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A note on once reinforced random walk on ladder $\mathbb{Z} \times \{0, 1\}$*

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Abstract

Given any $\delta \in (0, \infty)$, let $(X_n)_{n=0}^{\infty}$ be the δ-once reinforced random walk on ladder $\mathbb{Z} \times \{0, 1\}$ with the following edge weight function at the $(n+1)$-th step:

$$w_n(e) = 1 + (\delta - 1) \cdot I_{\{N(e,n)>0\}} = \begin{cases} 1 & \text{if } N(e, n) = 0, \\ \delta & \text{if } N(e, n) > 0. \end{cases}$$

Here $N(e, n) := \#\{i < n : X_i X_{i+1} = e\}$ is the number of times that edge e has been traversed by the walk before time n. It was proved that $(X_n)_{n=0}^{\infty}$ is almost surely recurrent for $\delta > 1/2$ (Vervoort (2002) [8] and Sellke (2006) [7]), while the a.s. recurrence for negative reinforcement factor $\delta \in (0, 1/2]$ remained open. In this note, we give an affirmative answer to this question.

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1 Introduction and main result

Let $\mathbb{Z}$ (resp. $\mathbb{N}$) be the set of all integers (resp. natural numbers). For any $n \in \mathbb{N}$, let $\mathbb{Z} \times \{0, \dots, n\}$ be a ladder graph with $n+1$ levels, where two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. In this note, we devote to investigate recurrence of once negatively reinforced random walk on ladder $\mathbb{Z} \times \{0, 1\}$.

To begin, let $G = (V, E)$ be a connected locally finite graph with vertex set V and edge set E. When two vertices u and v are adjacent, denoted by $u \sim v$, we denote by uv the undirected edge connecting u and v, $\overrightarrow{uv}$ the directed edge from u to v. The edge reinforced random walk (ERRW) on G is a stochastic process $\mathbf{X} = (X_n)_{n=0}^{\infty}$ in V with the following transition probability:

$$\mathbb{P}(X_{n+1} = u | \mathscr{F}_n) = \begin{cases} \frac{w_n(uv)}{\sum_{u' \sim v} w_n(u'v)}, & \text{on } \{X_n = v\}, u \sim v, \\ 0, & \text{otherwise}, \end{cases}$$

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where $(\mathscr{F}_n)_{n\geq 0}$ is the natural filtration generated by the history of $\mathbf{X}$, i.e. $\mathscr{F}_n = \sigma(X_0, \dots, X_n)$ for any integer $n \geq 0$, and $w_n(uv)$ is an $\mathscr{F}_n$-measurable random weight of edge uv at the $(n+1)$-th step.

The original model of reinforced random walk in literature was firstly introduced by Coppersmith and Diaconis in 1987 [1]. They considered the following weight function:

$$w_n(e) = 1 + N(e,n) \cdot \delta, \tag{1.1}$$

where $\delta > 0$ is the reinforcement factor and $N(e,n) = \#\{i < n : X_i X_{i+1} = e\}$ is the number of times that edge e has been traversed before time n.

Davis (1990) [3] introduced the δ-once reinforced random walk (ORRW), which is an interesting variant of ERRW with weight

$$w_n(e) = 1 + (\delta - 1) \cdot I_{\{N(e,n)>0\}} = \begin{cases} 1 & \text{if } N(e,n) = 0, \\ \delta & \text{if } N(e,n) > 0. \end{cases}$$

It takes value δ if and only if the edge has been crossed, and 1 otherwise. We call ORRW positively reinforced if $\delta > 1$ and negatively reinforced if $\delta \in (0,1)$. While it seems to be a simpler model than the other ERRWs since the weight function is simple, ORRW has less results and no general methods are developed for its study. We consider transience/recurrence of this stochastic process. A sample realization of a random walk is said to be *transient* (resp. *recurrent*) if every vertex is visited only finitely many times (resp. infinitely often) (see [8, Definition 2]). ORRW is recurrent a.s. on $\mathbb{Z}^1$ for any $\delta > 0$. However, no transience/recurrence result of ORRW is currently known on $\mathbb{Z}^d$ with $d \geq 2$. Kious and Sidoravicius (2018) [6] showed a transience/recurrence phase transition for ORRW on $\mathbb{Z}^d$-like trees. To our knowledge, it is the first example of phase transition for ORRW. Then Collevecchio, Kious and Sidoravicius (2020) [2] proved a very elegant result: ORRW on general trees $\mathcal{T}$ has a transience/recurrence phase transition. The critical point is exactly the following branching-ruin number:

$$\mathrm{br}_r(\mathcal{T}) = \sup\left\{\lambda > 0 : \inf_{\pi \in \Pi} \sum_{e \in \pi} |e|^{-\lambda} > 0\right\},$$

where Π is the set of cutsets separating the root from infinity, and $|e|$ is the distance between e and the root. It completes the whole work for transience/recurrence of the ORRW on trees.

When considering ORRW on $\mathbb{Z}^d$, Sidoravicius conjectured that it is recurrent with $d = 2$ and undergoes a phase transition for any $d \geq 3$. However, it is still an open problem and there is no result on transience/recurrence for any $\delta > 0$. Since $\mathbb{Z}^2$ is the asymptotic graph of $\mathbb{Z} \times \{-n, \dots, -1, 0, 1, \dots, n\}$ as $n \to \infty$, it leads to the study of ORRW on general ladders $\mathbb{Z} \times \Gamma$ with Γ being a finite connected graph. To the best of our knowledge, the complete depiction on transience/recurrence under this setting remains an open problem, even for the simplest case $\mathbb{Z} \times \{0,1\}$.

In 1994, Sellke [7] showed that the ORRW on $\mathbb{Z} \times \{0, 1, \dots, n\}$ is recurrent almost surely for any $\delta \in (1, n/(n-1))$; and in particular it is recurrent almost surely for any $\delta \in (1, \infty)$ when $n = 1$. Afterwards Vervoort (2002) [8] verified the recurrence of ORRW for $\delta \in (n/(n+1), 1)$, and then claimed that there exist $\delta_1, \delta_2 > 0$ with $\delta_1 < 1 - 1/(n+1) < 1 + 1/(n-1) < \delta_2$ such that the ORRW is a.s. recurrent for any $\delta \in (\delta_1, \delta_2)$. These results offered partial characterizations on the phase space when reinforcement parameters are small. Intuitively, when the reinforcement factor δ is large enough, the walk will have a strong tendency to cross the edge traversed before. That is to say, the ORRW prefers to stick around the origin, which naturally deduces recurrence. Kious, Schapira and Singh

(2018) [5] proved that there exists a constant $C > 0$ such that for any finite connected graph Γ, the ORRW on $\mathbb{Z} \times \Gamma$ is recurrent when $\delta \geq 1 + C|\Gamma|^{40}$.

