise diffusions driven by Brownian motion. These results have been extended by Budhiraja and Dupuis [9] to Hilbert space-valued Brownian motions, and later generalized by Zhang [26] to abstract Wiener spaces. In Boué–Dupuis [7], formula (1.1) is also applied to risk-sensitive stochastic control problems. Recently, the

formula has been used effectively by Barashkov and Gubinelli [2] in the study of the Φ_3^4 Gibbs measure in the quantum field theory (in fact, they employ an extended formula by Üstünel [24] to a class of unbounded functions F; see Remark 1.2(3)). Their idea is exploited further by Chandra, Gunaratnam and Weber in [10], and there seems to be growing interest in the formula in this direction; see, e.g., [19] among others. The work of Barashkov and Gubinelli also inspires approaches employed in [13, 8, 20], in which a variational argument is used in the construction of quasi-invariant measures for dispersive partial differential equations with random initial data.

One of the objectives of this paper is to show that the boundedness imposed on the functions F is removable when both $e^{F(B)}$ and $F(B)$ are integrable. We do this by slightly modifying reasoning by Lehec [18] based on deep understanding of the Gaussian relative entropy, which also allows $F(B)$ to be a functional of B over the whole time interval.

In order to state the result precisely, we prepare some of notation. We denote by $\mathbb{P}$ the probability measure of the probability space on which the Brownian motion B is defined. We set

$$\mathcal{F}_t^B := \sigma(B_s, 0 \leq s \leq t) \vee \mathcal{N}, \quad t \geq 0,$$

the filtration generated by B and augmented by the set $\mathcal{N}$ of all $\mathbb{P}$-null events. Let $v = \{v_t\}_{t \geq 0}$ be a d-dimensional process defined on the same probability space as B. We call v a *drift* if it is $\{\mathcal{F}_t^B\}$-progressively measurable. We denote by $\mathcal{V}$ the set of drifts v satisfying

$$\mathbb{E}\left[\int_0^\infty |v_t|^2 \, dt\right] < \infty. \tag{1.2}$$

Here and in what follows, $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$ and $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^d$.

Let $\mathbb{W} = C([0, \infty); \mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $[0, \infty)$ endowed with the topology of uniform convergence on compact subsets of $[0, \infty)$. We denote by $\mathcal{B}(\mathbb{W})$ the associated Borel σ-field and by $\mathcal{W}$ the Wiener measure on $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$. Let $F : \mathbb{W} \to \mathbb{R}$ be measurable. We assume:

$$(A1) \quad \int_{\mathbb{W}} e^F \, d\mathcal{W} < \infty; \qquad\qquad (A2) \quad \int_{\mathbb{W}} F_- \, d\mathcal{W} < \infty,$$

where we set $F_-(w) := \max\{-F(w), 0\}$, $w \in \mathbb{W}$. The main result of the paper is stated as follows:

Theorem 1.1. *Let $F : \mathbb{W} \to \mathbb{R}$ be a measurable function satisfying* (A1) *and* (A2). *Then it holds that*

$$\log \mathbb{E}\left[e^{F(B)}\right] = \sup_{v \in \mathcal{V}} \mathbb{E}\left[F\left(B + \int_0^\cdot v_t \, dt\right) - \frac{1}{2} \int_0^\infty |v_t|^2 \, dt\right]. \tag{1.3}$$

We may replace the supremum over $\mathcal{V}$ by that over a class of bounded drifts; see Corollary 2.11.

We give a remark on Theorem 1.1.

Remark 1.2. (1) Under assumption (A1), the right-hand side of (1.3) is well-defined in the sense that, for any $v \in \mathcal{V}$,

$$\mathbb{E}\left[F_+\left(B + \int_0^\cdot v_t \, dt\right)\right] < \infty \quad \text{with} \quad F_+ := \max\{F, 0\},$$

while $\mathbb{E}\left[F_-\left(B + \int_0^\cdot v_t \, dt\right)\right]$ may take value ∞ for some $v \in \mathcal{V}$; see the beginning of the proof of Proposition 2.1.

(2) Although a little involved argument is used in [6, Section 5], the extension of formula (1.1) to the case that F is only assumed to be bounded from below is immediate from the monotone convergence theorem. Indeed, for each positive real M, truncating F from above by M, we have, from (1.1),

$$\log \mathbb{E}\left[e^{F_M(B)}\right] = \sup_{v \in \mathcal{V}} \mathbb{E}\left[F_M\left(B + \int_0^{\cdot} v_t\, dt\right) - \frac{1}{2}\int_0^1 |v_t|^2\, dt\right],$$

where $F_M := \min\{F, M\}$; then, by the monotone convergence theorem, the left-hand side converges as $M \to \infty$ to the expression with F_M replaced by F, and so does the right-hand side since

$$\sup_{M > 0} \sup_{v \in \mathcal{V}} \mathbb{E}\left[F_M\left(B + \int_0^{\cdot} v_t\, dt\right) - \frac{1}{2}\int_0^1 |v_t|^2\, dt\right]$$

$$= \sup_{v \in \mathcal{V}} \sup_{M > 0} \mathbb{E}\left[F_M\left(B + \int_0^{\cdot} v_t\, dt\right) - \frac{1}{2}\int_0^1 |v_t|^2\, dt\right]$$

$$= \sup_{v \in \mathcal{V}} \mathbb{E}\left[F\left(B + \int_0^{\cdot} v_t\, dt\right) - \frac{1}{2}\int_0^1 |v_t|^2\, dt\right].$$

In this respect, what is essential in Theorem 1.1 is the removal of the boundedness of F from below.

