Electron. Commun. Probab. **27** (2022), article no. 37, 1–12.
https://doi.org/10.1214/22-ECP480
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Kinetic Dyson Brownian motion

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Abstract

We study the spectrum of the kinetic Brownian motion in the space of $d \times d$ Hermitian matrices, $d \geq 2$. We show that the eigenvalues stay distinct for all times, and that the process Λ of eigenvalues is a kinetic diffusion (i.e. the pair $(\Lambda, \dot{\Lambda})$ of Λ and its derivative is Markovian) if and only if $d = 2$. In the large scale and large time limit, we show that Λ converges to the usual (Markovian) Dyson Brownian motion under suitable normalisation, regardless of the dimension.

Keywords: random matrices; Markov process; stochastic differential equations.
MSC2020 subject classifications: 60B20; 60G53; 60J60.
Submitted to ECP on January 27, 2022, final version accepted on July 25, 2022.
Supersedes arXiv:2101.10426.

1 Introduction

In the space $\mathcal{H}_d$ of $d \times d$ complex Hermitian matrices, there is a natural Brownian process W, whose covariance is given by the Hilbert–Schmidt norm. It turns out, as was first observed by Dyson in 1962 [6], that the spectrum of W is Markovian: the law of the spectrum of $s \mapsto W_{t+s}$ depends on W_t only through its spectrum. In this paper, we show that for a natural smoothing of the Brownian motion, the so-called kinetic Brownian motion, the situation can be different.

Here, a kinetic motion with values in $\mathcal{H}_d$ is a process of regularity $\mathcal{C}^1$, such that the couple $(H, \dot{H})$ of the position and associated velocity is Markovian. Kinetic Brownian motion is the kinetic motion whose velocity $\dot{H}$ is a standard Brownian motion on the unit sphere. We say that it is a smoothing of Brownian motion because in the large scale limit, H looks like a Brownian motion; namely, the law of the process $t \mapsto \frac{1}{L} H_{L^2 t}$ converges to that of W as $L \to \infty$, up to a factor $4/d^2(d^2-1)$. In other words, $\frac{1}{L} H_{L^2 t}$ is very similar to a Brownian motion, with the important difference that it is actually $\mathcal{C}^1$. For a proof of this convergence, see [9, Theorem 1.1] or [2, Proposition 2.5]. See references below for more on kinetic Brownian motion.

Define Λ_t as the vector of eigenvalues of H_t, the order being irrelevant provided it depends continuously on time. Then a classical application of the inverse function theorem shows that Λ has to be $\mathcal{C}^1$ whenever the eigenvalues of H_t are distinct; in particular, Λ cannot be Markovian, otherwise it would be deterministic. The next natural hope would be for the process $(\Lambda, \dot{\Lambda})$ to be Markovian. The main objective of this paper is to prove it is the case if and only if $d = 2$.

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Theorem 1.1. *Let $(H, \dot{H})$ be a kinetic Brownian motion on the space $\mathcal{H}_d$ of $d \times d$ Hermitian matrices ($d \geq 2$), and $0 \leq \tau \leq \infty$ the first time H has multiple eigenvalues. Let Λ be the process of eigenvalues of H, seen as a continuous process with values in $\mathbb{R}^d$.*

Then $\tau = \infty$ whenever H_0 has distinct eigenvalues. Moreover, $(\Lambda_t, \dot{\Lambda}_t)_{0 \leq t < \tau}$ is well-defined, and is Markovian if and only if $d = 2$.

In the case $d = 2$, the stochastic differential equations describing the eigenvalues of H are given in Lemma 2.5, Section 2.4. In the general case, we introduce in Section 2.3 a subdiffusion (Λ, A) of $(H, \dot{H})$. Since this process is Markovian in any dimension (see Lemma 2.3 for defining equations), one may want to consider it as a suitable approach to kinetic Dyson Brownian motion, rather than the more restrictive $(\Lambda, \dot{\Lambda})$.

As discussed above, it is known that the process $t \mapsto \frac{1}{L} H_{L^2 t}$ converges in law to a Brownian motion. From this known fact, we will deduce that Λ converges to a standard Dyson Brownian motion, in the following sense.

Proposition 1.2. *Let $(H, \dot{H})$ be a kinetic Brownian motion on the space $\mathcal{H}_d$ of $d \times d$ complex Hermitian matrices ($d \geq 2$) such that H_0 has distinct eigenvalues almost surely. Let Λ be the process of eigenvalues of H in non-decreasing order, seen as a continuous process from an interval of $\mathbb{R}_+$ to $\mathbb{R}^d$. Let D be the process of eigenvalues of a standard Brownian motion in $\mathcal{H}_d$ starting at zero, with the same conventions.*

Then we have the following convergence in law as L goes to infinity:

$$(t \mapsto \Lambda_{L^2 t})_{0 \leq t \leq 1} \xrightarrow{\mathcal{L}} \frac{4}{d^2(d^2 - 1)} \cdot D.$$

We give a sketch of proof of Theorem 1.1 in Sections 2.1 to 2.4, using various lemmas proved in Part 3. Proposition 1.2 is proved in Section 2.5.

Kinetic Brownian motion was first introduced by Li in [8], where Theorem 4.3 proves a stronger convergence theorem to Brownian motion than stated above. A self-contained proof by the same author appeared later in [9]. This result was generalised by Angst, Bailleul and Tardif [2] and the author [11]. Kinetic Brownian motion has been given different names in the literature, for instance velocity spherical Brownian motion by Baudoin and Tardif [4] or circular Langevin diffusion by Franchi [7] in the context of heat kernels. See also [1] for considerations in an infinite dimensional setting.

2 Definitions and proof outline

Let $\mathcal{H}_d$ be the space of $d \times d$ complex Hermitian matrices. We assume $d \geq 2$, and endow it with the Hilbert–Schmidt inner product:

$$\|H\|_{\mathcal{H}}^2 := \operatorname{tr}(H^2) = \operatorname{tr}(H^* H) = \sum_{ij} |H_{ij}|^2,$$

where H^* is the conjugate transpose of H, $(H_{ij})_{ij}$ are the coefficients of H, and $\langle \cdot, \cdot \rangle$ is the Hermitian product on $\mathbb{C}^d$, with the convention that it is linear in its second argument. It is isometric to the standard Euclidean matrix space $\mathbb{R}^{d \times d}$, via $(m_{ij}) \mapsto (h_{ij})$,

$$h_{ij} = \begin{cases} m_{ij} & \text{for } i = j, \\ \frac{m_{ij} + \mathrm{i}\, m_{ji}}{\sqrt{2}} & \text{for } i < j, \\ \frac{m_{ji} - \mathrm{i}\, m_{ij}}{\sqrt{2}} & \text{for } j < i. \end{cases}$$

A Brownian motion W in $\mathcal{H}_d$ associated to this Euclidean structure can be described as a matrix $H = (h_{ij})$ with coefficients as above, where the m_{ij}'s are independent (real standard) Brownian motions.

