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Generalized Peano problem with Lévy noise

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Abstract

We revisit the zero-noise Peano selection problem for Lévy-driven stochastic differential equation considered in [Pilipenko and Proske, Statist. Probab. Lett., 132:62–73, 2018] and show that the selection phenomenon pertains in the multiplicative noise setting and is robust with respect to certain perturbations of the irregular drift and of the small jumps of the noise.

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1 Introduction, setting, and the main result

The well known Peano existence theorem [7, Theorem II.2.1] states that an ordinary differential equation (ODE) $\mathrm{d}x = a(x)\,\mathrm{d}t$ with a continuous function $a\colon \mathbb{R}\to\mathbb{R}$ has a local solution which however may be not unique.

On the contrary, an addition of a noise term allows to obtain unique solutions of stochastic differential equations (SDE) with irregular or even just measurable coefficients, see, e.g. [3, 5, 10, 12, 13, 15, 16, 19, 20, 21, 23, 25].

Let us consider an SDE with a drift a and assume that the underlying ODE $\mathrm{d}x = a(x)\,\mathrm{d}t$ has multiple solutions. A natural question arises, what happens when the random perturbation vanishes. Heuristically, solutions of the small noise SDE should converge to one of the various deterministic solutions and the *selection problem* consists in description of this limit behaviour.

Originally, the selection problem was treated in [1], where the authors considered the SDE $\mathrm{d}X^\varepsilon = a(X^\varepsilon)\,\mathrm{d}t + \varepsilon b(X^\varepsilon)\,\mathrm{d}W$ with an irregular drift at $x=0$. It was shown that the limit law $\mathrm{Law}(X^\varepsilon|X^\varepsilon(0)=0)$ is supported by the deterministic maximal and minimal solutions of the ODE $\mathrm{d}x = a(x)\,\mathrm{d}t$ starting at zero with the *selection probabilities* $p_\pm$ that can be explicitly determined, see [1, Theorem 4.1]. The uniqueness of the limit in the case of odd continuous concave drift and additive noise was proven in [24]. Recently a new proof of these results was given in [4] for the piece-wise power drift

$$\bar a(x) = A_+ x^\beta \mathbb{I}_{[0,\infty)}(x) - A_-|x|^\beta \mathbb{I}_{(-\infty,0)}(x) \tag{1.1}$$

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with $\beta \in (0,1)$ and $A_\pm > 0$ in the case of additive Brownian perturbations. Further results concerning this equation can be found in [6, 22, 9].

In [14], the authors considered a class of SDEs $\mathrm{d}X^\varepsilon = \bar{a}(X^\varepsilon)\,\mathrm{d}t + \varepsilon \mathrm{d}B^{(\alpha)}$ with a piece-wise power drift (1.1) with $\beta \in (-1,1)$ driven by α-self-similar processes $B^{(\alpha)}$, e.g. by a fractional Brownian motion or a strictly stable Lévy process. They showed that under some natural assumptions X^ε also selects the maximal and minimal solutions of the ODE $\mathrm{d}x = \bar{a}(x)\,\mathrm{d}t$. Unfortunately, the selection probabilities cannot be always determined explicitly.

In this paper we address the selection problem for a Lévy driven SDE with multiplicative noise

$$X^\varepsilon(t) = \int_0^t a(X^\varepsilon(s))\,\mathrm{d}s + \varepsilon \int_0^t b(X^\varepsilon(s-))\,\mathrm{d}Z(s), \quad t \geq 0, \quad \varepsilon \to 0, \tag{1.2}$$

whose drift $a = a(x)$ has an irregular point at $x = 0$ but does not have the exact piece-wise power form (1.1). The small jumps of the driving Lévy process Z remind of those of an α-stable Lévy process. In other words we answer the question whether the selection dynamics are robust w.r.t. perturbations of the drift and the noise.

Let us formulate the precise assumptions. As usual, we write $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$; $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$, $a,b \in \mathbb{R}$; $\|f\|$ denotes the supremum norm of a function f; "$\Rightarrow$" denotes weak convergence in the Skorokhod space D; functions of slow and regular variation are understood in Karamata's sense, see [2, Sections 1.2.1 and 1.4.2].

$\mathbf{A}_Z$: Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process without a Gaussian component and the jump measure ν such that for some $\alpha \in (1,2)$ and some constants $C_\pm \geq 0$, $C_- + C_+ > 0$,

$$\nu([z,+\infty)) \sim C_+ z^{-\alpha} l_\nu\Big(\frac{1}{z}\Big), \qquad \nu((-\infty,-z]) \sim C_- z^{-\alpha} l_\nu\Big(\frac{1}{z}\Big), \quad z \to +0, \tag{1.3}$$

for a function l_ν slowly varying at infinity.

$\mathbf{A}_a$: Let $x \mapsto a(x)$ be a real valued continuous function of linear growth such that $a(0) = 0$ and that for $\beta \in (0,1)$

$$a(x) = x^\beta L_+(x) \quad \text{for} \quad x > 0 \quad \text{and} \quad a(x) = -|x|^\beta L_-(|x|) \quad \text{for} \quad x < 0, \tag{1.4}$$

with continuous functions $L_\pm \colon (0,\infty) \to (0,\infty)$ that satisfy

$$L_\pm(x) \sim A_\pm l_a\Big(\frac{1}{x}\Big) \quad \text{as } x \to +0, \tag{1.5}$$

for a function l_a slowly varying at infinity, and $A_\pm > 0$.

$\mathbf{A}_b$: Let $x \mapsto b(x)$ be a bounded continuous real valued function such that $b(0) > 0$.

It follows from assumptions $\mathbf{A}_a$, $\mathbf{A}_b$ that equation (1.2) has a weak solution, see, e.g. Theorem 1 of §2 Chapter 5 in [5].

Remark 1.1. We will see in the main result that the weak limit of the sequence $\{X^\varepsilon\}$ as $\varepsilon \to 0$ is independent of the choice of a weak solution X^ε. Therefore from now on we assume that X^ε is any weak solution to (1.2). It should be also noticed that the presence of a noise often implies uniqueness of a solution and the strong Markov property, see references above.

A simple analysis shows that the limit deterministic equation

$$X^0(t) = \int_0^t a(X^0(s))\,\mathrm{d}s \tag{1.6}$$

has the maximal and the minimal solutions

$$x^{\pm}(t) := \pm A_{\pm}^{-1}(t), \quad t \geq 0, \tag{1.7}$$

where $A_{\pm}(\cdot)$ are continuous non-negative strictly increasing functions, $A_{\pm}(0) = 0$, given by

$$A_{+}(x) := \int_{0+}^{x} \frac{\mathrm{d}y}{a(y)},\ x > 0, \qquad A_{-}(x) := \int_{0-}^{x} \frac{\mathrm{d}y}{a(y)},\ x < 0,$$

and $A_{\pm}^{-1}(\cdot)\colon [0,\infty) \to [0,\infty)$ are their inverses. All these functions are well defined because of assumption $\mathbf{A}_a$.

It is intuitively clear that under the influence of noise the selection occurs very quickly, so that it is determined only by the small jumps of Z and the local behaviour of $a(\cdot)$ in the vicinity of zero. Omitting the functions l_ν and l_a in (1.3) and (1.5) we introduce the auxiliary *model* SDE

$$\bar X^{\varepsilon}(t) = \int_0^t \bar a(\bar X^{\varepsilon}(s))\,\mathrm{d}s + \varepsilon Z^{(\alpha)}(t) \tag{1.8}$$

with the piece-wise power drift $\bar a$ defined in (1.1) and driven by a strictly α-stable Lévy process $Z^{(\alpha)}$, $\alpha \in (1,2)$, with the generating triplet $(0, \nu^{(\alpha)}, 0)_1$ (see [18, Section 8]) where

$$\nu^{(\alpha)}([z,+\infty)) = C_+ z^{-\alpha}, \quad \nu^{(\alpha)}((-\infty,-z]) = C_- z^{-\alpha},\ z > 0. \tag{1.9}$$

The model equation (1.8) has a unique strong Markov solution due to Theorem 3.1 from [21] (although in [21] the drift is supposed to be bounded, an extension of their results to $\bar a$ given by (1.1) follows easily from the sublinear growth of $\bar a$ at infinity).