(3) In the setting of the classical Wiener space, Üstünel proved formula (1.1) in [24, Theorem 7] under the condition that, for some $p, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$\mathbb{E}[|F(B)|^p] < \infty \quad \text{and} \quad \mathbb{E}\left[e^{qF(B)}\right] < \infty,$$

while, in [14, Theorem 1.1], the condition that

$$\mathbb{E}[|F(B)|^p] < \infty \text{ for some } p > 1 \quad \text{and} \quad \mathbb{E}\left[e^{F(B)}\right] < \infty$$

was imposed. Due to their methods, the restriction $p > 1$ seems inevitable: [24] uses its Lemma 1 and [14] its Lemma 2.10. Note that our assumption of Theorem 1.1 is equivalently rephrased as

$$\mathbb{E}[|F(B)|] < \infty \quad \text{and} \quad \mathbb{E}\left[e^{F(B)}\right] < \infty.$$

Theorem 7 of Üstünel [24] aims at characterizing in terms of the relative entropy the invertibility of path transformations of the form $B + \int_0^{\cdot} v_t\, dt$ with B denoting the canonical process in the classical Wiener space, and the essential assumption in his exploration is $\mathbb{E}\left[(1 + |F(B)|)\, e^{F(B)}\right] < \infty$. Generalizations of formula (1.1), such as ones to diffusions and conditional expectations, in the same spirit as [24] may be found in a series of papers by Hartmann; see [16] and references therein.

Independently of the work [6] by Boué–Dupuis, Borell [5] proved formula (1.1) when $F(B)$ is of the form $f(B_1)$ with $f : \mathbb{R}^d \to \mathbb{R}$ a bounded measurable function, and, among other applications, applied it to a simple derivation of the Prékopa–Leindler inequality. In the last part of the paper, we will show that it also yields readily the exponential version of the hypercontractivity of the Ornstein–Uhlenbeck semigroup in $\mathbb{R}^d$; the equivalence between the exponential hypercontractivity and the logarithmic Sobolev inequality then entails that the formula implies the Gaussian logarithmic Sobolev inequality.

We give an outline of the paper. In Section 2, we prove Theorem 1.1; the lower bound in formula (1.3) is proven in Subsection 2.1 while the upper bound is proven in Subsection 2.2, where the case of bounded drifts is also stated in Corollary 2.11.

The paper is concluded with Section 3 that explores the above-mentioned connection between the formula and the exponential hypercontractivity of the Ornstein–Uhlenbeck semigroup.

For every $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For a positive integer n, we denote by $C_b^\infty(\mathbb{R}^n)$ the set of real-valued bounded C^∞-functions on $\mathbb{R}^n$ whose partial derivatives are all bounded. Given a measured space $(\mathscr{X}, \mathscr{B}, m)$, for every $1 \leq p \leq \infty$, we denote by $L^p(m)$ the set of real-valued measurable functions f on the measurable space $(\mathscr{X}, \mathscr{B})$ such that

$$\left\{ \|f\|_{L^p(m)} \right\}^p := \int_{\mathscr{X}} |f(x)|^p \, m(dx) < \infty \qquad \text{for } p < \infty,$$

and that, with $\operatorname{ess\,sup}$ denoting the essential supremum with respect to m,

$$\|f\|_{L^\infty(m)} := \operatorname*{ess\,sup}_{x \in \mathscr{X}} |f(x)| < \infty \qquad \text{for } p = \infty.$$

Other notation will be introduced as needed.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Let $(\mathscr{X}, \mathscr{B})$ be a measurable space and $\mathcal{P}(\mathscr{X})$ the set of probability measures on it. For $\mu, \nu \in \mathcal{P}(\mathscr{X})$, recall that the relative entropy of ν with respect to μ is defined by

$$H(\nu \mid \mu) := \int_{\mathscr{X}} \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \, d\mu \quad \text{if } \nu \ll \mu,$$

and $H(\nu \mid \mu) := +\infty$ otherwise; see, e.g., [11, Section 1.4].

In the sequel, for every drift v, we denote

$$B_t^v = B_t + \int_0^t v_s \, ds, \quad t \geq 0,$$

the Brownian motion drifted by v and, whenever $v \in \mathcal{V}$,

$$\|v\|_{\mathcal{V}} = \left\{ \mathbb{E}\left[\int_0^\infty |v_t|^2 \, dt \right] \right\}^{1/2}.$$

A drift v is said to be *bounded* if it satisfies

$$\sup_{t \geq 0} \| |v_t| \|_{L^\infty(\mathbb{P})} < \infty.$$

For later use in Subsection 2.2, we set

$$\mathcal{V}_b := \left\{ v; \ v \text{ is a bounded drift satisfying (1.2)} \right\},$$

$$\mathcal{V}_{b,0} := \left\{ v; \ \begin{array}{l} v \text{ is a bounded drift satisfying that there exists} \\ K > 0 \text{ such that } v_t = 0 \text{ a.s. for all } t \geq K \end{array} \right\}.$$

Clearly, we have $\mathcal{V}_{b,0} \subset \mathcal{V}_b \subset \mathcal{V}$.