In this note, we prove the following result which completely confirms the recurrence on the simplest ladder $\mathbb{Z} \times \{0,1\}$.

Theorem 1.1. *Let $\mathbf{X} = (X_n)_{n=0}^{\infty}$ be an* ORRW *on ladder $\mathbb{Z} \times \{0,1\}$ with initial site $(0,0)$ and reinforcement factor $\delta \in (0,1)$. Then $\mathbf{X}$ is recurrent almost surely.*

This note is organized as follows: In Section 2, inspired by Vervoort [8, Lemma 11], we prove a criterion for recurrence of ORRWs on infinite connected locally finite graphs, see Theorem 2.1. In Section 3, based on Theorem 2.1, we prove Theorem 1.1 by a novel estimate of the ORRWs on $\mathbb{Z} \times \{0,1\}$ specified in Lemma 3.2. Section 4 is a short conclusion.

2 Criterion for recurrence of ORRWs

We start with some further notations. For h a function defined on V, and for any directed edge $\overrightarrow{vu}$ of G, define $\Delta_h(\overrightarrow{vu}) = h(u) - h(v)$. We say $h : V \to \mathbb{R}$ is a superharmonic (resp. harmonic) function if

$$\sum_{u \sim v} \Delta_h(\overrightarrow{vu}) \leq 0 \text{ (resp. } = 0), \ \forall v \in V.$$

Then for $t \in \mathbb{N}$, define random sets E_t and A_t (Vervoort [8, Definition 10]),

$$\begin{aligned} E_t &= \{X_s X_{s+1} : \ s < t\}, \\ A_t &= \left\{\overrightarrow{vu} : \ vu \in E_t, \overrightarrow{vu} = \overrightarrow{X_s X_{s+1}} \text{ for } s = \min\{s' < t : X_{s'} X_{s'+1} = vu\}\right\}. \end{aligned}$$

That is to say, E_t is an edge set containing the edges that have been traversed up to time t and A_t is an arc set obtained from E_t by orienting each edge according to the direction in which it was firstly traversed.

Theorem 2.1. *Let $G = (V,E)$ be an infinite connected locally finite graph and $h : \ V \to \mathbb{R}$ be a function satisfying that*

- *h is superharmonic everywhere except on a finite subset $F \subset V$,*
- *$h(v) \to +\infty$ as v goes to infinity.*

*Consider δ-*ORRW *$\mathbf{X}$ on G starting at a vertex v_0, and denote by $\eta_r := \inf\{t : \ h(X_t) \geq r\}$ and $\tau_{r'} := \inf\{t \geq \eta_r : h(X_t) \geq r' \text{ or } X_t \in F\}$. If for some $\varepsilon > 0$, any $r > h(v_0)$ and any $r_0 \in \mathbb{R}$, there exists a $r' > r_0$ (i.e., there exists a sequence of real numbers $r' \uparrow \infty$) such that*

$$(\delta - 1)\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r} \right) \geq -(1-\varepsilon) r', \ a.s. \tag{2.1}$$

then $\mathbf{X}$ is recurrent almost surely.

Remark 2.2. Theorem 2.1 inherits the spirit of [8, Lemma 11]. To the best of our knowledge, Vervoort's proof of [8, Lemma 11] is not completely precise and not easily to be corrected since he used the deterministic time t_0. Therefore, we cannot use this result directly and have to show Theorem 2.1 in details. In this theorem, we replace "fixed time t_0" and "$\sum_{\overrightarrow{vu} \in A_{\tau_r}}$" in (57) of [8, Lemma 11] by stopping time η_r and $\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}}$ respectively, and remove "$-c$" on the right hand side (RHS) of (57) of [8, Lemma 11].

Before proving this theorem, we show some lemmas.

Lemma 2.3. *For δ-ORRW $\mathbf{X}$ on finite connected graph $G_0 = (V_0, E_0)$, denote vertex cover time by*

$$\rho_{V_0} := \inf\{t: \ \forall v \in V_0, \ \exists s \le t, \ X_s = v\}.$$

Then $\rho_{V_0} < \infty$ a.s.

We omit the proof of Lemma 2.3 since it is straightforward. Inspired by [8, Lemma 6-7], we obtain the following lemma.

Lemma 2.4. *Given an infinite connected locally finite graph $G = (V, E)$. Let $\mathbf{X}$ be the δ-ORRW on G starting from $v_0 \in V$. For any finite connected subgraph with vertex set $F \subset V$, the following two statements are equivalent.*

(a) *F is visited infinitely often by $\mathbf{X}$ almost surely.*

(b) *$\mathbf{X}$ is recurrent almost surely.*

Proof. Since **(b)**$\Longrightarrow$**(a)** is straightforward, we thus concentrate on the converse direction.

Assume **(a)** holds. Due to F being finite, there is at least one random vertex $v \in F$ which is visited infinitely often by $\mathbf{X}$ almost surely. Hence, to prove **(b)**, it suffices to prove that

$$\mathbb{P}(v \text{ is visited infinitely often and } u \text{ only finitely often by } \mathbf{X}) = 0, \ v \in F, \ u \in V. \tag{2.2}$$

Moreover, once showing that (2.2) holds for all $u \sim v$, we may further verify the equation through (2.2) for any $v \in F$ and $u \in V$ by the connectivity of G and induction on the graph distance $d_G(v, u)$. Thus we are to concentrate on the case of adjacent vertices.

Let $\mathcal{A}^k_{t_0,t}(u, v)$ be the event that u is not visited by $\mathbf{X}$ in time interval $[t_0, t]$ and that v is visited by $\mathbf{X}$ for k times in $[t_0, t-1]$. For any visit time $s \in [t_0, t-1]$ of $\mathbf{X}$ to v,

$$\sum_{v' \sim v} w_s(v'v) \le (\#\{v' : v' \sim v\}) \cdot (\delta \vee 1) \text{ and } w_s(uv) \ge \delta \wedge 1.$$

Thus the probability of $\mathbf{X}$ not immediately traversing uv just after time s is at most $1 - c$, where

$$c = \frac{\delta \wedge 1}{(\#\{v' : v' \sim v\}) \cdot (\delta \vee 1)} \in (0, 1].$$

Therefore, by induction on k,

$$\mathbb{P}\left(\mathcal{A}^k_{t_0,t}(u, v)\right) \le (1 - c)^k.$$

If we first let $t \uparrow \infty$ and then $k \uparrow \infty$, we see that

$$\mathbb{P}(v \text{ is visited infinitely often and } u \text{ never by } \mathbf{X} \text{ after time } t_0) \le \lim_{k\uparrow\infty} (1 - c)^k = 0.$$

Summing over all $t_0 \ge 0$, we get the desired result. □

Note that

$$N(e, n) = \#\{i: \ X_i X_{i+1} = e, \ 0 \le i \le n-1\}$$

is the number of times that edge e has been traversed before time n and $w_n(e) = 1 + (\delta - 1) \cdot I_{\{N(e,n)>0\}}$. For any $V' \subset V$, define

$$M_t = \sum_{s=0}^{t-1} \begin{cases} \dfrac{\Delta_h\left(\overrightarrow{X_s X_{s+1}}\right)}{w_s(X_s X_{s+1})}, & X_s \in V', \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

We recall the following two lemmas from Vervoort [8], which are not hard to verify.