The *model* ODE $\mathrm{d}x = \bar a(x)\,\mathrm{d}t$ has the following maximal and minimal solutions starting at $x = 0$:

$$\bar x^{\pm}(t) = \pm\big(A_{\pm}\cdot(1-\beta)t\big)^{\frac{1}{1-\beta}}, \quad t \geq 0. \tag{1.10}$$

The selection problem for the model SDE (1.8) was solved in [14, Theorem 1.1], namely it was shown that $\operatorname{Law}\bar X^{\varepsilon} \Rightarrow \bar p_- \delta_{\bar x^-} + \bar p_+ \delta_{\bar x^+}$, $\varepsilon \to 0$, in $D([0,\infty),\mathbb{R})$, where the selection probabilities

$$\bar p_{\pm} = \mathbf{P}\Big(\lim_{t\to\infty} \bar X^{\varepsilon}(t) = \pm\infty\Big) \tag{1.11}$$

are independent of ε, and $\bar p_- + \bar p_+ = 1$.

Remark 1.2. It follows from the self-similarity of $Z^{(\alpha)}$ that for any $\varepsilon, \delta, \gamma > 0$ the rescaled process $\bar X^{\gamma,\delta,\varepsilon}(t) := \gamma \bar X^{\varepsilon}(\delta t)$, $t \geq 0$, satisfies the SDE

$$\bar X^{\gamma,\delta,\varepsilon}(t) = \int_0^t \gamma^{1-\beta}\delta\, \bar a(\bar X^{\gamma,\delta,\varepsilon}(s))\,\mathrm{d}s + \varepsilon\gamma\delta^{\frac{1}{\alpha}}\cdot \bar Z^{(\alpha)}(t),$$

where $\bar Z^{(\alpha)} \overset{\mathrm{d}}{=} Z^{(\alpha)}$. This implies that the selection probabilities $\bar p_{\pm}$ defined in (1.11) are the same for any model equation $\mathrm{d}X^{\varepsilon} = \bar a(X^{\varepsilon})\,\mathrm{d}t + \sigma \mathrm{d}Z^{(\alpha)}$ driven by a rescaled process $\sigma Z^{(\alpha)}$ with any $\sigma > 0$. Moreover, they are completely determined by the four parameters $\alpha \in (1,2)$, $C_+/C_- \in [0,+\infty]$, $\beta \in (0,1)$, and $A_+/A_- \in (0,\infty)$.

In the present paper we solve the generalized selection problem for the SDE (1.2). The main result of this paper is the following.

Theorem 1.3. *Let assumptions $\mathbf{A}_Z$, $\mathbf{A}_a$, and $\mathbf{A}_b$ hold true, and let X^{ε} be a solution to* (1.2). *Then*

$$\operatorname{Law} X^{\varepsilon} \Rightarrow \bar p_- \delta_{x^-} + \bar p_+ \delta_{x^+}, \quad \varepsilon \to 0, \tag{1.12}$$

in $D([0,\infty),\mathbb{R})$ where functions $x^{\pm}$ are defined in (1.7) *and the selection probabilities $\bar p_{\pm}$ are defined in* (1.11) *for the model equation* (1.8).

Remark 1.4. The question of how to determine the selection probabilities $\bar{p}_\pm$ is still open except of the case of Brownian perturbation, see [1, 4]:

$$\bar{p}_- = \frac{A_-^{\frac{1}{1+\beta}}}{A_-^{\frac{1}{1+\beta}} + A_+^{\frac{1}{1+\beta}}} \quad \text{and} \quad \bar{p}_+ = \frac{A_+^{\frac{1}{1+\beta}}}{A_-^{\frac{1}{1+\beta}} + A_+^{\frac{1}{1+\beta}}}.$$

To prove Theorem 1.3 it suffices to show two properties of the limit laws of X^ε as $\varepsilon \to 0$: i) a process X^ε can spend only infinitesimal time near zero and ii) the deterministic solutions $x^\pm$ are chosen with the probabilities $\bar{p}_\pm$ given in (1.11).

In particular we will show that the selection takes place in the infinitesimal time-space box $(t,x) \in [0, T_0\varepsilon'] \times [-R\varepsilon'', R\varepsilon'']$ with appropriately chosen bounds $\varepsilon', \varepsilon'' \to 0$ and $T_0, R > 0$ large enough. To achieve this, we introduce a rescaled process $Y^\varepsilon(t) := X^\varepsilon(\varepsilon' t)/\varepsilon''$ and show that Y^ε converges weakly to a solution of the model equation (1.8) driven by $b(0)Z^{(\alpha)}$. Hence the exit of X^ε from the infinitesimal time-space box $[0, T_0\varepsilon'] \times [-R\varepsilon'', R\varepsilon'']$ is equivalent to the exit of Y^ε from the ε-independent time-space box $[0, T_0] \times [0, R]$ which is the subject of Theorem 1.1 in [14].

Eventually we show that upon leaving the ε-dependent time-space box, a solution X^ε with high probability follows the maximal (minimal) solution $x^\pm$ as $\varepsilon \to 0$.

2 Preliminary considerations and time-space rescaling

Before starting the proof we make two technical assumptions that do not reduce the generality of the setting but simplify the arguments significantly.

Remark 2.1. To establish convergence (1.12) it suffices to show the weak convergence on the space $D([0,T], \mathbb{R})$ for each $T > 0$. After applying the standard "truncation of large jumps procedure", see, e.g. [11, Section 3.2] we can assume that for some $M > 0$ the Lévy measure ν defined in (1.3) additionally satisfies

$$\operatorname{supp} \nu \subseteq [-M, M] \quad \text{and} \quad \nu(\{\pm M\}) = 0. \tag{2.1}$$

Remark 2.2. We also note that for any two drifts a and $\tilde{a}$ both satisfying $\mathbf{A}_a$ and such that $a(x) = \tilde{a}(x)$, $|x| \le 1$, the corresponding solutions X^ε and $\tilde{X}^\varepsilon$ coincide up to the exit from $[-1, 1]$. Hence the selection probabilities for these solutions in the limit $\varepsilon \to 0$ are equal too. From now on we assume without loss of generality that

$$L_\pm(x) = L_\pm(x \wedge 1) \text{ for } x > 0 \tag{2.2}$$

to ensure the power growth of a at infinity.

Lemma 2.3. *Assume that assumptions $\mathbf{A}_a$ and $\mathbf{A}_b$ are satisfied. Then the family of distributions $\{\mathrm{Law}(X^\varepsilon)\}_{\varepsilon \in (0,1]}$ is tight in $D([0,\infty), \mathbb{R})$ and a limit of any weakly convergent subsequence satisfies the integral equation* (1.6).