Let $\dot{H}$ be a standard Brownian motion on the unit sphere $\mathbb{S}(\mathcal{H}_d)$ of $\mathcal{H}_d$, and H its integral. For instance, one may define $(H, \dot{H})$ as the solution of the stochastic differential equation

$$\mathrm{d}H_t = \dot{H}_t \mathrm{d}t, \tag{2.1}$$

$$\mathrm{d}\dot{H}_t = \underbrace{\circ\, \mathrm{d}W_t - \dot{H}_t \langle \dot{H}_t,\, \circ\, \mathrm{d}W_t \rangle_{\mathcal{H}}}_{\text{projection of } \circ \mathrm{d}W_t \text{ on } \dot{H}_t^{\perp}} = \mathrm{d}W_t - \dot{H}_t \operatorname{tr}(\dot{H}_t^* \mathrm{d}W_t) - \frac{d^2-1}{2} \dot{H}_t \mathrm{d}t, \tag{2.2}$$

where $\circ\mathrm{d}W$ (resp. $\mathrm{d}W$) denotes the Stratonovich (resp. Itô) integral. It is defined for all times, since $\dot{H}$ is the solution of a SDE with smooth coefficients on a compact manifold, and H is the integral of a process that is uniformly bounded. Let $0 \le \tau \le \infty$ be the first time H has multiple eigenvalues, with $\tau = \infty$ if its eigenvalues stay distinct for all times.

In the following, we will work with diagonal matrices; we write $\mathcal{H}_d^{\Delta}$ for the corresponding space, and $\pi^{\Delta} : \mathcal{H}_d \to \mathcal{H}_d^{\Delta}$ for the orthogonal projection. In simple terms, π^{Δ} replaces every off-diagonal entry by a zero.

The remainder of this section is a complete proof outline of Theorem 1.1, using a few lemmas proved in the next section.

2.1 Explosion time

By "$\tau = \infty$ whenever H_0 has distinct eigenvalues", we mean that the event of having both $\tau < \infty$ and H_0 without repeated eigenvalues has measure zero. It is known, see references in Section 3.1, that the subset of $\mathcal{H}_d$ consisting of matrices with multiple eigenvalues is contained in a finite collection of submanifolds of codimension 3. Then it is enough to prove the following result, of independent interest.

Proposition 2.1 (proved in Section 3.2)**.** *Let M be a complete Riemannian manifold, and $N \subset M$ a submanifold of codimension at least 2. Let $(H, \dot{H})$ be a kinetic Brownian motion in M, defined for all times $t \ge 0$. Then H never crosses N for any $t > 0$.*

2.2 Diagonalisation

Because H_t is Hermitian, there exists for all t a unitary matrix U_t such that $U_t^* H_t U_t$ is diagonal. Abstract geometric arguments show that it is possible to show that for a fixed realisation of H, we can find a U with regularity $\mathcal{C}^1$, at least as long as the eigenvalues stay distinct. However, we would like U to be described by an explicit stochastic differential equation. Let us look for a candidate.

Given a $\mathcal{C}^1$ process U with unitary values, we call

$$\dot{u}_t := U_t^{-1} \dot{U}_t = U_t^* \dot{U}_t$$

its derivative, seen in the Lie algebra of $U_d(\mathbb{C})$. Hence, $\dot{u}_t$ is skew-Hermitian: $\dot{u}_t^* = -\dot{u}_t$. If we define the $\mathcal{C}^1$ process

$$\Lambda : t \mapsto U_t^* H_t U_t, \tag{2.3}$$

then H_t will be diagonal in the frame U_t if and only if $\Lambda_t \in \mathcal{H}^{\Delta}$. It means that we are looking for a U such that the derivative $\dot{\Lambda}_t$ stays in $\mathcal{H}^{\Delta}$. We have

$$\dot{\Lambda}_t = \left(\dot{U}_t^* H_t U_t + U_t^* H_t \dot{U}_t \right) + U_t^* \dot{H}_t U_t = (\dot{u}_t^* \Lambda_t + \Lambda_t \dot{u}_t) + U_t^* \dot{H}_t U_t.$$

Assuming Λ is indeed diagonal, and since $\dot{u}_t$ is skew-Hermitian, the coefficients of the first term are

$$(\dot{u}_t^* \Lambda_t + \Lambda_t \dot{u}_t)_{ij} = (\dot{u}_t^*)_{ij} (\Lambda_t)_{jj} + (\Lambda_t)_{ii} (\dot{u}_t)_{ij} = \big((\Lambda_t)_{ii} - (\Lambda_t)_{jj} \big) (\dot{u}_t)_{ij}. \tag{2.4}$$

In other words, for Λ to stay diagonal, we have no choice for the off-diagonal coefficients of the velocity $\dot{u}_t$: setting

$$A : t \mapsto U_t^* \dot{H}_t U_t, \tag{2.5}$$

they have to be

$$-\frac{(U_t^* \dot{H}_t U_t)_{ij}}{(U_t^* H_t U_t)_{ii} - (U_t^* H_t U_t)_{jj}} = -\frac{(A_t)_{ij}}{(\Lambda_t)_{ii} - (\Lambda_t)_{jj}}.$$

It turns out that this choice works, as we will see.

For the sake of conciseness, define $\dot{u}(\Lambda, A)$ as

$$\big(\dot{u}(\Lambda, A)\big)_{ij} := \begin{cases} -\dfrac{A_{ij}}{\Lambda_{ii} - \Lambda_{jj}} & \text{for } i \neq j, \\ 0 & \text{else} \end{cases} \tag{2.6}$$

whenever Λ has distinct diagonal entries. As long as Λ stays diagonal with distinct eigenvalues, $u(\Lambda, A)$ stays well-defined.

Lemma 2.2 (proved in Section 3.3)**.** *Let $(H_t, \dot{H}_t)_{t\geq 0}$ be a kinetic Brownian motion on $\mathcal{H}_d$, and $0 \leq \tau \leq \infty$ the first time H has multiple eigenvalues. Let U_0 be a (random) unitary matrix such that $U_0^* H_0 U_0$ is diagonal, and define U_t as the solution of*

$$\mathrm{d}U_t = U_t \dot{u}_t \mathrm{d}t, \qquad \dot{u}_t = \dot{u}\big(U_t^* H_t U_t, U_t^* \dot{H}_t U_t\big), \tag{2.7}$$

where $\dot{u}(\Lambda, A)$ is defined in equation (2.6)*.*

Then U_t is defined for all $0 \leq t < \tau$, and $U_t^* H_t U_t$ is diagonal for all such t.