2.1 Lower bound

In this subsection, we give a proof of the lower bound in (1.3), that is, we prove

Proposition 2.1. *For any measurable function $F : \mathbb{W} \to \mathbb{R}$ satisfying* (A1)*, we have*

$$\log \mathbb{E}\left[e^{F(B)} \right] \geq \sup_{v \in \mathcal{V}} \left\{ \mathbb{E}[F(B^v)] - \frac{1}{2} \|v\|_{\mathcal{V}}^2 \right\}. \tag{2.1}$$

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Proposition 2.1 is immediate once the following lemma is at our disposal.

Lemma 2.2. *Under* (A1)*, the lower bound* (2.1) *holds when F is bounded from below.*

By using this lemma, Proposition 2.1 is proven as follows:

Proof of Proposition 2.1. First we verify that, under assumption (A1),

$$\mathbb{E}[F_+(B^v)] < \infty \quad \text{for any } v \in \mathcal{V}, \tag{2.2}$$

where $F_+(w) := F(w) \vee 0$, $w \in \mathbb{W}$. Fix $v \in \mathcal{V}$ arbitrarily and set $F_{+,M} = F_+ \wedge M$ for each $M > 0$. Then, by Lemma 2.2, we have in particular

$$\mathbb{E}[F_{+,M}(B^v)] \leq \log \mathbb{E}\left[e^{F_{+,M}(B)}\right] + \frac{1}{2}\|v\|_\mathcal{V}^2.$$

By letting $M \to \infty$, the monotone convergence theorem entails that

$$\mathbb{E}[F_+(B^v)] \leq \log \mathbb{E}\left[e^{F_+(B)}\right] + \frac{1}{2}\|v\|_\mathcal{V}^2$$
$$\leq \log \mathbb{E}\left[1 + e^{F(B)}\right] + \frac{1}{2}\|v\|_\mathcal{V}^2,$$

which is finite by (A1).

For every $N > 0$, we now define $F_N(w) := F(w) \vee (-N)$, $w \in \mathbb{W}$. Then, by Lemma 2.2, the lower bound (2.1) holds for F_N:

$$\log \mathbb{E}\left[e^{F_N(B)}\right] \geq \sup_{v \in \mathcal{V}}\left\{\mathbb{E}[F_N(B^v)] - \frac{1}{2}\|v\|_\mathcal{V}^2\right\}. \tag{2.3}$$

By assumption (A1), the random variable $\sup_{N>0} e^{F_N(B)}$ is integrable and, thanks to (2.2), so is $\sup_{N>0} F_N(B^v)$ for any $v \in \mathcal{V}$. Therefore, as $N \to \infty$, we may use the monotone convergence theorem on both sides of (2.3) to get

$$\log \mathbb{E}\left[e^{F(B)}\right] \geq \inf_{N>0}\sup_{v \in \mathcal{V}}\left\{\mathbb{E}[F_N(B^v)] - \frac{1}{2}\|v\|_\mathcal{V}^2\right\}$$
$$\geq \sup_{v \in \mathcal{V}}\inf_{N>0}\left\{\mathbb{E}[F_N(B^v)] - \frac{1}{2}\|v\|_\mathcal{V}^2\right\}$$
$$= \sup_{v \in \mathcal{V}}\left\{\mathbb{E}[F(B^v)] - \frac{1}{2}\|v\|_\mathcal{V}^2\right\},$$

which is the assertion. □

We proceed to the proof of Lemma 2.2. We prepare two lemmas, the first one of which is adapted from [11, Proposition 4.5.1]: set $\Delta(\mathbb{W}) := \{\mu \in \mathcal{P}(\mathbb{W}); H(\mu \mid \mathcal{W}) < \infty\}$.

Lemma 2.3. *Let $F : \mathbb{W} \to \mathbb{R}$ be a measurable function bounded from below. Then it holds that*

$$\log \int_\mathbb{W} e^F \, d\mathcal{W} = \sup_{\mu \in \Delta(\mathbb{W})}\left\{\int_\mathbb{W} F \, d\mu - H(\mu \mid \mathcal{W})\right\}.$$

The second one is taken from [18].

Lemma 2.4 ([18], Proposition 1)**.** *Let v be a drift and μ the law of B^v. Then it holds that*

$$H(\mu \mid \mathcal{W}) \leq \frac{1}{2}\mathbb{E}\left[\int_0^\infty |v_t|^2 \, dt\right].$$

In the setting of the classical Wiener space, the assertion of the above lemma goes back to [23, Theorem 8].

Combining the above two lemmas yields Lemma 2.2 readily.