Proof. Indeed, tightness of $\{\mathrm{Law}(X^\varepsilon)\}_{\varepsilon \in (0,1]}$ follows, e.g. from the sublinear growth of a, boundedness on b, Aldous' criterion and boundedness $\sup_{\varepsilon \in (0,1]} \mathbf{E} \sup_{t \in [0,T]} |X_t^\varepsilon|^2 < \infty$ for each $T > 0$, see (2.1). The weak convergence $X^{\varepsilon_n} \Rightarrow X$, $n \to \infty$ follows from the continuity of a and b. $\square$

Let X^ε be any solution of (1.2). For any $\varepsilon' = \varepsilon'(\varepsilon) > 0$ and $\varepsilon'' = \varepsilon''(\varepsilon) > 0$ we define

$$a_\varepsilon(y) = \frac{a(\varepsilon'' y)}{\varepsilon''/\varepsilon'}, \quad b_\varepsilon(y) = b(\varepsilon'' y), \quad Z_\varepsilon(t) = \frac{Z(\varepsilon' t)}{\varepsilon''/\varepsilon} \tag{2.3}$$

and consider a time-space rescaled process

$$Y^\varepsilon(t) = \frac{X^\varepsilon(\varepsilon' t)}{\varepsilon''}, \quad t \ge 0, \tag{2.4}$$

which satisfies the SDE

$$\begin{aligned} Y^\varepsilon(t) &= \frac{1}{\varepsilon''}\int_0^{\varepsilon' t} a(X^\varepsilon(s))\,\mathrm{d}s + \frac{\varepsilon}{\varepsilon''}\int_0^{\varepsilon' t} b(X^\varepsilon(s-))\,\mathrm{d}Z(s)\\ &= \int_0^t \frac{a(\varepsilon'' Y^\varepsilon(s))}{\varepsilon''/\varepsilon'}\,\mathrm{d}s + \int_0^t b(\varepsilon'' Y^\varepsilon(s-))\,\mathrm{d}\frac{Z(\varepsilon' s)}{\varepsilon''/\varepsilon}\\ &= \int_0^t a_\varepsilon(Y^\varepsilon(s))\,\mathrm{d}s + \int_0^t b_\varepsilon(Y^\varepsilon(s-))\,\mathrm{d}Z_\varepsilon(s). \end{aligned} \tag{2.5}$$

Lemma 2.4. *Let l_a and l_ν be slowly varying functions from assumptions $\mathbf{A}_Z$ and $\mathbf{A}_a$. Then there exist positive functions $\varepsilon' = \varepsilon'(\varepsilon)$ and $\varepsilon'' = \varepsilon''(\varepsilon)$ such that $\lim_{\varepsilon\to 0}\varepsilon' = \lim_{\varepsilon\to 0}\varepsilon'' = 0$,*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon''}{\varepsilon} = 0 \tag{2.6}$$

and

$$\frac{\varepsilon''}{\varepsilon'} \sim (\varepsilon'')^\beta l_a\Big(\frac{1}{\varepsilon''}\Big), \tag{2.7}$$

$$\varepsilon' \sim \Big(\frac{\varepsilon''}{\varepsilon}\Big)^\alpha \cdot \Big(l_\nu\Big(\frac{\varepsilon}{\varepsilon''}\Big)\Big)^{-1} \quad \textit{as} \quad \varepsilon\to 0. \tag{2.8}$$

Proof. The construction of the scalings ε', ε'' relies on the Karamata theory of regularly varying functions. Consider regularly varying at infinity functions $f_1(x) = x^{1-\beta}/l_a(x)$ and $f_2(x) = x^\alpha l_\nu(x)$, $x>0$, and let g_1, g_2 be their asymptotic inverses, see [2, Section 1.5.7]. The ratio

$$f_3(x) := \frac{g_1(x)}{g_2(x)}$$

is also regularly varying with index $\frac{1}{1-\beta} - \frac{1}{\alpha} > 0$. Let g_3 be its asymptotic inverse. We set

$$\frac{1}{\varepsilon'(\varepsilon)} := g_3\Big(\frac{1}{\varepsilon}\Big), \tag{2.9}$$

$$\frac{1}{\varepsilon''(\varepsilon)} := g_1\Big(\frac{1}{\varepsilon'(\varepsilon)}\Big). \tag{2.10}$$

It is easy to see that $\varepsilon\mapsto\varepsilon'(\varepsilon)$ and $\varepsilon\mapsto\varepsilon''(\varepsilon)$ are positive and converge to zero as $\varepsilon\to 0$. Applying f_1 to the both sides of (2.10) immediately yields (2.7):

$$f_1\Big(\frac{1}{\varepsilon''}\Big) = f_1\Big(g_1\Big(\frac{1}{\varepsilon'}\Big)\Big) \sim \frac{1}{\varepsilon'}.$$

Furthermore, due to (2.9) we have $f_3(\frac{1}{\varepsilon'}) = f_3(g_3(\frac{1}{\varepsilon})) \sim \frac{1}{\varepsilon}$ and hence

$$\frac{\varepsilon}{\varepsilon''} \sim \frac{g_1(\frac{1}{\varepsilon'})}{f_3(\frac{1}{\varepsilon'})} = g_2\Big(\frac{1}{\varepsilon'}\Big). \tag{2.11}$$

Since regularly varying functions preserve asymptotic equivalence we obtain (2.8) by application of f_2 to (2.11):

$$f_2\Big(g_2\Big(\frac{1}{\varepsilon'}\Big)\Big) \sim f_2\Big(\frac{\varepsilon}{\varepsilon''}\Big) \sim \frac{1}{\varepsilon'}. \quad \square$$

Let ν be the Lévy measure of the process Z satisfying $\mathbf{A}_Z$ and (2.1), and let ε', ε'' be the scalings chosen in Lemma 2.4. For $\varepsilon\in(0,1]$ let us define rescaled jump measures ν_ε by setting for $z>0$

$$\nu_\varepsilon([z,\infty)) = \varepsilon'\nu\Big(\Big[\frac{\varepsilon'' z}{\varepsilon},\infty\Big)\Big), \qquad \nu_\varepsilon((-\infty,-z]) = \varepsilon'\nu\Big(\Big(-\infty,-\frac{\varepsilon'' z}{\varepsilon}\Big]\Big). \tag{2.12}$$

Lemma 2.5. *For the family of jump measures* $\{\nu_\varepsilon\}_{\varepsilon\in(0,1]}$ *defined in* (2.12) *we have:*

1. *for each* $z>0$
$$\lim_{\varepsilon\to 0}\nu_\varepsilon([z,\infty))=\nu^{(\alpha)}([z,\infty))\quad\textit{and}\quad \lim_{\varepsilon\to 0}\nu_\varepsilon((-\infty,-z])\quad=\nu^{(\alpha)}((-\infty,z]),\tag{2.13}$$
where $\nu^{(\alpha)}$ *is defined in* (1.9);
2. *for each* $\delta>0$ *there is* $C>0$ *such that for all* $z>0$
$$\sup_{\varepsilon\in(0,1]}\Big(\nu_\varepsilon((-\infty,-z])+\nu_\varepsilon([z,\infty))\Big)\leq C\Big(\frac{1}{z^{\alpha-\delta}}\vee\frac{1}{z^{\alpha+\delta}}\Big).\tag{2.14}$$

Proof. Without loss of generality we consider only the right tail of ν_ε.