Lemma 2.5 (Vervoort [8] Lemma 9)**.** *For any infinite connected locally finite graph G and δ-ORRW $\mathbf{X}$ on G, if $h : V \to \mathbb{R}$ is superharmonic function, then $(M_t)_{t=0}^{\infty}$ is a supermartingale.*

Proof. If $X_t \in V'$ then $M_{t+1} = M_t$, otherwise

$$\begin{aligned} M_t &\geq M_t + \frac{1}{\sum_{u \sim X_t} w_t(X_t u)} \sum_{u \sim X_t} \Delta_h(\overrightarrow{X_t u}) \\ &= M_t + \sum_{v \sim X_t} \frac{w_t(X_t v)}{\sum_{u \sim X_t} w_t(X_t u)} \frac{\Delta_h(\overrightarrow{X_t v})}{w_t(X_t v)} \\ &= M_t + \sum_{v \sim X_t} \mathbb{P}(X_{t+1} = v | \mathscr{F}_t) \frac{\Delta_h(\overrightarrow{X_t v})}{w_t(X_t v)} \\ &= \mathbb{E}(M_{t+1} | \mathscr{F}_t). \end{aligned}$$

□

Lemma 2.6 (Vervoort [8] Lemma 10)**.** *Given an infinite connected locally finite graph G and the δ-ORRW $\mathbf{X}$ on G. Let M_t be given by (2.3) for some $h : V \to \mathbb{R}$. Then for any $t \geq s$,*

$$\delta(M_t - M_s) = h(X_t) - h(X_s) + (\delta - 1) \sum_{\overrightarrow{vu} \in A_t \setminus A_s} \Delta_h(\overrightarrow{vu}) \tag{2.4}$$

on the event that the set $V \setminus V'$ has not been visited in the interval $[s,t)$.

Proof. Since $V \setminus V'$ has not been visited at any time t' with $s \leq t' < t$, $M_t - M_s = \sum_{t'=s}^{t-1} \frac{\Delta_h(\overrightarrow{X_{t'} X_{t'+1}})}{w_{t'}(X_{t'} X_{t'+1})}$. Note that for $t' \in [s,t)$, $w_{t'}(X_{t'} X_{t'+1}) = 1$ if and only if $\overrightarrow{X_{t'} X_{t'+1}} \in A_t \setminus A_s$. Moreover, for every $\overrightarrow{vu} \in A_t \setminus A_s$ there exists a unique $t' \in [s,t)$ such that $\overrightarrow{X_{t'} X_{t'+1}} = \overrightarrow{vu}$, which establishes a map $\overrightarrow{vu} \mapsto t'$. In addition, it is a bijection from $A_t \setminus A_s$ to $\{t' \in [s,t) : \overrightarrow{X_{t'} X_{t'+1}} \in A_t \setminus A_s\}$. Hence

$$\begin{aligned} M_t - M_s &= \sum_{\substack{s \leq t' < t, \\ \overrightarrow{X_{t'} X_{t'+1}} \notin A_t \setminus A_s}} \frac{\Delta_h(\overrightarrow{X_{t'} X_{t'+1}})}{\delta} + \sum_{\substack{s \leq t' < t, \\ \overrightarrow{X_{t'} X_{t'+1}} \in A_t \setminus A_s}} \Delta_h(\overrightarrow{X_{t'} X_{t'+1}}) \\ &= \sum_{t'=s}^{t-1} \frac{\Delta_h(\overrightarrow{X_{t'} X_{t'+1}})}{\delta} + \left(1 - \frac{1}{\delta}\right) \sum_{\substack{s \leq t' < t, \\ \overrightarrow{X_{t'} X_{t'+1}} \in A_t \setminus A_s}} \Delta_h(\overrightarrow{X_{t'} X_{t'+1}}) \\ &= \sum_{t'=s}^{t-1} \frac{\Delta_h(\overrightarrow{X_{t'} X_{t'+1}})}{\delta} + \left(1 - \frac{1}{\delta}\right) \sum_{\overrightarrow{vu} \in A_t \setminus A_s} \Delta_h(\overrightarrow{vu}) \\ &= \frac{h(X_t) - h(X_s)}{\delta} + \left(1 - \frac{1}{\delta}\right) \sum_{\overrightarrow{vu} \in A_t \setminus A_s} \Delta_h(\overrightarrow{vu}), \end{aligned}$$

which implies the result. □

Proof of Theorem 2.1. By Lemma 2.4, we only need to show F is visited infinitely often by $\mathbf{X}$ almost surely. To this end, it is enough to show that there is constant $c > 0$ such that for all $r > 0$, there exists a $\hat{r} > r$ with

$$\mathbb{P}\big(X_{\tau_{\hat{r}}} \in F \big| \mathscr{F}_{\eta_r}\big) \geq c \ \ a.s. \tag{2.5}$$

In fact, choosing a sequence of $r_n \uparrow \infty$ such that $\mathbb{P}\big(X_{\tau_{r_{n+1}}} \in F \big| \mathscr{F}_{\eta_{r_n}}\big) \geq c \ \ a.s.$ successively, we can obtain the theorem by the conditional Borel-Cantelli lemma ([4] Theorem 5.3.2) since $\eta_{r_n} \to \infty$ as $n \to \infty$.

Now, we determine c and $\hat{r}$ satisfying (2.5) by (2.1). Without loss of generality, assume $h \geq 0$. Noting that $G = (V, E)$ is locally finite and $h(v) \to +\infty$ as v goes to infinity, by condition of Theorem 2.1, for any $r > h(v_0)$, we can find a deterministic $r' > \frac{2h(X_{\eta_r})}{\varepsilon} \vee (r + 1)$ such that (2.1) holds.