Proof of Lemma 2.2. For an arbitrary $v \in \mathcal{V}$, let μ be the law of B^v. Then, since $\mu \in \Delta(\mathbb{W})$ by the definition of $\mathcal{V}$ and Lemma 2.4, we see from Lemma 2.3 that

$$\log \int_{\mathbb{W}} e^F \, d\mathcal{W} \geq \int_{\mathbb{W}} F \, d\mu - H(\mu \mid \mathcal{W})$$
$$\geq \int_{\mathbb{W}} F \, d\mu - \frac{1}{2} \|v\|_{\mathcal{V}}^2 ,$$

where we used Lemma 2.4 again for the second line. The assertion is proven because μ is the law of B^v and v is arbitrary. $\qquad\square$

2.2 Upper bound

In this subsection, we prove the upper bound in (1.3):

Proposition 2.5. *For any measurable function $F : \mathbb{W} \to \mathbb{R}$ satisfying* (A1) *and* (A2)*, we have*

$$\log \mathbb{E}\left[e^{F(B)}\right] \leq \sup_{v \in \mathcal{V}} \left\{ \mathbb{E}[F(B^v)] - \frac{1}{2} \|v\|_{\mathcal{V}}^2 \right\}. \tag{2.4}$$

We denote by $\mathcal{F}C_b^\infty$ the set of functions Φ on $\mathbb{W}$ of the form

$$\Phi(w) = \phi\left(w(t_1), \ldots, w(t_m)\right), \quad w \in \mathbb{W}, \tag{2.5}$$

for some $m \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_m$ and $\phi \in C_b^\infty(\mathbb{R}^{d \times m})$. We also denote by $\mathcal{C}$ the set of cylinder subsets C of $\mathbb{W}$, namely, each C is of the form

$$C = \{w \in \mathbb{W}; (w(t_1), \ldots, w(t_m)) \in \Gamma\} \tag{2.6}$$

for some $m \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_m$, and for some Borel subset Γ of $\mathbb{R}^{d \times m}$. It is well known that

$$\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{W}) \tag{2.7}$$

(see, e.g., [17, Problem 2.4.2]). Let $F : \mathbb{W} \to \mathbb{R}$ be a measurable function and define the σ-finite measure ν_F on $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ by

$$d\nu_F := (1 + F_-) \, d\mathcal{W}.$$

If F fulfills (A2), then ν_F is a finite measure and the following lemma is standard but crucial to our argument.

Lemma 2.6. $\mathcal{F}C_b^\infty$ *is dense in* $L^2(\nu_F)$ *under assumption* (A2)*.*

For the completeness of the paper, we give a proof.

Proof of Lemma 2.6. It suffices to show that, for any $A \in \mathcal{B}(\mathbb{W})$, its indicator function $\mathbf{1}_A$ can be approximated in $L^2(\nu_F)$ by a sequence $\{\Phi_n\}_{n=1}^\infty$ in $\mathcal{F}C_b^\infty$. To this end, fix a positive integer n arbitrarily. In view of (2.7), the approximation property (e.g., [3, Theorem 5.7]) entails that there exists $C \in \mathcal{C}$ such that

$$\|\mathbf{1}_A - \mathbf{1}_C\|_{L^2(\nu_F)} < n^{-1},$$

because of the fact that ν_F is a finite measure and $|\mathbf{1}_A - \mathbf{1}_C|^2 = \mathbf{1}_{A \triangle C}$, where the symbol $\triangle$ stands for the symmetric difference. As C may be expressed as (2.6), it is now routine

to check that elements of $C_b^\infty(\mathbb{R}^{d \times m})$ approximate $\mathbf{1}_\Gamma$ in the sense of L^2 under the finite measure

$$\nu_F^m(\,\cdot\,) \equiv \nu_F^{t_1,\ldots,t_m}(\,\cdot\,) := \nu_F(\{w \in \mathbb{W}; \, (w(t_1),\ldots,w(t_m)) \in \,\cdot\,\})$$

on $\mathbb{R}^{d \times m}$. To see that, notice that ν_F^m is inner regular (cf. [3, Lemma 26.2]). Hence there exists a compact subset K of Γ such that

$$\|\mathbf{1}_\Gamma - \mathbf{1}_K\|_{L^2(\nu_F^m)} = \sqrt{\nu_F^m(\Gamma) - \nu_F^m(K)} < n^{-1}.$$

Convoluting $\mathbf{1}_K$ with the standard mollifier ([12, Subsection 4.2.1]), we may construct a family $\{\phi_\varepsilon\}_{\varepsilon>0} \subset C_b^\infty(\mathbb{R}^{d \times m})$ (in fact, each ϕ_ε is compactly supported) such that, as $\varepsilon \downarrow 0$, $\phi_\varepsilon \to \mathbf{1}_K$ a.e. Thanks to the finiteness of ν_F^m, this a.e. convergence entails that there exists $\phi \in C_b^\infty(\mathbb{R}^{d \times m})$ such that

$$\|\mathbf{1}_K - \phi\|_{L^2(\nu_F^m)} < n^{-1}$$

by the bounded convergence theorem. Therefore, setting

$$\Phi_n(w) := \phi\left(w(t_1),\ldots,w(t_m)\right), \quad w \in \mathbb{W},$$

we have the desired sequence $\{\Phi_n\}_{n=1}^\infty \subset \mathcal{F}C_b^\infty$ because

$$\|\mathbf{1}_A - \Phi_n\|_{L^2(\nu_F)} < 3n^{-1}$$

for each n by construction. $\qquad\qquad\square$

Following the notation of [18], we define

$$\mathcal{S} := \left\{\mu \in \mathcal{P}(\mathbb{W}); \, \mu \text{ has a density } \Phi \in \mathcal{F}C_b^\infty \text{ w.r.t. } \mathcal{W} \text{ such that } \inf_{w \in \mathbb{W}} \Phi(w) > 0\right\}.$$