1. For any $z>0$ we apply (1.3), (2.6) and (2.8) to get for $\varepsilon\to 0$ that
$$\begin{aligned}\nu_\varepsilon([z,\infty))&=\varepsilon'\nu\Big(\Big[\frac{\varepsilon''z}{\varepsilon},\infty\Big)\Big)\sim C_+\varepsilon'\Big(\frac{\varepsilon''z}{\varepsilon}\Big)^{-\alpha}l_\nu\Big(\frac{\varepsilon}{\varepsilon''z}\Big)\\&\sim\frac{C_+}{z^\alpha}\cdot\frac{l_\nu\big(\frac{\varepsilon}{\varepsilon''z}\big)}{l_\nu\big(\frac{\varepsilon}{\varepsilon''}\big)}\sim\frac{C_+}{z^\alpha}=\nu^{(\alpha)}([z,\infty)).\end{aligned}\tag{2.15}$$

2. Let $\delta>0$, $\varepsilon\in(0,1]$, $z>0$. If $\frac{\varepsilon''z}{\varepsilon}>M$ the $\nu_\varepsilon([z,\infty))=0$ see (2.1). If $0<\frac{\varepsilon''z}{\varepsilon}\leq M$ the we take into account (2.8) and (1.3) and apply Potter's theorem, see e.g. [2, Theorem 1.5.6] to get
$$\begin{aligned}\nu_\varepsilon([z,\infty))&=\varepsilon'\nu\Big(\Big[\frac{\varepsilon''z}{\varepsilon},\infty\Big)\Big)=\frac{\varepsilon'}{(\frac{\varepsilon''}{\varepsilon})^\alpha l_\nu(\frac{\varepsilon}{\varepsilon''})^{-1}}\cdot\frac{\nu([\frac{\varepsilon''z}{\varepsilon},\infty))}{C_+(\frac{\varepsilon''z}{\varepsilon})^{-\alpha}l_\nu(\frac{\varepsilon}{\varepsilon''z})}\cdot\frac{C_+(\frac{\varepsilon''z}{\varepsilon})^{-\alpha}l_\nu(\frac{\varepsilon}{\varepsilon''z})}{(\frac{\varepsilon''}{\varepsilon})^{-\alpha}l_\nu(\frac{\varepsilon}{\varepsilon''})}\\&\leq\sup_{\varepsilon\in(0,1]}\frac{\varepsilon'}{(\frac{\varepsilon''}{\varepsilon})^\alpha l_\nu(\frac{\varepsilon}{\varepsilon''})^{-1}}\cdot\sup_{y\in(0,M]}\frac{\nu([y,\infty))}{C_+y^{-\alpha}l_\nu(\frac{1}{y})}\cdot\frac{C_+}{z^\alpha}\cdot(z^{-\delta}\vee z^\delta)\\&=\frac{C(\delta,M)}{z^\alpha}\cdot(z^{-\delta}\vee z^\delta).\quad\square\end{aligned}$$

Theorem 2.6. *Suppose that* ε' *and* ε'' *satisfy conditions of Lemma* (2.4) *and let assumptions of Theorem 1.3 hold true. Then*

1. $$Z_\varepsilon\Rightarrow Z^{(\alpha)},\quad\varepsilon\to 0;\tag{2.16}$$
2. *there exists a weak limit*
$$Y^\varepsilon\Rightarrow Y,\quad\varepsilon\to 0,\tag{2.17}$$
which satisfies the SDE
$$Y(t)=\int_0^t\bar a(Y(s))\,\mathrm{d}s+b(0)Z^{(\alpha)}(t),\quad t\geq 0.\tag{2.18}$$
The process Y *diverges to* $\pm\infty$ *with the selection probabilities* $\bar p_\pm$ *defined in* (1.11).

Proof. 1. It is well known that in the case of Lévy processes convergence of marginal distributions implies the weak convergence in the Skorokhod space, see [8, Corollary VII.3.6].

For some $\mu\in\mathbb{R}$, the process Z has the Lévy–Khintchine representation
$$\ln\mathbf{E}\mathrm{e}^{\mathrm{i}\lambda Z(1)}=\mathrm{i}\mu\lambda+\int_{\mathbb{R}}\big(\mathrm{e}^{\mathrm{i}\lambda z}-1-\mathrm{i}\lambda z\big)\nu(\mathrm{d}z),\quad\lambda\in\mathbb{R},$$

whereas the rescaled process Z_ε has the Lévy–Khintchine representation

$$\begin{aligned}\ln \mathbf{E}\mathrm{e}^{\mathrm{i}\lambda Z_\varepsilon(1)} = \ln \mathbf{E}\mathrm{e}^{\mathrm{i}\frac{\lambda Z(\varepsilon')}{\varepsilon''/\varepsilon}} &= \frac{\mathrm{i}\mu\lambda\varepsilon'}{\varepsilon''/\varepsilon} + \varepsilon'\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}\frac{\lambda z}{\varepsilon''/\varepsilon}} - 1 - \frac{\mathrm{i}\lambda z}{\varepsilon''/\varepsilon}\Big)\nu(\mathrm{d}z)\\ &=: \mathrm{i}\mu_\varepsilon\lambda + \int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}\lambda z} - 1 - \mathrm{i}\lambda z)\nu_\varepsilon(\mathrm{d}z),\end{aligned}$$

with the jump measures ν_ε defined in (2.12).

Hence, the integration by parts formula, Lebesgue's dominated convergence theorem, (2.13) and (2.14) yield that for each $\lambda \in \mathbb{R}$

$$\begin{aligned}\int_{(0,\infty)}(\mathrm{e}^{\mathrm{i}\lambda z} - 1 - \mathrm{i}\lambda z)\nu_\varepsilon(\mathrm{d}z) &= -\mathrm{i}\lambda\int_{(0,\infty)}(\mathrm{e}^{\mathrm{i}\lambda z} - 1)\nu_\varepsilon([z,\infty))\,\mathrm{d}z\\ &\to -\mathrm{i}\lambda\int_{(0,\infty)}(\mathrm{e}^{\mathrm{i}\lambda z} - 1)\nu^{(\alpha)}([z,\infty))\,\mathrm{d}z\\ &= \int_{(0,\infty)}(\mathrm{e}^{\mathrm{i}\lambda z} - 1 - \mathrm{i}\lambda z)\nu^{(\alpha)}(\mathrm{d}z), \quad \varepsilon \to 0.\end{aligned} \tag{2.19}$$

The same convergence holds analogously for the negative tail.

Eventually it follows from the choice of ε' and ε'' (see the proof of Lemma 2.4) that

$$\mu_\varepsilon = \mu\cdot\frac{\varepsilon\cdot\varepsilon'}{\varepsilon''} \sim \mu\varepsilon' g_2\Big(\frac{1}{\varepsilon'}\Big) \to 0, \quad \varepsilon \to 0. \tag{2.20}$$

Therefore we obtain convergence of the characteristic functions

$$\mathbf{E}\mathrm{e}^{\mathrm{i}\lambda Z_\varepsilon(1)} \to \mathbf{E}\mathrm{e}^{\mathrm{i}\lambda Z^{(\alpha)}(1)},\ \varepsilon \to 0.$$

2. To show (2.17), we note that the SDE (2.18) has a unique weak solution (see [21, Theorem 3.1]) and the corresponding martingale problem is well-posed. The convergence $a_\varepsilon(y) \to \bar{a}(y)$, $\mu_\varepsilon b_\varepsilon(y) \to 0$, and $b_\varepsilon(y) = b(0) > 0$ as $\varepsilon \to 0$ holds point-wise and uniformly on compact intervals. Hence due to Lemma 2.5 we also have locally uniform convergence

$$\lim_{\delta\to 0}\sup_{\varepsilon\in(0,1]}\int_{|z|\le\delta}|zb_\varepsilon(y)|^2\,\nu_\varepsilon(\mathrm{d}z) = 0 \quad\text{and}\quad \lim_{\varepsilon\to 0}\int g(zb_\varepsilon(y))\,\nu_\varepsilon(\mathrm{d}z) = \int g(zb(0))\,\nu^{(\alpha)}(\mathrm{d}z)$$

for each continuous bounded g vanishing in a neighbourhood of 0. Eventually the weak convergence $Y^\varepsilon \Rightarrow Y$ follows by [8, Theorem IX.4.8, p. 556] from the convergence of the semimartingale characteristics.