In particular, it means that Λ is the process of eigenvalues of H, so the potential kinetic Dyson Brownian motion is $(\Lambda, \dot{\Lambda})$. If one wishes, we can take U_0 so that the diagonal entries of $U_0^* H_0 U_0$ are in a given order, for instance non-decreasing. Then, using the fact that the eigenvalues of H_t stay distinct for all $t > 0$ (see Section 2.1), we see that the diagonal entries $U_t^* H_t U_t$ stay in the same order. In the following, we will not need nor assume that U_0 satisfies this property.

2.3 The Markovian process (Λ, A)

From $(H, \dot{H})$, we construct U, Λ and A using equations (2.7), (2.3) and (2.5). By Lemma 2.2, Λ is in fact diagonal. We also notice that A takes values in the sphere $\mathbb{S}(\mathcal{H}_d)$, since $\dot{H}$ does, and conjugation by a fixed unitary matrix is an isometry.

Then we see that the triple (U_t, Λ_t, A_t) satisfies the system of equations

$$\begin{aligned}
\mathrm{d}U_t &= U_t \dot{u}_t \mathrm{d}t \\
\mathrm{d}\Lambda_t &= (\dot{u}_t^* \Lambda_t + \Lambda_t \dot{u}_t)\mathrm{d}t + A_t \mathrm{d}t \\
\mathrm{d}A_t &= (\dot{u}_t^* A_t + A_t \dot{u}_t)\mathrm{d}t + U_t^* \mathrm{d}\dot{H}_t U_t
\end{aligned}$$

for $0 \leq t < \tau$, where $\dot{u}_t = \dot{u}(\Lambda_t, A_t)$. These $\dot{u}(A, \Lambda)$ and $\dot{u}_t$ are still the same, defined respectively in equations (2.6) and (2.7); we are merely emphasising the fact that $\dot{u}_t$ depends on $(H, \dot{H})$ only through (Λ, A). Note that the above is really a stochastic differential equation describing (U, Λ, A) with a driving noise $\dot{H}$, rather than an abstract functional of the couple $(H, \dot{H})$.

We are interested in the dynamics of $(\Lambda, \dot{\Lambda})$. In fact, this pair is nothing but the projection $(\Lambda, \pi^{\Delta}(A))$; indeed, this is a direct consequence of the fact that in the driving equation for $\mathrm{d}\Lambda_t$, the first term is zero on the diagonal as seen in (2.4). As explained in the following lemma, it turns out that (Λ, A) is Markovian.

Lemma 2.3 (proved in Section 3.3)**.** *Let $(H, \dot{H})$ be a kinetic Brownian motion in $\mathcal{H}_d$, $0 \leq \tau \leq \infty$ the first time H has multiple eigenvalues, and define U, Λ and A as in equations* (2.7), (2.3) *and* (2.5)*. Then, up to enlarging the underlying probability space, there exists a standard Brownian motion B with values in $\mathcal{H}_d$ such that*

$$\begin{aligned} \mathrm{d}\Lambda_t &= \pi^{\Delta}(A_t)\mathrm{d}t \\ \mathrm{d}A_t &= \big(\dot{u}_t^* A_t + A_t \dot{u}_t\big)\mathrm{d}t + \mathrm{d}B_t - A_t \operatorname{tr}\big(A_t^* \mathrm{d}B_t\big) - \frac{d^2-1}{2} A_t \mathrm{d}t \end{aligned}$$

for all $0 \leq t < \tau$, where $\dot{u}_t = u_t(\Lambda_t, A_t)$ as defined in equation (2.6)*. In particular, the process (Λ, A) is Markovian.*

Here, we could replace $\dot{u}_t$ everywhere by its actual expression, which makes it clear that (Λ, A) satisfies a self-contained stochastic differential equation.

Since (Λ, A) is Markovian in all dimensions whereas (as we will show later) $(\Lambda, \dot{\Lambda})$ is not, we made the remark in the introduction that one might view the former as a natural definition of kinetic Dyson Brownian motion. Instead of containing only the information about the derivative of the eigenvalues of Λ as $\dot{\Lambda}$ might do, A is the matrix $\dot{H}$ as seen in a referential that makes H diagonal; in particular, it contains some hints about the motion of the eigenspaces in relation to each other. As we will see below, at least some of this additional information is needed to describe the motion of Λ entirely when $d \geq 3$.

2.4 A criterion for a Markovian kinetic Dyson Brownian motion

It is worth noticing that the equation for A describes a Brownian motion with drift on a sphere; compare for instance the definition of $\dot{H}_t$ in (2.2), describing a Brownian motion without drift. In fact, if it were not for the first term $(\dot{u}^* A + A\dot{u})\mathrm{d}t$, A would be precisely a standard Brownian motion on the unit sphere of $\mathcal{H}_d$.

But it is known, and not too difficult to see, that the projection of a spherical Brownian motion X is Markovian. Indeed, once one fixes a subset of coordinates $(X^0, \ldots, X^k)$ of norm r, then the remaining coordinates $(X^{k+1}, \ldots, X^n)$ can always be reduced to $\big(\sqrt{1-r^2}, 0, \ldots, 0\big)$, up to a rotation fixing the first coordinates; see Section 3.4 for details and references. In particular, if we continue to ignore this drift term, it would be clear at this point that $(\Lambda, \dot{\Lambda}) = (\Lambda, \pi^{\Delta}(A))$ is Markovian. So any obstruction for $(\Lambda, \dot{\Lambda})$ to be Markovian must come from the additional term $(\dot{u}^* A + A\dot{u})$. The following lemma describes the situation with a precise criterion.

Lemma 2.4 (proved in Section 3.5)**.** *Define $\Phi = \Phi^{(d)} : \mathcal{H}_d^{\Delta} \times \mathbb{S}(\mathcal{H}_d) \to \mathcal{H}_d^{\Delta}$ by*

$$\Phi(\Lambda, A) := \pi^{\Delta}(\dot{u}^* A + A\dot{u}),$$

where π^{Δ} is the projection on $\mathcal{H}_d^{\Delta}$ and $\dot{u} = \dot{u}(\Lambda, A)$ as defined in (2.6)*.*

Let $(H, \dot{H})$ be a kinetic Brownian motion in $\mathcal{H}_d$, and define Λ the continuous process of its eigenvalues. Then $(\Lambda, \dot{\Lambda})$ is Markovian if and only if Φ factors through the projection map $(\Lambda, A) \mapsto (\Lambda, \pi^{\Delta}(A))$.

