Probability to be positive for the membrane model in dimensions 2 and 3

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Dedicated to Amir Dembo on the occasion of his sixtieth birthday.

Abstract

We consider the membrane model on a box $V_N \subset \mathbb{Z}^n$ of size $(2N + 1)^n$ with zero boundary condition in the subcritical dimensions $n = 2$ and $n = 3$. We show optimal estimates for the probability that the field is positive in a subset D_N of V_N. In particular we obtain for $D_N = V_N$ that the probability to be positive on the entire domain is exponentially small and the rate is of the order of the surface area N^{n-1}.

1 Introduction

The membrane model is the centred Gaussian field indexed by (a subset of) $\mathbb{Z}^n, n \geq 1$, whose covariance matrix is given by the Green's function of the discrete Bilaplacian. It is closely related to the well-known discrete Gaussian free field, or gradient model, whose covariance is the Green's function of the discrete Laplacian. Both of these models are considered to describe interfaces in the context of statistical physics. The particular motivation for studying the membrane model stems from physical surfaces that tend to have constant curvature, [19, 13, 22]. The two models have many features in common.

One example is that there is a critical dimension ($n = 2$ for the gradient model, $n = 4$ for the membrane model), such that the variances are unbounded in the subcritical dimensions, logarithmically divergent in the critical dimension and bounded in the supercritical dimensions. See e.g. [27] for a more general overview.

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A particular feature of the gradient model is the existence of a random walk representation, which allows relatively easy estimates on the covariances, and provides proofs for correlation inequalities such as the FKG inequality. In the membrane model, such a random walk representation is present only in certain special cases, see [15]. This makes the derivation of bounds on the covariances much harder, and moreover, some widely used correlation inequalities do not hold for the membrane model. In [20], Müller and Schweiger obtained very precise estimates on the Green's function of the discrete Bilaplacian in the subcritical dimensions 2 and 3. These results in particular imply that the membrane model is Hölder continuous, [20, 6]. Here we use the estimates to provide bounds for the probability of the interface to be positive on certain subsets of its domain.

Such results are related to the phenomenon of entropic repulsion, which refers to the observation that some interfaces are repelled by a hard wall to a height which is determined by the fluctuations of the field. Mathematically speaking, this amounts to considering the field conditional on the event of being positive on a specified part of the domain. The field then needs to accommodate its fluctuations, so its local averages will increase. We speak of entropic repulsion if the order by which the field increases is strictly larger than the order of the square root of the variances of the original field, [17, 12].

For the Gaussian gradient model entropic repulsion was proved in [3, 2, 9, 10]. For the membrane model, entropic repulsion was shown for $n \geq 4$ by Sakagawa and by Kurt [23, 14, 15]. In dimension $n = 1$ the model corresponds to an integrated Gaussian random walk, see [5]. There the probability to be positive is of order $N^{-1/2}$ as was shown by Denisov and Wachtel [8]. See also [7] for the one-sided case.

We consider here the membrane model defined on a box of side-length $2N + 1$, $N \in \mathbb{N}$, and focus on dimensions $n = 2, 3$. In this case only a first result by Sakagawa [24] is available.

1.1 Main results

Let $V = [-1, 1]^n$ and $V_N = NV \cap \mathbb{Z}^n$ with $n \in \mathbb{N}^+$ and $N \in \mathbb{N}^+$. We consider the Hamiltonian $H_N(\psi) = \frac{1}{2} \sum_{x \in \mathbb{Z}^n} |\Delta \psi_x|^2$, where Δ is the discrete Laplacian and $\psi \in \mathbb{R}^{V_N}$ is a function on V_N, extended by 0 to all of $\mathbb{Z}^n$. The associated Gibbs measure

$$\mathbb{P}_N(\mathrm{d}\psi) = \frac{1}{Z_N} \exp(-H_N(\psi)) \prod_{x \in V_N} \mathrm{d}\psi_x \prod_{x \in \mathbb{Z}^n \setminus V_N} \delta_0(\mathrm{d}\psi_x) \tag{1.1}$$

is then the distribution of a Gaussian random field on $\mathbb{Z}^n$ with 0 boundary data, the so-called membrane model. We care about the subcritical case $n \in \{2, 3\}$, and we are interested in the event $\Omega_{D_N,+} = \{\psi : \psi_x \geq 0 \ \forall x \in D_N\}$, where $D_N \subset V_N$, as well as the behaviour of ψ conditioned on $\Omega_{D_N,+}$.

Our main result is the following.

Theorem 1.1. *Let $n = 2$ or $n = 3$. There are constants C, c such that for all $N \in \mathbb{N}^+$, $0 \leq L \leq N$,*

$$e^{-C \frac{N^{n-1}}{(L+1)^{n-1}}} \leq \mathbb{P}_N\left(\Omega_{V_{N-L},+}\right) \leq e^{-c \frac{N^{n-1}}{(L+1)^{n-1}}}. \tag{1.2}$$

A first result in this direction was already established by Sakagawa [24] who proved that for every $x \in V$ there is a small neighbourhood B_x such that $\mathbb{P}_N(\Omega_{NB_x,+}) > c$ for some (universal) constant c.

Let us emphasize two important special cases of our theorem, which will help motivate its statement. We first consider the case $D_N = V_{\delta N}$ for $\delta \in (0, 1)$, where the hard wall stays away from the boundary. In that case the fact that the membrane model is Hölder

continuous suggests that the field has a decent chance to be positive if it is uniformly positive on a sufficiently dense set of lattice points of bounded cardinality. Thus the probability that ψ is positive on $D_N = V_{\delta N}$ should be comparable to the probability of uniform positivity on that dense set, and hence bounded away from zero. Indeed, Theorem 1.1 implies the following corollary.