Due to [14, Theorem 1.1] and Remark 1.2, the limiting process Y diverges to $\pm\infty$ with the selection probabilities $\bar{p}_\pm$. □

3 Estimates for the noise

In this section we get estimates for a growth rate of the noise term $\int_0^t b_\varepsilon(Y^\varepsilon(s-))\,\mathrm{d}Z_\varepsilon(s)$ as $t \to \infty$ that are uniform in ε. We start with the the following general result.

Lemma 3.1. *Let $\tilde{Z}$ be a zero mean Lévy process without a Gaussian component and with a jump measure ν such that for some $C > 0$ and $\gamma \in (1,2)$ it satisfies*

$$\int_{|z|>x}\nu(\mathrm{d}z) \le \frac{C}{x^\gamma}, \quad x \ge 1. \tag{3.1}$$

and

$$\int_{|z|\le 1} z^2\nu(\mathrm{d}z) \le C. \tag{3.2}$$

Then for any $\theta > 0$ and $\delta > 0$ there exists a generic constant $K = K(C, \gamma, \delta, \theta)$ such that for any predictable process $\{\sigma(t)\}_{t\geq 0}$, $|\sigma(t)| \leq 1$ a.s., we have

$$\mathbf{P}\Big(\sup_{t\geq 0}\frac{\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s)}{1+t^{\frac{1}{\gamma}+\delta}} > K\Big) \geq \theta. \tag{3.3}$$

Proof. Denote $T(x) := \nu((-x,x)^c)$, $x \geq 1$. With the help of the integration by parts and (3.1) we get for $x \geq 1$ that

$$\begin{aligned}\int_{|z|>x} |z|\,\nu(\mathrm{d}z) &= -\int_x^\infty z\,\mathrm{d}T(z) = -zT(z)\Big|_x^\infty + \int_x^\infty T(z)\,\mathrm{d}z\\ &\leq Cx^{1-\gamma} + \frac{C}{\gamma-1}x^{1-\gamma} = \frac{C\gamma}{\gamma-1}x^{1-\gamma}.\end{aligned} \tag{3.4}$$

Analogously, for $x \geq 1$

$$\int_{0<|z|\leq x} z^2\,\nu(\mathrm{d}z) \leq \int_{0<|z|\leq 1} z^2\,\nu(\mathrm{d}z) + \int_{1<|z|\leq x} z^2\,\nu(\mathrm{d}z) \leq 2C\frac{3-\gamma}{2-\gamma}x^{2-\gamma}. \tag{3.5}$$

To show (3.3), we follow the reasoning of [17]. Let us use the Lévy–Itô representation of the zero mean pure jump Lévy process process $\tilde Z$, namely for a Poisson random measure N with the compensator $\nu(\mathrm{d}z)\mathrm{d}t$ and the compensated Poisson random measure $\tilde N(\mathrm{d}z,\mathrm{d}s) = N(\mathrm{d}z,\mathrm{d}s) - \nu(\mathrm{d}z)\mathrm{d}t$ we write

$$\tilde Z(t) = \int_0^t \int z\,\tilde N(\mathrm{d}z,\mathrm{d}s).$$

For arbitrary $A \geq 1$ and $T > 0$ we estimate

$$\begin{aligned}&\mathbf{P}\Big(\sup_{t\in[0,T]}\Big|\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s)\Big| > A\Big) \leq \mathbf{P}\Big(\sup_{t\in[0,T]}\Big|\int_0^t \sigma(s)\int_{|z|\leq A} z\tilde N(\mathrm{d}z,\mathrm{d}s)\Big| > \frac{A}{3}\Big)\\ &+\mathbf{P}\Big(\int_0^T\int_{|z|>A} N(\mathrm{d}z,\mathrm{d}s) > 0\Big) + \mathbf{P}\Big(\int_0^T\int_{|z|>A}|z|\nu(\mathrm{d}z)\mathrm{d}s > \frac{A}{3}\Big) = I_1 + I_2 + I_3.\end{aligned} \tag{3.6}$$

By Doob's inequality and (3.5) we obtain

$$I_1 \leq \frac{36\int_0^T \mathbf{E}\sigma^2(s)\int_{|z|\leq A} z^2\nu(\mathrm{d}z)\mathrm{d}s}{A^2} \leq \frac{36T\int_{|z|\leq A} z^2\nu(\mathrm{d}z)}{A^2} \leq \frac{3-\gamma}{2-\gamma}\frac{72CT}{A^\gamma}. \tag{3.7}$$

The inequality $1 - \mathrm{e}^{-x} \leq x$, $x \geq 0$, and (3.1) imply that

$$I_2 = 1 - \exp\Big(-T\int_{|z|>A}\nu(\mathrm{d}z)\Big) \leq T\int_{|z|>A}\nu(\mathrm{d}z) \leq \frac{CT}{A^\gamma}. \tag{3.8}$$

The item I_3 equals 0 if $T\int_{|z|>A}|z|\,\nu(\mathrm{d}z)\,\mathrm{d}s \leq A/3$. By (3.4) this is true if $3CT\frac{\gamma}{\gamma-1}A^{-\gamma} \leq 1$. Hence for each $K > 0$ we have

$$\begin{aligned}&\mathbf{P}\Big(\sup_{t\geq 0}\frac{\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s)}{1+t^{\frac{1}{\gamma}+\delta}} > K\Big)\\ &\quad\leq \mathbf{P}\Big(\sup_{t\in[0,1]}\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s) > K\Big) + \sum_{n=0}^\infty \mathbf{P}\Big(\sup_{t\in[2^n,2^{n+1}]}\frac{\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s)}{1+t^{\frac{1}{\gamma}+\delta}} > K\Big)\\ &\quad\leq \mathbf{P}\Big(\sup_{t\in[0,1]}\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s) > K\Big) + \sum_{n=0}^\infty \mathbf{P}\Big(\sup_{t\in[2^n,2^{n+1}]}\frac{\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s)}{2^{n(\frac{1}{\gamma}+\delta)}} > K\Big)\\ &\quad\leq \mathbf{P}\Big(\sup_{t\in[0,1]}\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s) > K\Big) + \sum_{n=0}^\infty \mathbf{P}\Big(\sup_{t\in[0,2^{n+1}]}\int_0^t \sigma(s)\,\mathrm{d}\tilde Z(s) > K2^{n(\frac{1}{\gamma}+\delta)}\Big).\end{aligned}$$

Let us apply (3.6), (3.7), (3.8) to the terms in the last line. Note that all the respective items I_3 are zero if $K > K_0 = (6C\frac{\gamma}{\gamma-1})^{1/\gamma}$. Therefore for $C_1 = C(1+72\frac{3-\gamma}{2-\gamma})$ and $K > K_0$ we get

$$\mathbf{P}\Big(\sup_{t\geq 0}\frac{\int_0^t \sigma(s)\,\mathrm{d}\tilde{Z}(s)}{1+t^{\frac{1}{\gamma}+\delta}} > K\Big) \leq \frac{C_1}{K^\gamma} + \sum_{n\geq 0}\frac{C_1 2^{n+1}}{K^\gamma 2^{n\gamma(\frac{1}{\gamma}+\delta)}} = \frac{C_1}{K^\gamma}\Big(1+\frac{2^{2+\gamma\delta}}{2^{1+\gamma\delta}-1}\Big).$$

Choosing $K = K(C,\gamma,\delta,\theta)$ large enough we make the last probability less than θ. □