There is a concise expression for the coefficients of $\Phi(\Lambda, A)$. It is of course zero out of the diagonal, and we have

$$\Phi(\Lambda, A)_{ii} = \sum_{j \neq i} \Big(- \frac{\overline{A}_{ji}}{\Lambda_{jj} - \Lambda_{ii}} \cdot A_{ji} - A_{ij} \cdot \frac{A_{ji}}{\Lambda_{jj} - \Lambda_{ii}} \Big) = 2 \sum_{j \neq i} \frac{|A_{ij}|^2}{\Lambda_{ii} - \Lambda_{jj}}. \tag{2.8}$$

In dimension 2, since $|A| = 1$, we get directly

$$\Phi(\Lambda, A)_{11} = -\Phi(\Lambda, A)_{22} = \frac{|A_{12}|^2 + |A_{21}|^2}{\Lambda_{11} - \Lambda_{22}} = \frac{1 - |A_{11}|^2 - |A_{22}|^2}{\Lambda_{11} - \Lambda_{22}}.$$

This depends only on Λ and the on-diagonal coefficients of A, so $(\Lambda, \dot\Lambda)$ is indeed Markovian. In dimension $d \geq 3$, it is not obvious that one could use a similar trick, and in fact we can show that it is not possible and that the process is not Markovian.

In Sections 3.6 and 3.7 we carry out the computations in dimension $d = 2$ and $d \geq 3$ respectively, and we conclude as follows.

Lemma 2.5 (proved in Sections 3.6 and 3.7)**.**

- *If $d = 2$, the eigenvalues λ, μ of H make up a kinetic diffusion, and satisfy the equations*

$$\mathrm{d}\lambda_t = \dot\lambda_t \mathrm{d}t, \qquad \mathrm{d}\dot\lambda_t = +\frac{1 - \dot\lambda_t^2 - \dot\mu_t^2}{\lambda_t - \mu_t}\mathrm{d}t + \mathrm{d}M_t^\lambda - \frac{d^2 - 1}{2}\dot\lambda_t \mathrm{d}t,$$
$$\mathrm{d}\mu_t = \dot\mu_t \mathrm{d}t, \qquad \mathrm{d}\dot\mu_t = -\frac{1 - \dot\lambda_t^2 - \dot\mu_t^2}{\lambda_t - \mu_t}\mathrm{d}t + \mathrm{d}M_t^\mu - \frac{d^2 - 1}{2}\dot\mu_t \mathrm{d}t,$$

  where M^λ and M^μ are martingales with brackets

$$\mathrm{d}\langle M^\lambda, M^\lambda\rangle_t = \big(1 - \dot\lambda_t^2\big)\mathrm{d}t, \qquad \mathrm{d}\langle M^\mu, M^\mu\rangle_t = \big(1 - \dot\mu_t^2\big)\mathrm{d}t,$$
$$\mathrm{d}\langle M^\lambda, M^\mu\rangle_t = -\dot\lambda_t\dot\mu_t \mathrm{d}t.$$

- *If $d \geq 3$, $\Phi^{(d)}$ does not factor, and the process $(\Lambda, \dot\Lambda)$ is not Markovian.*

2.5 Homogenisation

As stated in the introduction, if we write H^L for the normalised process $t \mapsto \frac{1}{L}H_{L^2 t}$, then $(H_t^L)_{0\leq t\leq 1}$ converges in law to a standard Brownian motion, up to a constant scaling factor $4/d^2(d^2-1)$; see [9, Theorem 1.1] or [2, Proposition 2.5]. In this section we prove Proposition 1.2, namely that the process Λ, although not Markovian, is somehow almost Markovian in large scales, in the sense that a similar limit converges to a (Markovian) Dyson Brownian motion. We assume for simplicity that we chose U_0 so that the diagonal entries of Λ_0 are in non-decreasing order.

Define the map $\Lambda : \mathcal{H}_d \to \mathcal{H}_d^\Delta$ sending a matrix H to the matrix Λ whose diagonal entries are the eigenvalues of H with multiplicities according to a chosen order (e.g. non-decreasing); it should lead to no confusion of notation, since we then have $\Lambda_t = \Lambda(H_t)$. It is continuous, which means that Λ_t can be described as a continuous function of H_t up to time τ. Such operations preserve convergence in law, so given a standard Brownian motion W in $\mathcal{H}_d$, we have the following convergence in law:

$$(H_t^L)_{0\leq t\leq 1} \xrightarrow{\mathcal{L}} \frac{4}{d^2(d^2-1)} \cdot W, \qquad (\Lambda_t^L)_{0\leq t\leq 1} \xrightarrow{\mathcal{L}} \frac{4}{d^2(d^2-1)} \cdot \Lambda(W).$$

Since $\Lambda(W)$ is the spectrum of a Brownian motion in $\mathcal{H}_d$ in the form of a diagonal matrix, it is nothing but a Dyson Brownian motion. As stated above, $\Lambda = \Lambda(H)$ looks very much like a Dyson Brownian motion at large scales. Recall that Dyson Brownian motion is Markovian, so the hidden information preventing Λ to be Markovian (the off-diagonal coefficients of A, but also the derivative $\dot\Lambda$) vanishes in the limit.

It might be interesting to see if one could prove the convergence of Λ^L towards the rescaled $\Lambda(W)$ using only the dynamics of (Λ, A) as given in Lemma 2.3. Although the author does not pretend it is impossible, it seems that the non-linearity in the vector field $\Phi(\Lambda, A)$ makes it more difficult to approach than the convergence of H^L, using for instance the methods of [2].

3 Proof of the Lemmas

3.1 Matrices with multiple eigenvalues

We claimed earlier that the set of Hermitian matrices with multiple eigenvalues is covered by finitely many submanifolds of codimension at least 3. We sketch a proof of this result; the approach is carried out in [3] with a detailed discussion of the underlying combinatorial structure. See also [5] and references therein. Let $d_1 + \cdots + d_k = d$ be a partition of d into k positive integers. Consider the space N of all Hermitian matrices H with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_d$ such that the first d_1 eigenvalues are equal but less than the next, the following d_2 are equal but less than the $(d_1 + d_2 + 1)$st, etc.

There is a one-to-one correspondence between such matrices and a choice of k (orthogonal) eigenspaces and associated (real) eigenvalues. There are k degrees of freedom for the choice of the eigenvalues. Using for instance the Gram–Schmidt algorithm, the choice of the eigenspaces is equivalent to the data of a flag $E_1 \subset \cdots \subset E_k = \mathbb{C}^d$ of subspaces of respective dimensions $i_1 \leq \cdots \leq i_k$ with $i_\ell = d_1 + \cdots + d_\ell$. The space of these flags is known to be a manifold of complex dimension

$$d_1(d - i_1) + d_2(d - i_2) + \cdots + d_{k-1}(d - i_{k-1}) = \frac{1}{2}\Big(d^2 - \sum_\ell d_\ell^2\Big).$$

All in all, the set of matrices satisfying this constraint is a manifold of real dimension

$$d^2 - (d_1^2 - 1) - \cdots - (d_k^2 - 1)$$

(the restriction to a space of dimension d_ℓ could have been any matrix of $\mathcal{H}_{d_\ell}$, but is instead scalar), so it has codimension at least 3 when a given d_ℓ is not one. Considering all partitions of d except the trivial $1 + \cdots + 1$, we see that matrices with multiple eigenvalues belong to a finite collection of manifolds of codimension at least 3.