Set M_t to be the supermartingale in (2.3) with $V' := V \setminus F$. Due to Lemma 2.3, it is clear that $\eta_r, \tau_{r'} < \infty$ almost surely. Since $A_{\eta_r} \subset \{\overrightarrow{vu} : \ h(v) < r\}$ and

$$A_t \setminus A_{\eta_r} \subset \{\overrightarrow{vu} : \ v \notin F, \ h(v) < r'\} \text{ for any } \eta_r \leq t \leq \tau_{r'},$$

Lemma 2.6 implies that M_t is bounded for any $\eta_r \leq t \leq \tau_{r'}$. Thus by Lemma 2.5 and the optional stopping time theorem we know that $\mathbb{E}\left(\delta M_{\tau_{r'}} \middle| \mathscr{F}_{\eta_r}\right) \leq \delta M_{\eta_r}$ a.s.. Again, by Lemma 2.6, we obtain

$$\mathbb{E}\left(h(X_{\tau_{r'}}) \middle| \mathscr{F}_{\eta_r}\right) \leq h(X_{\eta_r}) + (\delta - 1) \sum_{\overrightarrow{vu} \in A_{\eta_r}} \Delta_h(\overrightarrow{vu}) - (\delta - 1)\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}}} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r}\right).$$

Noting that $\mathbb{E}\left(h(X_{\tau_{r'}}) \middle| \mathscr{F}_{\eta_r}\right) \geq \left[1 - \mathbb{P}\left(X_{\tau_{r'}} \in F \middle| \mathscr{F}_{\eta_r}\right)\right] r'$, thus we have

$$\begin{aligned} \mathbb{P}\left(X_{\tau_{r'}} \in F \middle| \mathscr{F}_{\eta_r}\right) &\geq 1 - \frac{h(X_{\eta_r})}{r'} + \frac{\delta - 1}{r'}\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r}\right) \\ &\geq 1 - \frac{\varepsilon}{2} - (1 - \varepsilon) = \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, we verify (2.5) by taking $c = \frac{\varepsilon}{2}$ and $\hat{r} = r'$ as above, and then finish the proof. □

3 Proof of Theorem 1.1

Now we apply Theorem 2.1 to prove Theorem 1.1. With each vertex v of $\mathbb{Z} \times \{0, 1\}$, we can associate it with coordinates $\mathbf{x}(v) \in \mathbb{Z}, \mathbf{y}(v) \in \{0, 1\}$, in the canonical fashion. Set

$$h(v) = |\mathbf{x}(v)|, \ F = \{v : \ \mathbf{x}(v) = 0\}, \ C_a = \{(a, i)(a + 1, i) : \ i = 0, 1\}, \ a \in \mathbb{Z}, \tag{3.1}$$

where *column* C_a denotes the collection of all horizontal edges connecting all vertices in $\{v : \mathbf{x}(v) = a\}$ with those in $\{v : \mathbf{x}(v) = a + 1\}$. One may see $\Delta_h(\overrightarrow{vu}) = 0$ when edge vu is not horizontal, which indicates that we only need to concentrate on edges in column C_a. Relying on Theorem 2.1, we choose r' large enough and separate the left hand side (LHS) of (2.1) into three parts: $\sum_{a=0}^{r-1}$, $\sum_{a=\tilde{r}}^{r'-1}$ and $\sum_{a=r}^{\tilde{r}-1}$, where $r < \tilde{r} < r'$ and the exact values of $\tilde{r}$ and r' will be specified later. The estimates of the first two parts are similar to those in [8]. We will use a novel technique to show the third part larger than $-(1 - \varepsilon')r'$ for some $\varepsilon' > 0$ based on an observation to be detailed in Lemma 3.4, where the estimate of $\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}, vu \in C_a} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r}\right)$ is transferred to the estimate of $\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_\infty} \setminus A_{\eta_r}, vu \in C_a} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r}\right)$ when r' is large enough. The latter is easier to estimate than the former through the iteration technique since the path before τ_∞ has translation invariance property (for details, see Lemma 3.2). Thus, we can determine ε', $\tilde{r}$ and r' by the specific estimates in the lemmas below.

Note $\eta_n = \inf\{t \geq 0 : \ h(X_t) = n\}$ for any $n \in \mathbb{N}$, since $h(v) = |\mathbf{x}(v)|$ on $\mathbb{Z} \times \{0, 1\}$. Set $\kappa_{n-1} = \inf\{t \geq \eta_n : \ h(X_t) = n - 1\}, \ n \in \mathbb{N}$ and $\Omega_n^i = \{\mathbf{y}(X_{\kappa_{n-1}}) = i, \ \kappa_{n-1} < \infty\}$.

Definition 3.1. *Let* $\mathbf{X}$ *be an* ORRW *on ladder* $\mathbb{Z} \times \{0, 1\}$ *with reinforcement factor* $\delta \in (0, 1)$*. Then for any* $n, n_0 \in \mathbb{Z}$ *and* $v_0 \in V$ *satisfying* $n > n_0 \geq 0$ *and* $h(v_0) = n_0$*, if* $X_0 = v_0$*, we define*

$$p^1 = \mathbb{P}\left(\Omega_n^{1-\mathbf{y}(X_{\eta_n})} \middle| \mathscr{F}_{\eta_n}\right), \; p^0 = \mathbb{P}\left(\Omega_n^{\mathbf{y}(X_{\eta_n})} \middle| \mathscr{F}_{\eta_n}\right).$$

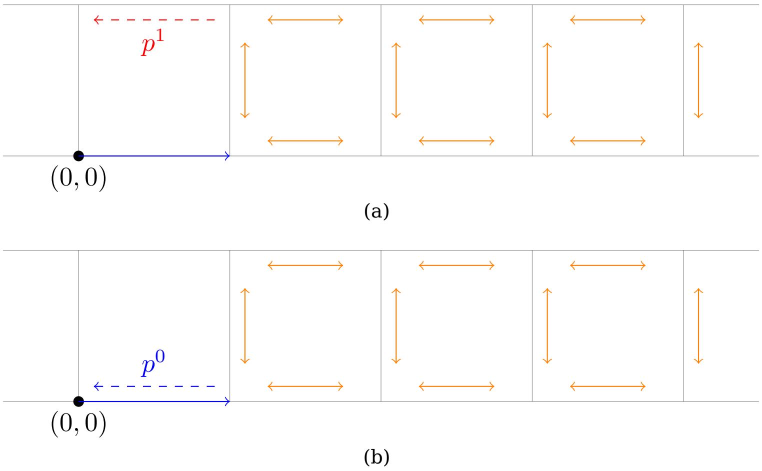

Figure 1: (a) represents the trajectory of $\mathbf{X}$ in Ω_1^1, the first step of which is from $(0, 0)$ to $(1, 0)$ and is indicated by the black real arrow. The red dashed arrow stands for the last step before the hitting time of the board $\{v : h(v) = 0\}$. (b) represents the trajectory of $\mathbf{X}$ in Ω_1^0, the first step of which is from $(0, 0)$ to $(1, 0)$ and is indicated by the black real arrow. The black dashed arrow is the last step before the hitting time of the board $\{v : h(v) = 0\}$.