Corollary 3.2. *Let $\theta > 0$. Let $\mathbf{A}_b$, (2.1), (2.7) and (2.8) be satisfied. Let X^ε be a solution to (1.2) with any starting point, and let $Y^\varepsilon(t) = X^\varepsilon(\varepsilon' t)/\varepsilon''$, $t\geq 0$, be the rescaled process. Then for any $\theta > 0$, $T > 0$ and $\delta > 0$ there exists a generic constant $K = K(\alpha,\delta,\theta,T)$ such that for any $\varepsilon\in(0,1]$ we have*

$$\mathbf{P}\Big(\sup_{t\in[0,\frac{T}{\varepsilon'}]}\Big|\frac{\int_0^t b_\varepsilon(Y^\varepsilon(s-))\,\mathrm{d}Z_\varepsilon(s)}{1+t^{\frac{1}{\alpha}+\delta}}\Big| > K\Big) = \mathbf{P}\Big(\sup_{t\in[0,T]}\Big|\frac{\frac{\varepsilon}{\varepsilon''}\int_0^t b(X^\varepsilon(s-))\,\mathrm{d}Z(s)}{1+(\frac{t}{\varepsilon'})^{\frac{1}{\alpha}+\delta}}\Big| > K\Big) \leq \theta. \tag{3.9}$$

Proof. The uniform estimate (2.14) from Lemma 2.5 implies that for any $\gamma\in(1,\alpha)$ there is a constant $C > 0$ such that the inequalities

$$\int_{|z|>x}\nu_\varepsilon(\mathrm{d}z) \leq \frac{C}{x^\gamma},\ x\geq 1 \quad\text{and}\quad \int_{|z|\leq 1} z^2\nu_\varepsilon(\mathrm{d}z)\leq C,$$

hold uniformly over $\varepsilon\in(0,1]$.

The only difference between the statement of Lemma 3.1 and this corollary is that the processes $\{Z_\varepsilon\}$ and the process Z respectively are not necessarily centered and that the supremum is taken over a finite ε-dependent interval. Hence we have to estimate the impact of the deterministic drift. It is more convenient to treat the deterministic linear mean value component μt of Z, $\mu\in\mathbb{R}$. Indeed, for $\delta > 0$ due to (2.11) there is a constant $C_1 = C_1(\alpha,\delta)$ such that $\varepsilon/\varepsilon'' \leq C_1\cdot(\varepsilon')^{-\frac{1}{\alpha}-\delta}$ for $\varepsilon\in(0,1]$. Therefore we have

$$\begin{aligned}\sup_{t\in[0,T]}\Big|\frac{\frac{\varepsilon}{\varepsilon''}\mu\int_0^t b(X^\varepsilon(s-))\,\mathrm{d}s}{1+(\frac{t}{\varepsilon'})^{\frac{1}{\alpha}+\delta}}\Big| &\leq \sup_{t\in[0,T]}\frac{t\cdot\frac{\varepsilon}{\varepsilon''}\cdot|\mu|\cdot\|b\|}{1+(\frac{t}{\varepsilon'})^{\frac{1}{\alpha}+\delta}} \leq C_1\cdot|\mu|\cdot\|b\|\cdot\sup_{t\in[0,T]}\frac{t(\varepsilon')^{-\frac{1}{\alpha}-\delta}}{1+(\frac{t}{\varepsilon'})^{\frac{1}{\alpha}+\delta}}\\ &= C_1\cdot|\mu|\cdot\|b\|\cdot\sup_{t\in[0,T]}\frac{t^{1-\frac{1}{\alpha}-\delta}(\frac{t}{\varepsilon'})^{\frac{1}{\alpha}+\delta}}{1+(\frac{t}{\varepsilon'})^{\frac{1}{\alpha}+\delta}}\\ &\leq C_1\cdot|\mu|\cdot\|b\|\cdot T^{1-\frac{1}{\alpha}-\delta}\cdot\sup_{s\geq 0}\frac{s^{\frac{1}{\alpha}+\delta}}{1+s^{\frac{1}{\alpha}+\delta}} =: K_0(\alpha,\delta,T),\end{aligned}$$

that gives us the lower bound for K in (3.9). □

4 Exit of X^ε from the time-space box $[0,T_0\varepsilon']\times[-R\varepsilon'',R\varepsilon'']$

In the following Lemma we estimate the exit time of X^ε from a small neighborhood of 0. Here we essentially use the representation of X^ε in terms of Y^ε and establish the proper relations between its small time and small space behaviour. For $R > 0$ and a stochastic process X we denote the first exit times

$$\tau_R^X = \inf\{t\geq 0\colon X(t) > R\},\quad \tau_{-R}^X = \inf\{t\geq 0\colon X(t) < -R\}.$$

Lemma 4.1. *For any $\theta > 0$ and any $R > 0$ there is $T_0 = T_0(R) > 0$ such that*

$$\liminf_{\varepsilon\to 0}\mathbf{P}\Big(\tau_{R\varepsilon''}^{X^\varepsilon}\wedge\tau_{-R\varepsilon''}^{X^\varepsilon}\leq T_0\varepsilon'\Big)\geq 1-\theta. \tag{4.1}$$

Proof. Recall that $\varepsilon' = \varepsilon'(\varepsilon)$ and $\varepsilon'' = \varepsilon''(\varepsilon)$ are chosen according to Lemma 2.4. Note that due to rescaling (2.4)

$$\tau^{X^\varepsilon}_{\pm R\varepsilon''} = \varepsilon' \tau^{Y^\varepsilon}_{\pm R}, \qquad X^\varepsilon(\tau^{X^\varepsilon}_{\pm R\varepsilon''}) = \varepsilon'' Y^\varepsilon(\varepsilon' \tau^{Y^\varepsilon}_{\pm R}). \tag{4.2}$$

Let $R > 0$ and choose $\varepsilon_0 \in (0, 1]$ be such that

$$\begin{aligned} 0 < \frac{b(0)}{2} \leq \inf_{|y|\leq R,\, \varepsilon\in(0,\varepsilon_0]} b(\varepsilon'' y) &= \inf_{|y|\leq R,\, \varepsilon\in(0,\varepsilon_0]} b_\varepsilon(y) \\ &\leq \sup_{|y|\leq R,\, \varepsilon\in(0,\varepsilon_0]} b_\varepsilon(y) = \sup_{|y|\leq R,\, \varepsilon\in(0,\varepsilon_0]} b(\varepsilon'' y) \leq 2b(0). \end{aligned}$$

Also recall that $Z_\varepsilon \Rightarrow Z^{(\alpha)}$ by Theorem 2.6 so that Z_ε has unbounded jumps in the limit as $\varepsilon \to 0$. Let σ_ε be the first jump time such that $|\Delta Z_\varepsilon(\sigma_\varepsilon)| > 6R/b(0)$. Then $|\Delta Y^\varepsilon(\sigma_\varepsilon)| > 3R$ and hence $\tau^{Y^\varepsilon}_R \leq \sigma_\varepsilon$. Eventually (2.16) yields

$$\lim_{\varepsilon\to 0} \mathbf{E}\sigma_\varepsilon = \Big(\int_{|z|>6R/b(0)} \nu^{(\alpha)}(\mathrm{d}z) \Big)^{-1}$$

and the statement of the Lemma follows from (4.2) and Chebyshev's inequality. □

Corollary 4.2. *For any $\theta > 0$ there exist $R > 0$ large enough and $T_0 > 0$ such that*

$$\limsup_{\varepsilon\to 0} \Big| \mathbf{P}\Big(\tau^{X^\varepsilon}_{\mp R\varepsilon''} < \tau^{X^\varepsilon}_{\pm R\varepsilon''} \leq T_0 \varepsilon' \Big) - \bar{p}_\pm \Big| \leq \theta.$$

Proof. The result follows from (4.1), (4.2), (1.11) and Theorem 2.6. □

5 Behaviour of X^ε upon exit from the time-space box $[0, T_0\varepsilon'] \times [-R\varepsilon'', R\varepsilon'']$. Proof of the main result

For definiteness, let us consider only dynamics on the positive spatial half line $x \geq 0$.