3.2 Proof of Proposition 2.1

Let us turn to the proof of Proposition 2.1. Let (M, g) be a complete Riemannian manifold of dimension n, and $N \subset M$ a submanifold of codimension at least 2. We suppose N is an embedded manifold without boundary, although it will be clear that the proof may be adapted to the more general case of immersed manifolds with boundary. Given a kinetic Brownian motion $(H, \dot{H})$ on M, we want to show that the event of H ever hitting N after $t = 0$ has probability zero.

We call embedded (closed) disc of dimension k a subset D of M for which there exists an open set $\mathcal{U} \supset D$ and a diffeomorphism $\phi : \mathcal{U} \to B_0(1)$ to the unit ball of $\mathbb{R}^n$ such that $\phi(D)$ is the intersection $(\mathbb{R}^k \times \{0\}^{n-k}) \cap \overline{B}_0(1/2)$. Then, because N is second countable, it can be covered by countably many embedded discs of codimension 2, say $N \subset \bigcup_{i \geq 0} D_i$. It means that

$$\mathbb{P}(H_t \in N \text{ for some } t > 0) \leq \sum_{j,i \geq 0} \mathbb{E}\big[\mathbb{P}(H_t \in D_i \text{ for some } t \in [2^{-j}, 2^j])\big|(H_0, \dot{H}_0)\big].$$

We are left to show that for a given compact interval $[a, b] \subset (0, \infty)$, starting point $(H_0, \dot{H}_0)$ and embedded disc D of codimension 2, we have

$$\mathbb{P}\left(H_t \in D \text{ for some } t \in [a, b]\right) = 0.$$

Fix some $\delta > 0$, and write D_δ for the set of points at distance at most δ from D. Since the position process of the kinetic Brownian motion has velocity one, if we have $H_t \in D$ for a given $t > 0$, then there must exist $t' \in 2\delta\mathbb{N}$ such that $H_{t'} \in D_\delta$. It means that

$$\mathbb{P}\left(H_t \in D \text{ for some } t \in I\right) \leq \sum_{\ell = \lfloor a/2\delta \rfloor}^{\lceil b/2\delta \rceil} \mathbb{P}(H_{2\delta\ell} \in D_\delta). \tag{3.1}$$

We will prove that

$$\limsup_{\delta \to 0} \sup_{a/2 \leq t \leq 2b} \frac{\mathbb{P}(H_t \in D_\delta)}{\delta^2} < \infty,$$

which will show that the sum in (3.1) is bounded by a constant multiple of δ, so the left hand side is zero upon taking the limit $\delta \to 0$.

Let $\phi : \mathcal{U} \to B_0(1)$ be a map compatible with D, in the sense described above. It induces a diffeomorphism $T^1\phi$ from $T^1\mathcal{U}$ to $T^1B_0(1) \simeq B_0(1) \times \mathbb{S}^{n-1}$, sending $(h, \dot{h})$ to $(\phi(h), \mathrm{d}\phi_h(\dot{h})/|\mathrm{d}\phi_h(\dot{h})|)$. Since D is compact, there is some small $\varepsilon > 0$ such that $D_\varepsilon \subset \mathcal{U}$. The fact that $(H, \dot{H})$ is a hypoelliptic diffusion means that the density of $(H_t, \dot{H}_t)$ is smooth for any given $t > 0$, and moreover depends smoothly on t. In particular, there exists a smooth function p depending on $t > 0$ and $(x, \dot{x}) \in T^1B_0(1)$ such that

$$\mathbb{P}\Big(H_t \in (T^1\phi)^{-1}(A)\Big) = \int_A p_t(x, \dot{x})\mathrm{d}x\mathrm{d}\dot{x},$$

where the integral is considered with respect to Lebesgue measure. Since p is smooth and D_ε is compact, there exists a constant $\|p\| > 0$ such that we have $p_t(x, \dot{x}) \leq \|p\|$ for all $t \in [a/2, 2b]$, $x \in \phi(D_\varepsilon)$, $\dot{x} \in \mathbb{S}^{n-1}$. It means that for any such t and $0 < \delta \leq \varepsilon$,

$$\mathbb{P}(H_t \in D_\delta) \leq \|p\| \cdot \mathrm{Vol}(\phi(D_\delta)) \cdot \mathrm{Vol}(\mathbb{S}^{n-1}).$$

Let $\phi^* g$ be the metric on $B_0(1)$ induced by the identification with $\mathcal{U}$, seen as a $n \times n$ matrix. Using smoothness and compactness again, here exists a constant $C > 0$ such that $\phi^* g$ is bounded above by $C\,\mathrm{id}$ in D_ε. In particular, given $0 < \delta \leq \varepsilon$ and a point y in D_δ, there exists (by compactness) a point x in D and a smooth curve γ of length at most δ with endpoints x and y. The points of γ belong to D_ε, so the Euclidean length of $\phi \circ \gamma$ is at most C times the length of γ, which means that $\phi(y)$ is included in $\phi(D) + B_0(C\delta)$. All in all,

$$\phi(D_\delta) \subset \phi(D) + B_0(C\delta) \subset \big(B_0(1) \cap \mathbb{R}^{n-2}\big) \times \big(B_0(C\delta) \cap \mathbb{R}^2\big)$$

for all δ small enough, and the Euclidean volume of $\phi(D_\delta)$ is bounded by δ^2 up to a constant factor, which concludes.

Note that Proposition 2.1 is obviously optimal in terms of dimension. If a given kinetic motion $(H, \dot{H})$ were to avoid codimension 1 manifolds, then it wouldn't be able to reach the boundary of small balls around its initial point, so it would have to be stationary: $H_t = H_0$, $\dot{H}_t = 0$.

3.3 Proof of Lemmas 2.2 and 2.3

Suppose $(H, \dot{H})$ is defined as in (2.1) and (2.2), driven by a standard Brownian motion W with values in $\mathcal{H}_d$. We want to show that the processes U, Λ and A are well-defined as long as H has distinct eigenvalues, that they take values in the spaces $U_d(\mathbb{C})$, $\mathcal{H}_d^\Delta$ and $\mathbb{S}(\mathcal{H}_d)$ respectively, and that (Λ, A) satisfies the equations stated in Lemma 2.3.