The next lemma is also adapted from [18].

Lemma 2.7 ([18], Theorem 7)**.** *For every $\mu \in \mathcal{S}$, there exists $v \in \mathcal{V}$ such that B^v has law μ and*

$$H(\mu \mid \mathcal{W}) = \frac{1}{2}\|v\|_\mathcal{V}^2. \tag{2.8}$$

Remark 2.8. With $u : [0,\infty) \times \mathbb{W} \to \mathbb{W}$ the *Föllmer process* associated with μ, as constructed in the proof of [18, Theorem 2], one of v's fulfilling (2.8) is given by $v = \{u(t,X)\}_{t \geq 0}$, where $X = \{X_t\}_{t \geq 0}$ is the unique strong solution to the stochastic differential equation

$$dX_t = dB_t + u(t,X)\,dt, \quad t \geq 0, \, X_0 = 0,$$

whose pathwise uniqueness is ensured by $\mu \in \mathcal{S}$. The above choice of v is in $\mathcal{V}_{b,0}$; indeed, supposing that $\mu \in \mathcal{S}$ has a density Φ given by (2.5), we see that, by construction,

$$|v_t| \leq \frac{1}{\inf\limits_{x \in \mathbb{R}^{d \times m}} \phi(x)} \sum_{i=1}^m \sup_{x \in \mathbb{R}^{d \times m}} |\nabla_{x^i}\phi(x)| \quad \text{a.s.,}$$

for $0 \leq t \leq t_m$ and $v_t = 0$ for $t > t_m$. Here, for each $1 \leq i \leq m$, $\nabla_{x^i}\phi$ is the gradient of $\phi(x) \equiv \phi(x^1,\ldots,x^m)$ with respect to the variable $x^i \in \mathbb{R}^d$.

Combining Lemmas 2.6 and 2.7, we immediately obtain

Proposition 2.9. *The upper bound* (2.4) *holds for any measurable function $F : \mathbb{W} \to \mathbb{R}$ that is bounded from above and satisfies* (A2)*.*

Proof. Set $G := e^F$. Without loss of generality, we may assume $\|G\|_{L^1(\mathcal{W})} = 1$. As $G \in L^2(\nu_F)$ thanks to the boundedness of G, there exists a sequence $\{\Phi_n\}_{n=1}^{\infty} \subset \mathcal{F}C_b^{\infty}$ such that

$$\lim_{n\to\infty} \|\Phi_n - G\|_{L^2(\nu_F)} = 0 \tag{2.9}$$

by Lemma 2.6. For every n, truncating Φ_n if necessary, we may assume $\inf_{w\in\mathbb{W}} \Phi_n(w) > 0$. For each n, define $G_n := \Phi_n / \|\Phi_n\|_{L^1(\mathcal{W})}$ so that $d\mu_n := G_n\, d\mathcal{W}$ is in $\mathcal{S}$. It is clear that

$$\lim_{n\to\infty} \|G_n - G\|_{L^2(\nu_F)} = 0 \tag{2.10}$$

by (2.9); indeed,

$$\|G_n - G\|_{L^2(\nu_F)} \leq \frac{1}{\|\Phi_n\|_{L^1(\mathcal{W})}} \|\Phi_n - G\|_{L^2(\nu_F)} + \left| \frac{1}{\|\Phi_n\|_{L^1(\mathcal{W})}} - 1 \right| \|G\|_{L^2(\nu_F)},$$

which tends to 0 because (2.9) also entails that $\lim_{n\to\infty} \|\Phi_n\|_{L^1(\mathcal{W})} = \|G\|_{L^1(\mathcal{W})} = 1$. As $\{G_n\}_{n=1}^{\infty}$ is bounded in $L^2(\mathcal{W})$ by (2.10) and the definition of ν_F, the sequence $\{G_n \log G_n\}_{n=1}^{\infty}$ is uniformly integrable under $\mathcal{W}$, whence, by Vitali's convergence theorem (see, e.g., [21, Theorem 22.7]),

$$\lim_{n\to\infty} \int_{\mathbb{W}} G_n \log G_n\, d\mathcal{W} = \int_{\mathbb{W}} G \log G\, d\mathcal{W} \tag{2.11}$$

because (2.10) also implies $G_n \to G$ in probability under $\mathcal{W}$. Moreover, it follows that

$$\lim_{n\to\infty} \int_{\mathbb{W}} G_n F_-\, d\mathcal{W} = \int_{\mathbb{W}} G F_-\, d\mathcal{W}.$$