Lemma 5.1. *1. For each $\gamma \in (0, \beta)$ there is $K_\gamma > 0$ such that for all $x \geq 1$ and $\varepsilon \in (0, 1]$*

$$a_\varepsilon(x) \geq K_\gamma x^{\beta-\gamma}. \tag{5.1}$$

2. For any $\kappa \in (0, 1)$ there exists $\mu \in (0, 1)$ such that

$$\inf_{\frac{x}{y}\in[1-\mu,1+\mu]} \frac{a_\varepsilon(x)}{a_\varepsilon(y)} > 1 - \kappa. \tag{5.2}$$

Proof. 1. Recall that according to Assumption $\mathbf{A}_a$ and Remark 2.2, $a(x) = x^\beta L_+(x \wedge 1)$, $x > 0$, and $a(0) = 0$. Hence

$$\begin{aligned} a_\varepsilon(x) = \frac{a(\varepsilon'' x)}{\varepsilon''/\varepsilon'} &= \varepsilon' \cdot (\varepsilon'')^{\beta-1} \cdot x^\beta L_+((\varepsilon'' x) \wedge 1) \\ &= \varepsilon' \cdot (\varepsilon'')^{\beta-1} \cdot l\Big(\frac{1}{\varepsilon''}\Big) \cdot x^\beta \cdot \frac{L_+((\varepsilon'' x) \wedge 1)}{A_+ l_a\big(\frac{1}{x\varepsilon''} \vee 1\big)} \cdot \frac{A_+ l_a\big(\frac{1}{x\varepsilon''} \vee 1\big)}{l_a\big(\frac{1}{\varepsilon''}\big)}. \end{aligned} \tag{5.3}$$

The equivalence (2.7) guarantees that $\varepsilon' \cdot (\varepsilon'')^{\beta-1} \cdot l_a(\frac{1}{\varepsilon''}) \geq C_1 > 0$ for some $C_1 > 0$ and $\varepsilon \in (0, 1]$.

Let $\gamma \in (0, \beta)$, $x \geq 1$ and $\varepsilon \in (0, 1]$. We consider two cases.

a) For $x\varepsilon'' < 1$, with the help of Potter's theorem [2, Theorem 1.5.6 (ii)] applied to the function l_a we get

$$a_\varepsilon(x) \geq C_1 \cdot x^\beta \cdot \inf_{0<y<1} \frac{L_+(y)}{A_+ l_a\big(\frac{1}{y}\big)} \cdot A_+ \cdot \frac{l_a\big(\frac{1}{x\varepsilon''}\big)}{l_a\big(\frac{1}{\varepsilon''}\big)} \geq C_2 \cdot x^{\beta-\gamma}$$

for some $C_2 = C_2(\gamma) > 0$.

b) For $x\varepsilon'' \geq 1$ applying Potter's theorem again we get

$$a_\varepsilon(x) \geq C_1 \cdot x^\beta \cdot \frac{L_+(1)}{l_a(1)} \cdot \frac{l_a(1)}{l_a\big(\frac{1}{\varepsilon''}\big)} \geq C_2 \cdot x^\beta \cdot (\varepsilon'')^\gamma \geq C_3 \cdot x^{\beta-\gamma}$$

for some $C_3 = C_3(\gamma)$, and (5.1) follows with $K_\gamma = C_2 \wedge C_3$.

2. To prove (5.2) we note that

$$\inf_{\frac{x}{y}\in[1-\mu,1+\mu]} \frac{a_\varepsilon(x)}{a_\varepsilon(y)} = \inf_{\frac{x}{y}\in[1-\mu,1+\mu]} \frac{a(x)}{a(y)} = (1-\mu)^\beta \cdot \inf_{\frac{x}{y}\in[1-\mu,1+\mu]} \frac{L_+(x\wedge 1)}{L_+(y\wedge 1)} = C(\mu)\cdot(1-\mu)^\beta,$$

where $0 < C(\mu) \to 1$ as $\mu \to 0$ by continuity of L_+ and Potter's bounds. Hence for any $\kappa \in (0,1)$, the estimate (5.2) holds for μ small enough. □

Lemma 5.2. *Let $\gamma \in (0,\beta)$. Then for any $y \geq 1$, $\kappa \in (0,1)$ and any $\varepsilon \in (0,1]$ the solution of the ODE*

$$\zeta^\varepsilon_\kappa(t;y) = y + (1-\kappa)\int_0^t a_\varepsilon(\zeta^\varepsilon_\kappa(s;y))\,\mathrm{d}s$$

satisfies

$$\zeta^\varepsilon_\kappa(t;y) \geq y + Kt^{\frac{1}{1-\beta+\gamma}},\ t\geq 0, \tag{5.4}$$

with a constant $K = K(\beta,\gamma,\kappa) > 0$.

Proof. Let $\gamma \in (0,\beta)$ be fixed. For $y \geq 1$ we use (5.1) and compare $\zeta^\varepsilon_\kappa(\cdot;y)$ with the solution of the auxiliary ODE

$$z_\kappa(t;y) = y + (1-\kappa)K_\gamma \int_0^t (z_\kappa(s;y))^{\beta-\gamma}\,\mathrm{d}s,\ t\geq 0.$$

This solution has the explicit form

$$z_\kappa(t;y) = \Big(y^{1-\beta+\gamma} + (1-\kappa)(1-\beta+\gamma)K_\gamma t\Big)^{\frac{1}{1-\beta+\gamma}}.$$

Hence the application of an elementary inequality $(a+b)^p \geq a^p + b^p$, $a,b\geq 0$, $p\geq 1$, yields (5.4) with some $K>0$. □

We need the following comparison theorem for solutions of integral equations.

Lemma 5.3. *Let for $T>0$ and $i=1,2$, the functions u_i be solutions (not necessarily unique) to the equations*

$$u_i(t) = u_i(0) + \int_0^t U_i(s,u_i(s))\,\mathrm{d}s,\ t\in[0,T].$$

Assume that $u_1(0) \geq u_2(0)$, $U_1(t,u_2(t)) > U_2(t,u_2(t))$, $t \in [0,T]$, and functions $t \mapsto U_i(t,u_i(t))$ are right-continuous. Then $u_1(t) \geq u_2(t)$, $t\in[0,T]$.

Proof. The proof of this Lemma is quite standard. Assume that there is $\tau = \inf\{t > 0\colon u_1(t) < u_2(t)\} \in [0,T]$. Then by continuity $u_1(\tau) = u_2(\tau)$ we necessarily have the inequality $D^+u_1(\tau) \leq D^+u_2(\tau)$ for the right Dini derivatives of the solutions. However since $t\mapsto U_i(t,u_i(t))$ are right-continuous, by assumption

$$D^+u_1(\tau) = U_1(\tau,u_1(\tau)) = U_1(\tau,u_2(\tau)) > U_2(\tau,u_2(\tau)) = D^+u_2(\tau),$$

and we obtain a contradiction. □

In the next Lemma we determine a lower bound for the process Y^ε starting sufficiently far from zero.