In the first steps, we can actually work with a fixed realisation of $(H, \dot{H})$. There is nothing to prove if H_0 has repeated eigenvalues, so we assume it is not the case. Writing

$$\dot{u}_t = \dot{u}(U_t^* H_t U_t, U_t^* \dot{H}_t U_t)$$

and since $\mathrm{d}U_t = U_t \dot{u}_t \mathrm{d}t$ by definition, we can use the usual Picard-Lindelöf theorem to see that U is uniquely well-defined for a maximal time interval $[0, T)$. Note that $\dot{u}(\Lambda, A)$ is skew-Hermitian whenever it is well-defined, even if Λ is not diagonal, which means that U is in $U_d(\mathbb{C})$ until T. This directly implies that A takes values in the sphere $\mathbb{S}(\mathcal{H}_d)$.

If $U_t^* H_t U_t$, up to time T, stays uniformly away from the closed set of matrices with repeated diagonal entries, then $\dot{u}_t$ is bounded for $t \in [0, T)$, which means that U_t converges to a limit as $t \to T$. Since $U_T^* H_T U_T$ has distinct diagonal entries, we can then apply Picard-Lindelöf again at T and get a solution defined on a larger interval $[0, T + \varepsilon)$, which is a contradiction. Therefore, T must be at least as large as the stopping time when at least two diagonal coefficients of $U_t^* H_t U_t$ become equal, or more precisely arbitrarily close. It implies that Λ and A are well-defined up to this time as well. Moreover, if we show that Λ is diagonal up to T, then in fact the diagonal entries of $\Lambda_t = U_t^* H_t U_t$ are precisely the eigenvalues of H_t, and the collapse of the diagonal entries of the former corresponds to that of the eigenvalues of the latter, i.e. $T = \tau$.

Using the representation of $(H, \dot{H})$ involving W, we must have, up to T,

$$\begin{aligned}
\mathrm{d}U_t &= U_t \dot{u}_t \mathrm{d}t, \\
\mathrm{d}\Lambda_t &= \big(\dot{u}_t^* \Lambda_t + \Lambda_t \dot{u}_t\big)\mathrm{d}t + A_t \mathrm{d}t, \\
\mathrm{d}A_t &= \big(\dot{u}_t^* A_t + A_t \dot{u}_t\big)\mathrm{d}t + U_t^* \Big(\mathrm{d}W_t - \dot{H}_t \operatorname{tr}(\dot{H}_t^* \mathrm{d}W_t) - \frac{d^2-1}{2}\dot{H}_t \mathrm{d}t\Big) U_t \\
&= \big(\dot{u}_t^* A_t + A_t \dot{u}_t\big)\mathrm{d}t + U_t^* \mathrm{d}W_t U_t - A_t \operatorname{tr}(A_t^* U_t^* \mathrm{d}W_t U_t) - \frac{d^2-1}{2} A_t \mathrm{d}t.
\end{aligned}$$

Since U is unitary, the integral

$$B : t \mapsto \int_0^t U_s^* \mathrm{d}W_s U_s$$

defines a standard Brownian motion in $\mathcal{H}_d$, and A satisfies the equation described in Lemma 2.3. Moreover, if Λ stays diagonal, then in fact

$$\mathrm{d}\Lambda_t = \pi^\Delta\big(\mathrm{d}\Lambda_t\big) = \pi^\Delta\big(\dot{u}_t \Lambda_t + \Lambda_t \dot{u}_t\big)\mathrm{d}t + \pi^\Delta(A_t)\mathrm{d}t = \pi^\Delta(A_t)\mathrm{d}t$$

according to equation (2.4), so Λ satisfies the equation given in Lemma 2.3. So the last thing we need to prove is that Λ stays diagonal for all $t < T$.

This last fact is essentially a consequence of uniqueness for strong solutions of stochastic differential equations. Indeed, we can define (Λ^B, A^B) as the solution of

$$\begin{aligned}
\mathrm{d}\Lambda_t^B &= \big((\dot{u}_t^B)^* \Lambda_t^B + \Lambda_t^B \dot{u}_t^B\big)\mathrm{d}t + A_t^B \mathrm{d}t, \\
\mathrm{d}A_t^B &= \big((\dot{u}_t^B)^* A_t^B + A_t^B \dot{u}_t^B\big)\mathrm{d}t + \mathrm{d}B_t - A_t^B \operatorname{tr}((A_t^B)^* \mathrm{d}B_t) - \frac{d^2-1}{2} A_t^B \mathrm{d}t
\end{aligned}$$

for $\dot{u}^B = \dot{u}(\Lambda^B, A^B)$ and initial condition $(\Lambda^B, A^B)_0 = (\Lambda, A)_0$, seen as a process with values in the open set of $\mathcal{H}_d^\Delta \times \mathcal{H}_d$ where the first component has distinct eigenvalues. It is defined on a (random) maximal interval $[0, T^B)$. The pairs (Λ, A) and (Λ^B, A^B) are solution to the same stochastic differential equation for all times before T and T^B, so they are equal and Λ is actually diagonal over $[0, T \wedge T^B)$. However, over the event $\{T^B < T\}$, the limit $(\Lambda^B, A^B)_{T^B}$ is well-defined in the large space where (Λ, A) takes values, namely it is $(\Lambda, A)_{T^B}$ where $\Lambda_{T^B} \in \mathcal{H}_d$ with distinct diagonal entries and $A_{T^B} \in \mathcal{H}_d$. Since $\mathcal{H}_d^\Delta$ is closed, then in fact (Λ^B, A^B) admits a limit in the small space as t approaches T^B. But this event has measure zero according to the classical explosion criterion for equations with smooth coefficients, so the event $\{T^B < T\}$ has measure zero and $T \wedge T^B = T$, hence Λ is diagonal for all times $t < T$.

As discussed above, this concludes the proof of Lemmas 2.2 and 2.3.

3.4 Projections of spherical Brownian motions

Let X be a standard Brownian motion on the sphere $\mathbb{S}(\mathbb{R}^n)$, and $X^{[k]}$ its projection $(X^1, \dots, X^k)$. The case we have in mind is $n = d^2$ and $k = d$. One way to define such an X is to fix a standard Brownian motion B with values in $\mathbb{R}^n$ and set X the solution of

$$\mathrm{d}X_t = \circ\, \mathrm{d}B_t - X_t X_t^* \circ \mathrm{d}B_t = \mathrm{d}B_t - X_t X_t^* \mathrm{d}B_t - \frac{n-1}{2} X_t \mathrm{d}t.$$

We want to show that $X^{[k]}$ is Markovian.