The conditional probabilities above are well defined since the path from time η_n to κ_{n-1} cannot be influenced by the path before time $\eta_n - 1$. The performance of the path from time η_n to κ_{n-1} can be seen as an ORRW on half ladder $\mathbb{Z}^+ \times \{0, 1\}$ with the first step from $(0, 0)$ to $(0, 1)$. What we consider, actually, is the probability the process $\mathbf{X}$ going back to the board $\{v : h(v) = 0\}$ from below or above. (See Fig. 1)

Specifically, with $y = 0$ or 1,

$$\begin{aligned} \mathbb{P}\left(\Omega_n^{1-y} \middle| \mathscr{F}_m\right) = p^1 \quad & a.s. \text{ on } \{\mathbf{y}(X_m) = y, \eta_n = m\}, \\ \mathbb{P}\left(\Omega_n^{y} \middle| \mathscr{F}_m\right) = p^0 \quad & a.s. \text{ on } \{\mathbf{y}(X_m) = y, \eta_n = m\}. \end{aligned} \tag{3.2}$$

Here we propose our key lemma whose proof is based on an iterative scheme.

Lemma 3.2. *Let* $\mathbf{X}$ *be an* ORRW *on ladder* $\mathbb{Z} \times \{0, 1\}$ *with reinforcement factor* $\delta \in (0, 1)$ *and* p^1, p^0 *be the probabilities given in Definition 3.1. Then*

$$p^0 - p^1 < \frac{\delta}{1 - \delta}. \tag{3.3}$$

Proof. Take $n = 1, v_0 = (0, 0)$ in Definition 3.1. In fact, $p^1 = \mathbb{P}(\Omega_1^1 | X_1 = (1, 0))$ and $p^0 = \mathbb{P}(\Omega_1^0 | X_1 = (1, 0))$. Then by the total probability formula, we know that

$$\begin{aligned} p^1 \;=\; & \mathbb{P}\left(X_2 = (2, 0) | X_1 = (1, 0)\right) \mathbb{P}\left(\Omega_1^1 \middle| X_2 = (2, 0), X_1 = (1, 0)\right) + \\ & \mathbb{P}\left(X_2 = (1, 1) | X_1 = (1, 0)\right) \mathbb{P}\left(\Omega_1^1 \middle| X_2 = (1, 1), X_1 = (1, 0)\right) + \\ & \mathbb{P}\left(X_2 = (0, 0) | X_1 = (1, 0)\right) \mathbb{P}\left(\Omega_1^1 \middle| X_2 = (0, 0), X_1 = (1, 0)\right) \\ =\; & \frac{1}{2 + \delta}(P_\alpha + P_\beta), \end{aligned}$$

where $P_\alpha := \mathbb{P}\left(\Omega_1^1 \middle| X_2 = (2, 0), X_1 = (1, 0)\right)$, $P_\beta := \mathbb{P}\left(\Omega_1^1 \middle| X_2 = (1, 1), X_1 = (1, 0)\right)$ and $\mathbb{P}\left(\Omega_1^1 \middle| X_2 = (0, 0), X_1 = (1, 0)\right) = 0$. Note that $\{X_2 = (2, 0), X_1 = (1, 0)\} \subset \{\mathbf{y}(X_2) = 0, \eta_2 = 2\}$. By (3.2) we obtain

$$\begin{aligned}
P_\alpha &\geq \mathbb{P}\left(\Omega_2^1 \cap \Omega_1^1 \middle| X_2 = (2, 0), X_1 = (1, 0)\right) \\
&= \mathbb{P}\left(\Omega_2^1 \middle| X_2 = (2, 0), X_1 = (1, 0)\right) \mathbb{P}\left(\Omega_1^1 \middle| X_2 = (2, 0), X_1 = (1, 0), \Omega_2^1\right) \\
&= p^1 \cdot \mathbb{P}\left(\Omega_1^1 \middle| X_2 = (2, 0), X_1 = (1, 0), X_{\kappa_1} = (1, 1)\right) \\
&\geq p^1 \cdot \mathbb{P}\left(X_{\kappa_1+1} = (0, 1) \middle| X_2 = (2, 0), X_1 = (1, 0), X_{\kappa_1} = (1, 1)\right) \\
&= \frac{1}{2+\delta} p^1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P_\beta &\geq \frac{1}{2+\delta}\left[\mathbb{P}\left(\Omega_1^1 \middle| X_3 = (0, 1), X_2 = (1, 1), X_1 = (1, 0)\right) + \right. \\
&\qquad \left. \mathbb{P}\left(\Omega_1^1 \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0)\right)\right] \\
&= \frac{1}{2+\delta}\left[1 + \mathbb{P}\left(\Omega_1^1 \cap \Omega_2^1 \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0)\right) + \right. \\
&\qquad \left. \mathbb{P}\left(\Omega_1^1 \cap \Omega_2^0 \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0)\right)\right] \\
&= \frac{1}{2+\delta}\left[1 + p^0 \cdot \mathbb{P}\left(\Omega_1^1 \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0), \Omega_2^1\right) + \right. \\
&\qquad \left. p^1 \cdot \mathbb{P}\left(\Omega_1^1 \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0), \Omega_2^0\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}\left(\Omega_1^1 \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0), \Omega_2^1\right) \\
&\quad \geq \mathbb{P}\left(X_{\kappa_1+1} = (0, 1) \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0), X_{\kappa_1} = (1, 1)\right) \\
&\quad = \frac{1}{1+2\delta}, \\
&\mathbb{P}\left(\Omega_1^1 \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0), \Omega_2^0\right) \\
&\quad \geq \mathbb{P}\left(X_{\kappa_1+2} = (0, 1), X_{\kappa_1+1} = (1, 1) \middle| X_3 = (2, 1), X_2 = (1, 1), X_1 = (1, 0), X_{\kappa_1} = (1, 0)\right) \\
&\quad = \frac{1}{1+2\delta} \cdot \frac{1}{3},
\end{aligned}$$

which implies $P_\beta \geq \frac{1}{2+\delta}\left[1 + p^0 \frac{1}{1+2\delta} + p^1 \frac{1}{1+2\delta} \cdot \frac{1}{3}\right]$. Therefore,

$$\begin{aligned}
p^1 &\geq \frac{1}{(2+\delta)^2}\left[p^1 + 1 + p^0 \frac{1}{1+2\delta} + p^1 \frac{1}{1+2\delta}\frac{1}{3}\right] \\
&\geq \frac{1}{(2+\delta)^2}\left[p^1 + (p^0 + p^1) + p^0 \frac{1}{1+2\delta} + p^1 \frac{1}{1+2\delta}\frac{1}{3}\right],
\end{aligned}$$

which implies that

$$p^1 \geq \frac{2+2\delta}{(\delta^2+4\delta+3)(1+2\delta)+\frac{2}{3}}(p^0 + p^1).$$