Lemma 5.4. *For any $\theta > 0$, $\kappa \in (0,1)$ and $T > 0$ there are $\mu = \mu(\kappa) \in (0,1)$ and $R = R(T,\kappa,\theta) \geq 1$ such that for any $\mathcal{F}_0$-measurable initial condition $Y^\varepsilon(0) > R$ a.s. and all $\varepsilon \in (0,1]$*

$$\mathbf{P}\Big(Y^\varepsilon(t) \geq (1-\mu)\zeta^\varepsilon_\kappa(t;Y^\varepsilon(0)),\ t\in[0,T/\varepsilon']\Big) \geq 1-\theta.$$

A similar estimate from above also holds for $Y^\varepsilon(0) < -R$ a.s.

Proof. For $\varepsilon \in (0,1]$ let

$$g^\varepsilon(t) := \int_0^t b_\varepsilon(Y^\varepsilon(s-))\,\mathrm{d}Z_\varepsilon(s), \quad \tilde{Y}^\varepsilon(t) := Y^\varepsilon(t) - g^\varepsilon(t).$$

Then $\tilde{Y}^\varepsilon(t)$ satisfies the integral equation

$$\tilde{Y}^\varepsilon(t) = Y^\varepsilon(0) + \int_0^t a_\varepsilon(\tilde{Y}^\varepsilon(s) + g^\varepsilon(s))\,\mathrm{d}s.$$

Choose $\gamma \in (0,\beta)$ small enough such that $\frac{1}{1-\beta+\gamma} > \frac{1}{\alpha}$. For $\theta > 0$ fixed, we apply Corollary 3.2 and find a constant $K_1 = K_1(T,\beta,\gamma,\theta) > 0$ such that for all $\varepsilon \in (0,1]$ and any initial starting point $Y^\varepsilon(0) \in \mathbb{R}$

$$\mathbf{P}\Big(\sup_{t\in[0,\frac{T}{\varepsilon'}]}\Big|\frac{g^\varepsilon(t)}{1+t^{\frac{1}{1-\beta+\gamma}}}\Big| \leq K_1\Big) \geq 1-\theta. \tag{5.5}$$

Consequently, for any $\kappa \in (0,1)$ and any $y \geq 1$ with the help of (5.4) we get

$$\mathbf{P}\Big(\sup_{t\in[0,\frac{T}{\varepsilon'}]}\Big|\frac{g^\varepsilon(t)}{\zeta^\varepsilon_\kappa(t;y)}\Big| \leq \frac{K_1(1+t^{\frac{1}{1-\beta+\gamma}})}{y+Kt^{\frac{1}{1-\beta+\gamma}}}\Big) \geq 1-\theta.$$

Let $\mu = \mu(\kappa) \in (0,1)$ be such that (5.2) holds. For this μ choose $R \geq 1$ such that $\sup_{t\geq 0} \frac{K_1(1+t^{\frac{1}{1-\beta+\gamma}})}{R+Kt^{\frac{1}{1-\beta+\gamma}}} \leq \mu$. Then

$$\mathbf{P}\Big(\sup_{t\in[0,\frac{T}{\varepsilon'}]}\Big|\frac{g^\varepsilon(t)}{\zeta^\varepsilon_\kappa(t;R\vee Y^\varepsilon(0))}\Big| \leq \mu\Big) \geq 1-\theta.$$

In other words, for $Y^\varepsilon(0) \geq R$ a.s. we have

$$\mathbf{P}\Big(a_\varepsilon(\zeta^\varepsilon_\kappa(t;Y^\varepsilon(0)) + g^\varepsilon(t)) > (1-\kappa)a_\varepsilon(\zeta^\varepsilon_\kappa(t;Y^\varepsilon(0))),\ t\in[0,T/\varepsilon']\Big) \geq 1-\theta.$$

Therefore the comparison Lemma 5.3 applied to $u_1 = \tilde{Y}^\varepsilon$ and $u_2 = \zeta^\varepsilon_\kappa(\cdot;Y^\varepsilon(0))$ yields

$$\mathbf{P}\Big(\tilde{Y}^\varepsilon(t) \geq \zeta^\varepsilon_\kappa(t;Y^\varepsilon(0)),\ t\in[0,T/\varepsilon']\Big) \geq 1-\theta$$

and hence

$$\mathbf{P}\Big(Y^\varepsilon(t) \geq (1-\mu)\zeta^\varepsilon_\kappa(t;Y^\varepsilon(0)),\ t\in[0,T/\varepsilon']\Big) \geq 1-\theta. \quad \square$$

Proof of Theorem 1.3. Notice that for each $\kappa \in (0,1)$, $\varepsilon \in (0,1]$ and $y > 0$ the function $\hat{\zeta}^\varepsilon_\kappa(t;y) := \varepsilon''\zeta^\varepsilon_\kappa(t/\varepsilon';y)$, $t \geq 0$, satisfies the equation

$$\hat{\zeta}^\varepsilon_\kappa(t;y) = \varepsilon'' y + (1-\kappa)\int_0^t a(\hat{\zeta}^\varepsilon_\kappa(s;y))\,\mathrm{d}s.$$

Hence according to (1.6) and (1.7)

$$\hat{\zeta}^{\varepsilon}_{\kappa}(t;y) = X^0_{\varepsilon''y}((1-\kappa)t) \geq x^+((1-\kappa)t), \quad t\geq 0. \tag{5.6}$$

Let $\mu=\mu(\kappa)\in(0,1)$ be chosen to satisfy (5.2).

Since the Lévy process Z_ε is strong Markov, analogously to Lemma 3.1 and Corollary 3.2 we have the following. For any T, δ, $\theta>0$ there exists a generic constant $K=K(T,\alpha,\delta,\theta)$ such that for any $\varepsilon\in(0,1]$ the estimate

$$\mathbf{P}\Big(\sup_{t\in[0,\frac{T}{\varepsilon'}]}\Big|\frac{\int_\tau^{\tau+t} b_\varepsilon(Y^\varepsilon(s-))\,\mathrm{d}Z_\varepsilon(s)}{1+t^{\frac{1}{\alpha}+\delta}}\Big| \leq K\Big)\geq 1-\theta$$

holds for any stopping time τ. It follows from Corollary 3.2, Lemma 4.1, Corollary 4.2, Lemma 5.4, and (5.6) that for any $\theta>0$ and $T>0$ there are $R>0$ and $T_0>0$ large enough such that

$$\liminf_{\varepsilon\to 0}\mathbf{P}\Big(\tau^{X^\varepsilon}_{\pm R\varepsilon''}<\tau^{X^\varepsilon}_{\mp R\varepsilon''}\leq T_0\varepsilon', X^\varepsilon(\tau^{X^\varepsilon}_{\pm R\varepsilon''}+t)\geq(1-\mu)x^\pm((1-\kappa)t),\ t\in[0,T]\Big)\geq \bar{p}_\pm-\theta.$$

In the last formula, Lemma 5.4 is applied to the process $Y^\varepsilon(t+\tau^{Y^\varepsilon}_{-R}\wedge\tau^{Y^\varepsilon}_{R})$, $t\geq 0$, whose initial value belongs to the set $[-R,R]^c$, see (4.2). Corollary 3.2 holds true since $\tau^{Y^\varepsilon}_{\pm R}$ are stopping times.

Since $\bar{p}_-+\bar{p}_+=1$ and any limit law of $\{X^\varepsilon\}$ is supported by the solutions $x^\pm$ (see Lemma 2.3) we get that for each $\delta>0$

$$\limsup_{\varepsilon\to 0}\Big|\mathbf{P}\Big(\sup_{t\in[0,T_0\varepsilon'+T]}|X^\varepsilon(t)-x^\pm(t)|\leq\delta\Big)-p_\pm\Big|\leq\theta,$$

and the proof is finished. □

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