One can use abstract invariance arguments to see that this is the case, see for instance the arXiv version of this paper. However, it is not much more difficult to exhibit an explicit stochastic differential equation for the dynamics; namely, let us sketch an argument to show that the decomposition $X = (r\theta, \sqrt{1-r^2}\phi)$, for $r \in \mathbb{R}$, $\theta \in \mathbb{S}(\mathbb{R}^k)$ and $\phi \in \mathbb{S}(\mathbb{R}^{n-k})$, is solution to

$$\begin{aligned} \mathrm{d}(r^2)_t &= 2\sqrt{(1-r_t^2)r_t^2}\,\mathrm{d}B_t^r + (k - nr_t^2)\mathrm{d}t \\ \mathrm{d}\theta_t &= \frac{1}{\sqrt{r_t^2}}(\mathrm{d}B_t^\theta - \theta_t\theta_t^*\mathrm{d}B_t^\theta) - \frac{1}{r_t^2}\frac{k-1}{2}\theta_t\mathrm{d}t \\ \mathrm{d}\phi_t &= \frac{1}{\sqrt{1-r_t^2}}(\mathrm{d}B_t^\phi - \phi_t\phi_t^*\mathrm{d}B_t^\phi) - \frac{1}{1-r_t^2}\frac{n-k-1}{2}\phi_t\mathrm{d}t \end{aligned}$$

for (B^r, B^θ, B^ϕ) a standard Brownian motion on $\mathbb{R}^{n+1}$. Note that θ and ϕ are time changes of spherical Brownian motions. A (different) complete proof, as well as pathwise uniqueness, is described by Mijatović, Mramor and Uribe in [10].

Set $\Pi^{i:j} : \mathbb{R}^n \to \mathbb{R}^{j-i+1}$ the projection on the ith to jth coordinates. A direct application of the Itô formula shows that the processes r^2, θ and ϕ have drift term as described, and respective Brownian increments

$$\mathrm{d}M^r = 2\sqrt{(1-r_t^2)r_t^2}\Big(\sqrt{1-r_t^2}\;\theta_t^*\Pi^{1:k} - r_t\phi_t^*\Pi^{k+1:n}\Big)\mathrm{d}B_t,$$

$$\mathrm{d}M^\theta = \frac{1}{\sqrt{r_t^2}}\big(\Pi^{1:k} - \theta_t\theta_t^*\Pi^{1:k}\big)\mathrm{d}B_t, \quad \mathrm{d}M^\phi = \frac{1}{\sqrt{1-r_t^2}}\big(\Pi^{k+1:n} - \phi_t\phi_t^*\Pi^{k+1:n}\big)\mathrm{d}B_t.$$

Since the brackets of the martingales coincide with those in the equation driven by (B^r, B^θ, B^ϕ), the result follows up to enlarging the space by the martingale representation theorem.

3.5 Proof of Lemma 2.4

We want to show that $(\Lambda, \dot{\Lambda})$ is Markovian if and only if the vector field Φ depends on A only through its diagonal $\pi^\Delta(A)$. The indirect implication is clear: if $\Phi(\Lambda, A)$ rewrites as $\overline{\Phi}(\Lambda, \pi^\Delta(A))$, then

$$\begin{aligned} \mathrm{d}\Lambda_t &= \pi^\Delta(A_t)\mathrm{d}t \\ \mathrm{d}\pi^\Delta(A)_t &= \overline{\Phi}\big(\Lambda_t, \pi^\Delta(A)_t\big)\mathrm{d}t + b^\Delta\big(\pi^\Delta(A_t)\big)\mathrm{d}t + \sigma^\Delta\big(\pi^\Delta(A_t)\big)\mathrm{d}B_t, \end{aligned}$$

where

$$\mathrm{d}X_t^\Delta = b^\Delta(X_t^\Delta)\mathrm{d}t + \sigma^\Delta(X_t^\Delta)\mathrm{d}B_t$$

is the equation describing the projection $X^\Delta = \pi^\Delta(X)$ on $\mathcal{H}_d^\Delta$ of a spherical Brownian motion X in $\mathbb{S}(\mathcal{H}_d)$, as discussed in the previous section. Then $(\Lambda, \dot{\Lambda}) = (\Lambda, \pi^\Delta(A))$ is the solution of a self-contained SDE, so it is Markovian.

Conversely, suppose that $(\Lambda, \dot{\Lambda})$ is Markovian. Let L^Δ be its generator, L that of (Λ, A). For $f : \mathcal{H}_d^\Delta \times \mathcal{H}_d^\Delta \to \mathbb{R}$ regular enough, we should have

$$L\big(f \circ (\mathrm{id}, \pi^\Delta)\big)(\Lambda_0, A_0) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathbb{E}_{\Lambda_0, A_0}[f(\Lambda_t, \pi(A_t))] = (L^\Delta f)\big(\Lambda_0, \pi^\Delta(A_0)\big).$$

For instance, one can see that this holds for f smooth with compact support.

Let $X^\Delta = \pi^\Delta(X)$, as above, be the projection of a spherical Brownian motion on $\mathbb{S}(\mathcal{H}_d)$, and set I its integral (i.e. $\mathrm{d}I_t = X_t^\Delta \mathrm{d}t$). According to the previous section, (I, X^Δ) is Markovian. In particular, (I, X) and (I, X^Δ) both admit generators $\widetilde{L}$ and $\widetilde{L}^\Delta$, and they are linked by the same relation

$$\widetilde{L}\big(f \circ (\mathrm{id}, \pi^\Delta)\big)(I_0, X_0) = (\widetilde{L}^\Delta f)\big(I_0, \pi^\Delta(X_0)\big),$$

for instance when $f:\mathcal{H}_d^\Delta\times\mathcal{H}_d^\Delta\to\mathbb{R}$ is smooth with compact support.

As mentioned above, the only difference between L and $\widetilde{L}$ is the additional vector field Φ acting on the second component:

$$(L-\widetilde{L})g(\Lambda_0,A_0)=D_A\,g(\Lambda_0,A_0)(\Phi(\Lambda_0,A_0))=:(\Phi\cdot\nabla_A\,g)(\Lambda_0,A_0),$$

where $D_A\,g$ is the differential of $g:\mathcal{H}_d^\Delta\times\mathcal{H}_d\to\mathbb{R}$ with respect to its second variable. In particular,

$$\big(\Phi\cdot\nabla_A(f\circ(\mathrm{id},\pi^\Delta))\big)(\Lambda_0,A_0)=(L^\Delta f-\widetilde{L}^\Delta f)\big(\Lambda_0,\pi^\Delta(A_0)\big).$$

The right hand side depends on A_0 only through $\pi^\Delta(A_0)$, so the left hand side must be a function of $(\Lambda,\pi^\Delta(A))$. Since we can deduce a given vector field Ψ with values in $\mathcal{H}_d^\Delta$ by the action of the operator $\Psi\cdot\nabla_A$ on functions of the form $f\circ(\mathrm{id},\pi^\Delta)$ (choose for instance a collection of f smooth with compact support such that $f(\Lambda,\dot\Lambda)=\dot\Lambda_{ii}$ on a small open set), Φ actually factors through $(\Lambda,A)\mapsto(\Lambda,\pi^\Delta(A))$ as expected.