Note that $0 < \delta < 1$ and

$$\frac{2+2\delta}{(\delta^2+4\delta+3)(1+2\delta)+\frac{2}{3}} - \frac{1-2\delta}{2(1-\delta)} = \frac{4\delta^4 + 16\delta^3 + 7\delta^2 - \frac{8}{3}\delta + \frac{1}{3}}{[(\delta^2+4\delta+3)(1+2\delta)+\frac{2}{3}] \cdot 2(1-\delta)},$$

where $4\delta^4 + 16\delta^3 > 0$ and $7\delta^2 - \frac{8}{3}\delta + \frac{1}{3} > 0$ for any $\delta > 0$. This implies $\frac{2+2\delta}{(\delta^2+4\delta+3)(1+2\delta)+\frac{2}{3}} > \frac{1-2\delta}{2(1-\delta)}$ as $0 < \delta < 1$. Thus, $p^1 > \frac{1-2\delta}{2(1-\delta)}(p^0 + p^1)$, i.e. as $0 < \delta < 1$,

$$p^1 > (1-2\delta)p^0.$$

The inequality above implies $p^0-p^1<2\delta p^0$ and $(2-2\delta)p^0<p^0+p^1$. Noting that $p^0+p^1\le 1$, we obtain that $p^0-p^1<2\delta\frac{p^0+p^1}{2-2\delta}\le\frac{\delta}{1-\delta}$ as $0<\delta<1$. ☐

Recall $\eta_n=\inf\{t\ge 0: h(X_t)=n\}$, and let

$$T_{M,n}=\inf\left\{t\ge\eta_n:\ X_t\in\{v:h(v)=n-1 \text{ or } n+M\}\right\},\ M\in\mathbb{N}. \tag{3.4}$$

Definition 3.3. *Let* $\mathbf{X}$ *be an* ORRW *on ladder* $\mathbb{Z}\times\{0,1\}$ *with reinforcement factor* $\delta\in(0,1)$*. Then for any* $n,n_0\in\mathbb{Z}$ *and* $v_0\in V$ *satisfying* $n>n_0\ge 0$ *and* $h(v_0)=n_0$*, if* $X_0=v_0$ *we can define*

$$\begin{aligned}p_M^1&=\mathbb{P}\left(X_{T_{M,n}}=(n-1,1-\mathbf{y}(X_{\eta_n})) \text{ or } (1-n,1-\mathbf{y}(X_{\eta_n}))\big|\mathscr{F}_{\eta_n}\right),\\ p_M^0&=\mathbb{P}\left(X_{T_{M,n}}=(n-1,\mathbf{y}(X_{\eta_n})) \text{ or } (1-n,\mathbf{y}(X_{\eta_n}))\big|\mathscr{F}_{\eta_n}\right),\end{aligned}$$

where $T_{M,n}$ *is defined in (3.4).*

Note that p_M^0 and p_M^1 are independent of n. Moreover,

$$\begin{aligned}&\mathbb{P}\left(X_{T_{M,n}}=(n-1,1-y)\big|\mathscr{F}_1\right)=p_M^1,\ a.s. \text{ on } \{X_1=(n,y),X_0=(n-1,y)\},\\ &\mathbb{P}\left(X_{T_{M,n}}=(n-1,y)\big|\mathscr{F}_1\right)=p_M^0,\ a.s. \text{ on } \{X_1=(n,y),X_0=(n-1,y)\}.\end{aligned} \tag{3.5}$$

Now we show the asymptotic property for these probabilities. Since

$$\begin{aligned}p_M^1&=\mathbb{P}\big(X_{T_{M,1}}=(0,1)\big|X_1=(1,0),X_0=(0,0)\big),\\ p_M^0&=\mathbb{P}\big(X_{T_{M,1}}=(0,0)\big|X_1=(1,0),X_0=(0,0)\big),\end{aligned}$$

noting that $X_{T_{M,1}}(\omega)=(0,1)$ implies $X_{T_{M+1,1}}(\omega)=(0,1)$, we have the following lemma.

Lemma 3.4. *Let* $\mathbf{X}$ *be an* ORRW *on ladder* $\mathbb{Z}\times\{0,1\}$ *with reinforcement factor* $\delta\in(0,1)$*. Let* p_M^1,p_M^0 *are probabilities in Definition 3.3. Then as* $M\uparrow\infty$*,*

$$p_M^1\uparrow p^1,\ p_M^0\uparrow p^0. \tag{3.6}$$

For reader's convenience, we give the following simple property of conditional expectation.

Lemma 3.5. *Set* $(\Omega,\mathscr{F},P)$ *be a probability space equipped with a filtration* $(\mathscr{F}_n)_{n\in\mathbb{N}}$*. For any* $(\mathscr{F}_n)_{n\in\mathbb{N}}$ *stopping times* S_1*,* S_2 *and any integrable random variable* $f\in\sigma(\cup_{n=0}^{\infty}\mathscr{F}_n)$*,*

$$\mathbb{E}(fI_{\{S_1=S_2\}}|\mathscr{F}_{S_1})=\mathbb{E}(fI_{\{S_1=S_2\}}|\mathscr{F}_{S_2}),\ a.s.$$

At this point, we are ready to conclude the proof of our main theorem.

Proof of Theorem 1.1. $h(v)=|\mathbf{x}(v)|,\ v\in\mathbb{Z}\times\{0,1\}$ in (3.1) is harmonic (therefore, superharmonic) except on the finite set $F=\{v:h(v)=0\}$. Without loss of generality, let r be a positive integer. Note that C_a is given in (3.1), and recall the definition of $\tau_{r'}$ in Theorem 2.1.