Since $\{G_n\}_{n=1}^{\infty}$ also converges to G in $L^1(\mathcal{W})$ and F_+ is bounded, we have

$$\lim_{n\to\infty} \int_{\mathbb{W}} G_n F_+\, d\mathcal{W} = \int_{\mathbb{W}} G F_+\, d\mathcal{W}$$

as well, and hence

$$\lim_{n\to\infty} \int_{\mathbb{W}} F G_n\, d\mathcal{W} = \int_{\mathbb{W}} F G\, d\mathcal{W}. \tag{2.12}$$

Combining (2.11) and (2.12), we see that

$$\int_{\mathbb{W}} F\, d\mu_n - H(\mu_n \mid \mathcal{W}) = \int_{\mathbb{W}} F G_n\, d\mathcal{W} - \int_{\mathbb{W}} G_n \log G_n\, d\mathcal{W}$$
$$\xrightarrow[n\to\infty]{} 0$$

by the definition of G. Therefore, for any $\varepsilon > 0$, there exists $\mu \in \mathcal{S}$ such that

$$\log \mathbb{E}\left[e^{F(B)} \right] < \int_{\mathbb{W}} F\, d\mu - H(\mu \mid \mathcal{W}) + \varepsilon$$

because of $\mathbb{E}\left[e^{F(B)} \right] = 1$. The right-hand side of the last inequality is dominated by

$$\sup_{v\in\mathcal{V}} \left\{ \mathbb{E}[F(B^v)] - \frac{1}{2} \|v\|_{\mathcal{V}}^2 \right\} + \varepsilon \tag{2.13}$$

in view of Lemma 2.7, which proves the proposition as $\varepsilon > 0$ is arbitrary. □

Remark 2.10. If we let $v_n \in \mathcal{V}$ be as in Remark 2.8 for each μ_n, what is in fact proven is

$$\log \mathbb{E}\left[e^{F(B)}\right] = \lim_{n \to \infty} \left\{ \mathbb{E}[F(B^{v_n})] - \frac{1}{2} \|v_n\|_{\mathcal{V}}^2 \right\}.$$

We finish the proof of Proposition 2.5.

Proof of Proposition 2.5. For a measurable function $F : \mathbb{W} \to \mathbb{R}$ satisfying (A1) and (A2), we set, for each $M > 0$, $F_M(w) := F(w) \wedge M$, $w \in \mathbb{W}$. Then, for any M, we have, by Proposition 2.9,

$$\log \mathbb{E}\left[e^{F_M(B)}\right] \leq \sup_{v \in \mathcal{V}} \left\{ \mathbb{E}[F_M(B^v)] - \frac{1}{2} \|v\|_{\mathcal{V}}^2 \right\}$$
$$\leq \sup_{v \in \mathcal{V}} \left\{ \mathbb{E}[F(B^v)] - \frac{1}{2} \|v\|_{\mathcal{V}}^2 \right\},$$

the last expression being well-defined by (2.2). Letting $M \to \infty$ on the leftmost side completes the proof by the dominated/monotone convergence theorem. $\square$

Since domination (2.13) is valid if we replace the supremum over $\mathcal{V}$ by that over $\mathcal{V}_b$ or $\mathcal{V}_{b,0}$ in view of Remark 2.8, we have the following corollary, which we think is useful in some of applications; see, e.g., [10, Remarks 4.8 and 4.9].

Corollary 2.11. *The supremum in* (1.3) *may be replaced by that over drifts v in $\mathcal{V}_b$ or $\mathcal{V}_{b,0}$; that is, for any measurable function $F : \mathbb{W} \to \mathbb{R}$ satisfying* (A1) *and* (A2)*, we have*

$$\log \mathbb{E}\left[e^{F(B)}\right] = \sup_{v \in \mathcal{V}_b} \left\{ \mathbb{E}[F(B^v)] - \frac{1}{2} \|v\|_{\mathcal{V}}^2 \right\}$$
$$= \sup_{v \in \mathcal{V}_{b,0}} \left\{ \mathbb{E}[F(B^v)] - \frac{1}{2} \|v\|_{\mathcal{V}}^2 \right\}.$$

We end this section with a remark on the proof of Theorem 1.1 and related facts.

Remark 2.12. (1) Since both sides of (1.3) are well-defined only under assumption (A1) as noted in Remark 1.2(1), it is plausible that formula (1.3) holds true without any assumptions on F from below; however, we have not succeeded in proving it. The difficulty is to prove the upper bound (2.4) without assuming (A2).