3.6 The case $d=2$

We have seen at the end of Section 2.4 that in dimension $d=2$, the process $(\Lambda,\dot\Lambda)$ is Markovian, using the fact that

$$\Phi(\Lambda,A)_{11}=-\Phi(\Lambda,A)_{22}=\frac{|A_{12}|^2+|A_{21}|^2}{\Lambda_{11}-\Lambda_{22}}=\frac{1-|A_{11}|^2-|A_{22}|^2}{\Lambda_{11}-\Lambda_{22}}.$$

In fact, we can use this expression and the equation satisfied by (Λ,A), given in Lemma 2.3, to get the equation for the evolution. Write λ and μ for the eigenvalues Λ_{11} and Λ_{22} of H. For B a standard Brownian motion on $\mathcal{H}_2$, define the martingales

$$M^\lambda:t\mapsto(B_{11})_t-\int_0^t\dot\lambda_s\,\mathrm{tr}(A_s^*\mathrm{d}B_s)\quad\text{and}\quad M^\mu:t\mapsto(B_{22})_t-\int_0^t\dot\mu_s\,\mathrm{tr}(A_s^*\mathrm{d}B_s).$$

Then

$$\begin{aligned}
\mathrm{d}\lambda_t&=\dot\lambda_t\mathrm{d}t, &\mathrm{d}\dot\lambda_t&=+\frac{1-\dot\lambda_t^2-\dot\mu_t^2}{\lambda_t-\mu_t}\mathrm{d}t+\mathrm{d}M_t^\lambda-\frac{d^2-1}{2}\dot\lambda_t\mathrm{d}t,\\
\mathrm{d}\mu_t&=\dot\mu_t\mathrm{d}t, &\mathrm{d}\dot\mu_t&=-\frac{1-\dot\lambda_t^2-\dot\mu_t^2}{\lambda_t-\mu_t}\mathrm{d}t+\mathrm{d}M_t^\mu-\frac{d^2-1}{2}\dot\mu_t\mathrm{d}t.
\end{aligned}$$

Writing $A_{12}^{\Re}$ and $A_{12}^{\Im}$ for the real and imaginary parts of A_{12}, and similarly for B_{12},

$$\begin{aligned}
\mathrm{d}M_t^\lambda&=(1-\dot\lambda_t^2)\mathrm{d}(B_{11})_t-\dot\lambda_t(\overline{A}_{12})_t\mathrm{d}(B_{12})_t-\dot\lambda_t(\overline{A}_{21})_t\mathrm{d}(B_{21})_t-\dot\lambda_t\dot\mu_t\mathrm{d}(B_{22})_t\\
&=(1-\dot\lambda_t^2)\mathrm{d}(B_{11})_t-2\dot\lambda_t(A_{12}^{\Re})_t\mathrm{d}(B_{12}^{\Re})_t-2\dot\lambda_t(A_{12}^{\Im})_t\mathrm{d}(B_{12}^{\Im})_t-\dot\lambda_t\dot\mu_t\mathrm{d}(B_{22})_t,\\
\mathrm{d}M_t^\mu&=-\dot\lambda_t\dot\mu_t\mathrm{d}(B_{11})_t-2\dot\mu_t(A_{12}^{\Re})_t\mathrm{d}(B_{12}^{\Re})_t-2\dot\mu_t(A_{12}^{\Im})_t\mathrm{d}(B_{12}^{\Im})_t+(1-\dot\mu_t^2)\mathrm{d}(B_{22})_t.
\end{aligned}$$

Since $\sum_{ij}|A_{ij}|^2=1$, we deduce $2|A_{12}^{\Re}|^2+2|A_{12}^{\Im}|^2=1-\dot\lambda^2-\dot\mu^2$, and we find the bracket of M^λ:

$$\mathrm{d}\langle M^\lambda,M^\lambda\rangle_t=(1-\dot\lambda_t^2)^2\mathrm{d}t+\dot\lambda_t^2(1-\dot\lambda_t^2-\dot\mu_t^2)\mathrm{d}t+\dot\lambda_t^2\dot\mu_t^2\mathrm{d}t=(1-\dot\lambda_t^2)\mathrm{d}t.$$

Similarly the bracket of M^μ grows as $(1-\dot\mu_t^2)\mathrm{d}t$. The covariance term is given by

$$\mathrm{d}\langle M^\lambda,M^\mu\rangle_t=-\dot\lambda_t\dot\mu_t(1-\dot\lambda_t^2)\mathrm{d}t+\dot\lambda_t\dot\mu_t(1-\dot\lambda_t^2-\dot\mu_t^2)\mathrm{d}t-\dot\lambda_t\dot\mu_t(1-\dot\mu_t^2)\mathrm{d}t=-\dot\lambda_t\dot\mu_t\mathrm{d}t,$$

as stated in Lemma 2.5.

Note that it corresponds to the diffusion term for the projection of a spherical Brownian motion, as described in [10]. Indeed, as explained in the proof outline, the only difference between A and a spherical Brownian motion is a drift term. Alternatively, one can also study the trace and determinant of H, and deduce the process satisfied by the eigenvalues, since they are the roots of the polynomial $X^2-\mathrm{tr}(H)X+\det(H)$.

3.7 The case $d \geq 3$

We show that in this case, the vector field $\Phi(\Lambda, A)$ depends on the off-diagonal elements of A. As stated in Lemma 2.4, this will show that $(\Lambda, \dot{\Lambda})$ cannot be Markovian. We will use the expression given in equation (2.8).

In dimension 3, it is a direct computation to see that for any Λ with distinct eigenvalues, the following two choices for A give different $\Phi(\Lambda, A)_{11}$, although they are equal on the diagonal:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In fact, one gets $\Phi(\Lambda, A)_{11} = 2/(\Lambda_{11} - \Lambda_{22})$, whereas $\Phi(\Lambda, \widetilde{A})_{11} = 2/(\Lambda_{11} - \Lambda_{33})$. In higher dimension, chose A and $\widetilde{A}$ to be zero except on the top left 3×3 minor, which is given by the above expressions. Similar constructions show that this can also be achieved for $\widetilde{A}$ arbitrarily close to any given (non-diagonal) A, so the issue is not local.

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Acknowledgments. The author would like to thank Thierry Lévy, who raised the question of existence and behaviour of kinetic Dyson Brownian motion during the former's PhD defence.