Firstly, we determine ε and r' on RHS of (2.1). Set

$$\varepsilon'=\frac{1-(1-\delta)(p^0-p^1+1)}{2},$$

then $\varepsilon'>0$ by Lemma 3.2. Due to Lemma 3.4, there exists an integer $N>0$ satisfying $\forall n\ge N$,

$$(\delta-1)(p_n^0-p_n^1+1)>(\delta-1)(p^0-p^1+1)-\varepsilon'\ge-(1-\varepsilon'). \tag{3.7}$$

Now we choose some $\varepsilon \in (0, \varepsilon')$. Moreover, for any positive integer r, we can take an integer $r' > N + r$ such that $(1-\varepsilon)r' > (1-\varepsilon')(r'-N-r) + 2(1-\delta)(r+N)$. Set $\tilde{r} = r' - N$.

Secondly, we verify that (2.1) holds almost surely on $\{\omega : X_{\eta_r}(\omega) = (r,0) \text{ or } (r,1)\}$. In this situation, X_n walks on $\mathbb{Z}^+ \times \{0,1\}$ during $n \in [\eta_r, \tau_{r'}]$. Specifically, the horizontal coordinate of X_n, $\mathbf{x}(X_n) = h(X_n)$, and belongs to $[0, r']$, if $\eta_r \le n \le \tau_{r'}$.

We now separate $\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r} \right)$ in (2.1) into three parts according to the subscript a of column C_a: $a \in [0, r-1]$, $a \in [\tilde{r}, r'-1]$ and $a \in [r, \tilde{r}-1]$ (see Fig. 2), and estimate them respectively.

Figure 2: An intuitive illustration of the three parts separated in LHS of (2.1) and the notation of $\tilde{r}$.

<u>*Part 1 and Part 2:*</u> Noting that each column C_a has only two horizontal edges and for any integer $a \in [0, r-1] \cup [\tilde{r}, r'-1]$, we have

$$(\delta - 1)\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}, vu \in C_a} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r} \right) \ge 2(\delta - 1). \tag{3.8}$$

<u>*Part 3:*</u> For any integer $a \in [r, \tilde{r}-1]$, $r - a - 1 \ge N$, thus $\tau_{a+1} \le \tau_{r'}$. Noting that $h(X_t) < a$ for any $t \le \eta_r$, we have $\eta_{a+1} = \inf\{t > \eta_r : h(X_t) = a+1\}$, which implies $\eta_r < \tau_{a+1} \le \eta_{a+1}$.

Let

$$D_a^k = \left\{ \omega : \sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}, vu \in C_a} \Delta_h(\overrightarrow{vu}) = k \right\}, \; k = 0, 1, 2.$$

One can see that $\cup_{k=0}^2 D_a^k = \Omega$ and $D_a^i \cap D_a^j = \emptyset$ for $i \neq j$. $\omega \in D_a^1 \cup D_a^2$ indicates that there is an edge in C_a traversed by $X_n(\omega)$ between stopping time η_r and $\tau_{r'}$, thus $D_a^1 \cup D_a^2 \subseteq \{h(X_{\tau_{a+1}}) = a+1\} = \{\tau_{a+1} = \eta_{a+1}\}$. Then Lemma 3.5 implies that

$$\begin{aligned}
&(\delta - 1)\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}, vu \in C_a} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\tau_{a+1}} \right) \\
&= (\delta - 1)\Big[2\mathbb{P}\big(D_a^2 \big| \mathscr{F}_{\tau_{a+1}}\big) + \mathbb{P}\left(D_a^1 \big| \mathscr{F}_{\tau_{a+1}}\right) + 0 \cdot \mathbb{P}\left(D_a^0 \big| \mathscr{F}_{\tau_{a+1}}\right) \Big] \\
&= (\delta - 1)\Big[2\mathbb{P}\big(D_a^2, \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\tau_{a+1}}\big) + \mathbb{P}\big(D_a^1, \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\tau_{a+1}}\big) \Big] \\
&\overset{a.s.}{=} (\delta - 1)\Big[2\mathbb{P}\big(D_a^2, \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\eta_{a+1}}\big) + \mathbb{P}\big(D_a^1, \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\eta_{a+1}}\big) \Big].
\end{aligned} \tag{3.9}$$

Note that the fact $T_{r'-a-1,a+1} = \inf\{t \ge \eta_{a+1} : h(X_t) = a \text{ or } r'\}$ implies $\{D_a^2, \; \tau_{a+1} = \eta_{a+1}\} \subset \{X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \; \tau_{a+1} = \eta_{a+1}\}$. Thus

$$\begin{aligned}
&\mathbb{P}\left(D_a^2, \tau_{a+1} = \eta_{a+1} \,\middle|\mathscr{F}_{\eta_{a+1}}\right) \\
&= \mathbb{P}\left(D_a^2, X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1} \,\middle|\mathscr{F}_{\eta_{a+1}}\right).
\end{aligned} \tag{3.10}$$

Meanwhile, $\{X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1}\}$, $\{h(X_{T_{r'-a-1,a+1}}) = r', \tau_{a+1} = \eta_{a+1}\}$ and $\{X_{T_{r'-a-1,a+1}} = (a, 1 - \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1}\}$ are disjoint sets, and their union is $\{\tau_{a+1} = \eta_{a+1}, h(X_{\eta_r}) = r\} \supset D_a^1$. Observing $\{X_{T_{r'-a-1,a+1}} = (a, 1-\mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1}\} \subset D_a^0$ and $D_a^0 \cap D_a^1 = \emptyset$, we obtain that

$$\begin{aligned}\mathbb{P}\big(D_a^1, \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\eta_{a+1}}\big) &= \mathbb{P}\big(D_a^1, X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\eta_{a+1}}\big) \\ &\quad + \mathbb{P}\big(D_a^1, h(X_{T_{r'-a-1,a+1}}) = r', \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\eta_{a+1}}\big) \\ &\le 2\mathbb{P}\big(D_a^1, X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\eta_{a+1}}\big) \\ &\quad + \mathbb{P}\big(h(X_{T_{r'-a-1,a+1}}) = r', \tau_{a+1} = \eta_{a+1} \big| \mathscr{F}_{\eta_{a+1}}\big). \qquad (3.11)\end{aligned}$$