(2) Using the notion of filtrations introduced by Üstünel and Zakai [25] on abstract Wiener spaces, Zhang [26] extended formula (1.1) of Boué–Dupuis for bounded Wiener functionals to the framework of abstract Wiener spaces as simplifying the original proof of the upper bound which relied on a complicated measurable selection argument. As for the case of the Wiener space $(\mathbb{W}, \mathcal{W})$, Lehec [18] further simplified the proof of the upper bound, based on deep analysis of the Gaussian relative entropy as exhibited in Lemmas 2.4 and 2.7. Note that Lehec's extension [18, Theorem 9] to the case with $F(B)$ a functional of B, assumed bounded from below, over the whole time interval may be seen as a particular case of Zhang's result [26, Theorem 3.2]; indeed, as discussed in [22, Section 8.1], by restricting $\mathcal{W}$ to the Banach space $\tilde{\mathbb{W}}$ consisting of paths $w \in \mathbb{W}$ such that $\lim_{t \to \infty} |w(t)|/t = 0$ normed by $\sup_{t \geq 0} |w(t)|/(1+t)$, the triple $(\tilde{\mathbb{W}}, \mathbb{H}, \mathcal{W})$ forms an abstract Wiener space, where $\mathbb{H}$ is the usual Cameron–Martin subspace of $\mathbb{W}$.

(3) One of the main differences between Lehec's proof and ours is that we appeal to the density of $\mathcal{F}C_b^{\infty}$ in $L^2(\nu_F)$ instead of $L^2(\mathcal{W})$; another is the employment of Vitali's convergence theorem in (2.11).

3 Application to the Ornstein–Uhlenbeck semigroup

In this section, we explore a connection between formula (1.3) and the exponential version of the hypercontractivity of the Ornstein–Uhlenbeck semigroup in $\mathbb{R}^d$. For this purpose, we begin with restating Theorem 1.1 when the functional $F(B)$ is a function of B_1.

We consider the set of d-dimensional $\{\mathcal{F}_t^B\}$-progressively measurable processes $v = \{v_t\}_{0\le t\le 1}$ satisfying

$$\mathbb{E}\left[\int_0^1 |v_t|^2\, dt\right] < \infty;$$

in order to specify notationally that v_t is a functional of B up to time t and the terminal time is 1, we denote this set by $\mathcal{V}_1(B)$. Let γ denote the standard Gaussian measure on $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Noting that conditions (A1) and (A2) are equivalent to both e^F and F being in $L^1(\mathcal{W})$ (see Remark 1.2(3)), we assume

(B) $e^f \in L^1(\gamma)$ and $f \in L^1(\gamma)$.

The following is immediate from Theorem 1.1 applied to $F(B) = f(B_1)$:

Proposition 3.1. *Under assumption* (B), *we have*

$$\log \mathbb{E}\left[e^{f(B_1)}\right] = \sup_{v\in\mathcal{V}_1(B)} \mathbb{E}\left[f\left(B_1 + \int_0^1 v_t\, dt\right) - \frac{1}{2}\int_0^1 |v_t|^2\, dt\right]. \tag{3.1}$$

Next we recall the exponential hypercontractivity of the Ornstein–Uhlenbeck semigroup $Q = \{Q_t\}_{t\ge 0}$ defined in the Gaussian space $(\mathbb{R}^d, \gamma)$.

For each $t \ge 0$, the operator Q_t acts on $L^1(\gamma)$ in such a way that, for $f \in L^1(\gamma)$,

$$(Q_t f)(x) = \int_{\mathbb{R}^d} f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \gamma(dy), \quad x \in \mathbb{R}^d.$$

It is well known that Q enjoys the hypercontractivity, which is also known (see [1, Proposition 4]) to be equivalent to the following property that we call the *exponential hypercontractivity*: for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying (B),

$$\|\exp(Q_t f)\|_{L^{e^{2t}}(\gamma)} \le \|e^f\|_{L^1(\gamma)} \quad \text{for all } t \ge 0. \tag{3.2}$$

We provide a simple derivation of (3.2) by means of Proposition 3.1; formula (3.1) for any bounded measurable function f was discovered by Borell [5] independently of Boué–Dupuis [6] and applied to a simple proof of the Prékopa–Leindler inequality among others. Our application, which seems to be new to our knowledge, serves as another instance of usefulness of the formula, often referred to as *Borell's formula*, in deriving existing functional inequalities.

Let $f \in L^1(\gamma)$ and observe the following identity in law for every $t \ge 0$:

$$(Q_t f, \gamma) \overset{(d)}{=} \left(\mathbb{E}\left[f(B_1) \mid \mathcal{F}_{e^{-2t}}^B\right], \mathbb{P}\right).$$

Indeed, by the independence of $B_1 - B_{e^{-2t}}$ and $B_{e^{-2t}}$, we have, a.s.,

$$\mathbb{E}\left[f(B_1) \mid \mathcal{F}_{e^{-2t}}^B\right] = \mathbb{E}[f(B_1 - B_{e^{-2t}} + x)]\big|_{x=B_{e^{-2t}}},$$

which has the same law as

$$\mathbb{E}\left[f\left(\sqrt{1 - e^{-2t}}N_2 + e^{-t}x\right)\right]\big|_{x=N_1},$$

where N_1 and N_2 are d-dimensional standard Gaussian random variables. Therefore the exponential hypercontractivity (3.2) is equivalently stated as

Proposition 3.2. *For every measurable function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying* (B)*, it holds that*

$$t \log \mathbb{E}\big[\exp\big\{t^{-1}\mathbb{E}\big[f(B_1) \mid \mathcal{F}_t^B\big]\big\}\big] \le \log \mathbb{E}\big[e^{f(B_1)}\big] \tag{3.3}$$

for all $0 < t \le 1$.