Combining (3.10), (3.11) and $\mathbb{P}\left(D_a^0, X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1} \,\middle|\, \mathscr{F}_{\eta_{a+1}}\right) \ge 0$, and noting $\delta - 1 < 0$, we have that

$$\begin{aligned}(3.9) \quad &\ge (\delta-1)\Bigg[2\sum_{k=0}^{2} \mathbb{P}\left(D_a^k, X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1} \middle| \mathscr{F}_{\eta_{a+1}}\right) \\ &\qquad + \mathbb{P}\left(h(X_{T_{r'-a-1,a+1}}) = r', \tau_{a+1} = \eta_{a+1} \middle| \mathscr{F}_{\eta_{a+1}}\right)\Bigg] \\ &= (\delta-1)\Bigg[2\mathbb{P}\left(X_{T_{r'-a-1,a+1}} = (a, \mathbf{y}(X_{\eta_{a+1}})), \tau_{a+1} = \eta_{a+1} \middle| \mathscr{F}_{\eta_{a+1}}\right) \\ &\qquad + \mathbb{P}\left(h(X_{T_{r'-a-1,a+1}}) = r', \tau_{a+1} = \eta_{a+1} \middle| \mathscr{F}_{\eta_{a+1}}\right)\Bigg] \\ &\overset{a.s.}{=} (\delta-1)\Big[2p^0_{r'-a-1} + (1 - p^0_{r'-a-1} - p^1_{r'-a-1})\Big] \cdot I_{\{\tau_{a+1}=\eta_{a+1}\}} I_{\{\eta_{a+1}<\infty\}} \\ \text{(by (3.7))} \quad &> -(1-\varepsilon').\end{aligned}$$

Therefore, summing up these three parts, we get that on $\{\omega : X_{\eta_r}(\omega) = (r,0) \text{ or } (r,1)\}$,

$$\begin{aligned}&(\delta-1)\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r}\right) \\ &= \left(\sum_{a=0}^{r-1} + \sum_{a=r'-N}^{r'-1}\right)(\delta-1)\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}, vu \in C_a} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\eta_r}\right) \\ &\quad + \sum_{a=r}^{r'-N-1} (\delta-1)\mathbb{E}\left[\mathbb{E}\left(\sum_{\overrightarrow{vu} \in A_{\tau_{r'}} \setminus A_{\eta_r}, vu \in C_a} \Delta_h(\overrightarrow{vu}) \middle| \mathscr{F}_{\tau_{a+1}}\right) \middle| \mathscr{F}_{\eta_r}\right] \\ &\overset{a.s.}{\ge} [-(1-\varepsilon')(r'-N-r) - 2(1-\delta)(r+N)] \\ &\ge -(1-\varepsilon)r'. \qquad (3.12)\end{aligned}$$

Finally, we can also verify (3.12) on $\{\omega : X_{\eta_r}(\omega) = (-r,0) \text{ or } (-r,1)\}$ by the same approach to work on $\mathbb{Z}^- \times \{0,1\}$ and finish proving Theorem 1.1 by Theorem 2.1. □

4 Conclusion

Given any $n \in \mathbb{N}$, it is expected that δ-ORRW on ladder $\mathbb{Z} \times \{0, \dots, n\}$ with $n+1$ levels is almost surely recurrent for any $\delta > 0$ ([7]). Theorem 1.1 together with the results in [7] and [8] confirm this is true for $n = 1$.

For $n \ge 2$, our method seems to have met obstacles. In this case, we can still apply Theorem 2.1 and separate LHS of (2.1) into three parts: $\sum_{a=0}^{r-1}$, $\sum_{a=\tilde{r}}^{r'-1}$ and $\sum_{a=r}^{\tilde{r}-1}$, where

$r < \tilde{r} < r'$. The estimation approach of the first two parts is still applicable. While for the third part, our proposed approach is to decompose the paths into D_a^k, $k = 0, 1, \ldots, n+1$ according to the numbers of the horizontal edges of column C_a traversed firstly from left to right, and then to compute probabilities related to each D_a^k. At present, we are only able to get the following type of weaker results: there are δ_1, δ_2 depending on n such that $0 < \delta_1 < \frac{n}{n+1} < \frac{n}{n-1} < \delta_2$, and δ-ORRW is a.s. recurrent for any $\delta \in (\delta_1, \delta_2)$. Such a result was already claimed in [8]. Fortunately, for $\mathbb{Z} \times \{0,1\}$, Lemma 3.2 holds and provides some accurate estimates of probabilities related to each D_a^k for any $\delta \in (0,1)$, and thus we can prove Theorem 1.1.

To handle the case $n \geq 2$, new tools need to be developed, for example, decomposing paths more delicately in a larger column (e.g. $(n+1) \times (n+1)$ square) to get inequality similar to Lemma 3.2.

Moreover, if we assume that ORRWs on $\mathbb{Z} \times \{-n, \ldots, -1, 0, 1, \ldots, n\}$ are a.s. recurrent for any $n \in \mathbb{N}$, an interesting problem will arise naturally: Can one deduce a.s. recurrence for ORRWs on $\mathbb{Z}^2$?

There are plenty of differences between reinforced random walks and Markov Chains. For instance, unlike Markov Chains, whether or not the transience/recurrence 0-1 law of reinforced random walks holds is a subtle problem. In particular, although the known results of ORRWs show the transience/recurrence 0-1 law, there seems to be lack of deep insights on this property of general cases, and it is unknown that δ-ORRW on $\mathbb{Z}^d$ with $d \geq 2$ satisfies the transience/recurrence 0-1 law for all $\delta > 0$.

References

[1] Coppersmith, D. and Diaconis, P.: Random walk with reinforcement. Unpublished, (1987).

[2] Collevecchio, A., Kious, D. and Sidoravicius, V.: The branching-ruin number and the critical parameter of once-reinforced random walk on trees. *Comm. Pure Appl. Math.* **73(1)**, (2020), 210-236. MR-4033893

[3] Davis, B.: Reinforced random walk. *Probab. Theory Related Fields.* **84(2)**, (1990), 203-229. MR-1030727

[4] Durrett, R.: Probability: Theory and Examples (4th edition). *Cambridge University Press*, Cambridge, 2010. x+428 pp. MR-2722836

[5] Kious, D., Schapira, B. and Singh, A.: Once reinforced random walk on $\mathbb{Z} \times \Gamma$. arXiv:1807.07167v1 (2018).

[6] Kious, D. and Sidoravicius, V.: Phase transition for the once-reinforced random walk on $\mathbb{Z}^d$-like trees. *Ann. Probab.* **46(4)**, (2018), 2121-2133. MR-3813987

[7] Sellke, T.: Recurrence of reinforced random walk on a ladder. *Electron. J. Probab.* **11**, (2006), 301-310. MR-2217818

[8] Vervoort, M. R.: Reinforced random walks. Unpublished, (2002), https://staff.fnwi.uva.nl/m.r.vervoort/walk.pdf.

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