We give a proof of the proposition via Proposition 3.1. To this end, given $f \in L^1(\gamma)$, we set

$$g(t,x) := \mathbb{E}[f(B_1 - B_t + x)], \quad 0 \le t \le 1, \ x \in \mathbb{R}^d,$$

so that, for every $0 \le t \le 1$,

$$\mathbb{E}\big[f(B_1) \mid \mathcal{F}_t^B\big] = g(t, B_t) \quad \text{a.s.} \tag{3.4}$$

Proof of Proposition 3.2. By appealing to the monotone convergence theorem, it suffices to prove (3.3) when $f \in L^1(\gamma)$ is bounded from above. Fix $0 < t \le 1$ and set

$$W_s := \frac{1}{\sqrt{t}} B_{ts}, \qquad\qquad \mathcal{F}_s^W := \sigma(W_u, 0 \le u \le s) \vee \mathcal{N},$$

for $0 \le s \le 1$, so that $W = \{W_s\}_{0 \le s \le 1}$ is a standard d-dimensional $\{\mathcal{F}_s^W\}$-Brownian motion. Note that $\sqrt{t}W_1 = B_t$ and $\mathcal{F}_1^W = \mathcal{F}_t^B$ by definition. Moreover, as $g(t, B_t)$ is integrable in view of (3.4), the function $t^{-1}g(t, \sqrt{t}x)$, $x \in \mathbb{R}^d$, fulfills assumption (B) since we have assumed that f is bounded from above. Therefore, noting (3.4) again, we may apply Proposition 3.1 to $t^{-1}g(t, \sqrt{t}W_1)$ to rewrite the left-hand side of (3.3) as

$$
\begin{aligned}
&t \log \mathbb{E}\Big[\exp\Big\{t^{-1}g\big(t, \sqrt{t}W_1\big)\Big\}\Big] \\
&= t \sup_{v \in \mathcal{V}_1(W)} \mathbb{E}\Big[t^{-1}g\Big(t, \sqrt{t}W_1 + \sqrt{t}\int_0^1 v_s\, ds\Big) - \frac{1}{2}\int_0^1 |v_s|^2\, ds\Big] \\
&= \sup_{v \in \mathcal{V}_1(W)} \mathbb{E}\Big[g\Big(t, \sqrt{t}W_1 + \int_0^1 v_s\, ds\Big) - \frac{1}{2}\int_0^1 |v_s|^2\, ds\Big] \\
&= \sup_{v \in \mathcal{V}_1(W)} \mathbb{E}\Big[f\Big(B_1 + \int_0^1 v_s\, ds\Big) - \frac{1}{2}\int_0^1 |v_s|^2\, ds\Big].
\end{aligned}
\tag{3.5}
$$

Here the second equality follows from the equivalence $\sqrt{t}v \in \mathcal{V}_1(W) \iff v \in \mathcal{V}_1(W)$; for the third, by recalling the definition of g, and by noting that the random variables

$$\sqrt{t}W_1 + \int_0^1 v_s\, ds, \quad \int_0^1 |v_s|^2\, ds$$

are independent of $B_1 - B_t$ because they are $\mathcal{F}_t^B$-measurable by the definition of W, the boundedness of f from above allowed us to apply Fubini's theorem. Due to the obvious inclusion $\mathcal{V}_1(W) \subset \mathcal{V}_1(B)$, the last expression in (3.5) is dominated by

$$\sup_{v \in \mathcal{V}_1(B)} \mathbb{E}\Big[f\Big(B_1 + \int_0^1 v_s\, ds\Big) - \frac{1}{2}\int_0^1 |v_s|^2\, ds\Big],$$

and hence, in virtue of Proposition 3.1 again, by $\log \mathbb{E}\big[e^{f(B_1)}\big]$. This proves (3.3). $\square$

Remark 3.3. We may start the proof with bounded measurable functions by truncating f as $(f \wedge M) \vee (-N)$ for $M, N > 0$. Then repeated use of the monotone convergence theorem as $N \to \infty$ and then as $M \to \infty$ completes the proof. The essential part of the above proof is how Borell's formula applies to (3.3).

By [1, Proposition 4], the exponential hypercontractivity (3.2) is equivalent to the Gaussian logarithmic Sobolev inequality in $\mathbb{R}^d$: for any weakly differentiable function f in $L^2(\gamma)$ with $|\nabla f| \in L^2(\gamma)$,

$$\int_{\mathbb{R}^d} |f|^2 \log |f| \, d\gamma \leq \||\nabla f|\|_{L^2(\gamma)}^2 + \|f\|_{L^2(\gamma)}^2 \log \|f\|_{L^2(\gamma)} \, ; \tag{3.6}$$

we also refer to [15, Subsection A.1] in this respect. It is known [4, Section 3] that the Prékopa–Leindler inequality implies the logarithmic Sobolev inequality; the above exploration provides another path from formula (3.1) to (3.6).

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