Corollary 1.2. *Let $n = 2$ or $n = 3$. For $\delta \in (0,1)$ there is a constant $c_\delta > 0$ such that*

$$c_\delta \leq \mathbb{P}_N(\Omega_{V_{\delta N},+}) \leq \frac{1}{2}. \tag{1.3}$$

When $D_N = V_N$, the situation is somewhat different. While the Hölder continuity holds up to the boundary, the ψ_x for x near the boundary are only weakly correlated and behave almost like independent random variables. This suggests that the probability to be positive on all of V_N can at best scale like $e^{-cN^{n-1}}$ (note that the number of points of distance 1 to the boundary is of the order N^{n-1}). On the other hand, if the field is positive at all near-boundary points it gets pushed up in the interior quite a bit, and so the probability to be positive everywhere should be of the same order.

Indeed, another particular case of Theorem 1.1 is an estimate for $\mathbb{P}_N(\Omega_{V_N,+})$.

Corollary 1.3. *Let $n = 2$ or $n = 3$. There are constants C, c such that*

$$e^{-CN^{n-1}} \leq \mathbb{P}_N\left(\Omega_{V_N,+}\right) \leq e^{-cN^{n-1}}. \tag{1.4}$$

We expect this result to be true for the membrane model and the gradient model in any dimension $n \geq 2$. For the gradient model a stronger result has been shown for $n \geq 3$ in [9, Theorem 4.1]. Note that the behaviour for general $L \geq 1$ in Theorem 1.1 is different for the gradient model in dimension $n \geq 3$.

We give a proof of the lower and upper bound in Theorem 1.1 in Section 3 and 4, respectively.

1.2 Implications for entropic repulsion

Corollary 1.2 has some easy implications on the behaviour of the field when conditioned on $\Omega_{V_{\delta N},+}$. To state them precisely we need some preparation.

We define the interpolation $I_N \colon \mathbb{R}^{V_N} \to C^{0,1}([-1,1]^n)$ by $I_N f(x) = N^{-\frac{4-n}{2}} f(Nx)$ for $x \in \left(\frac{1}{N}\mathbb{Z}\right)^n \cap [-1,1]^n$, and interpolated piecewise affinely on simplices for other values of x. As the proof of [6, Theorem 2.1] shows, the pushforward measures $I_N \# \mathbb{P}_N$ converge weakly in $C^{0,\alpha}([-1,1]^n)$ for any $\alpha < \frac{4-n}{2}$ to a limit law $\mathbb{P}_\infty$. The limit $\mathbb{P}_\infty$ is the continuum Bilaplace field, i.e., the centred Gaussian field whose covariance is the Green's function of the continuum Bilaplace operator on V. Now Corollary 1.2 implies that the laws $I_N \# \mathbb{P}_N$ still converge when one conditions on $\Omega_{V_{\delta N},+}$. Indeed, if we introduce the event $\Omega^*_{D,+} = \{u \in C^{0,\alpha}([-1,1]^n) \colon u(x) \geq 0 \ \forall x \in D\}$ for $D \subset [-1,1]^n$, we have the following result.

Corollary 1.4. *Let $n = 2$ or $n = 3$, and $\delta \in (0,1)$. Then $I_N \# \mathbb{P}_N(\cdot \mid \Omega_{V_{\delta N},+})$ converges weakly in $C^{0,\alpha}([-1,1]^n)$ for any $\alpha < \frac{4-n}{2}$ to $\mathbb{P}_\infty\left(\cdot \mid \Omega^*_{\delta V,+}\right)$. In particular, we have*

$$\lim_{N \to \infty} \mathbb{E}_N\left(N^{-\frac{4-n}{2}} \max_{x \in V_N} \psi_x \ \Big| \ \Omega_{V_{\delta N},+}\right) < \infty. \tag{1.5}$$

The corollary follows from the facts that $\mathbb{P}_\infty\left(\Omega^*_{\delta V,+}\right)$ is a continuous function of δ and that $\Omega^\infty_{\delta V,+}$ is a continuity set for $\mathbb{P}_\infty$ (these both follow from [1, Corollary 4.4.2]). Note that the second point combined with the convergence of $I_N \# \mathbb{P}_N \to \mathbb{P}_\infty$ and Corollary 1.2 implies that $\mathbb{P}_\infty\left(\Omega^*_{\delta V,+}\right) > 0$ so that the conditioned measure $\mathbb{P}_\infty\left(\cdot \mid \Omega^*_{\delta V,+}\right)$ is well-defined.

This corollary shows that there is no entropic repulsion when conditioning on $\Omega_{V_{\delta N},+}$.

We conjecture that a similar result remains true if we condition on $\Omega_{V_N,+}$. However, due to the fact that the probability of $\Omega_{V_N,+}$ is exponentially small this is a difficult problem even in dimension one.

Conjecture 1.5. *For $n = 2$ and $n = 3$ the measures $I_N \# \mathbb{P}_N \left(\cdot \mid \Omega_{V_N,+} \right)$ converge weakly in $C^{0,\alpha}([-1,1]^n)$ for any $\alpha < \frac{4-n}{2}$ to some limiting measure. In particular,*

$$\lim_{N \to \infty} \mathbb{E}_N \left(N^{-\frac{4-n}{2}} \max_{x \in V_N} \psi_x \, \middle| \, \Omega_{V_N,+} \right) < \infty. \tag{1.6}$$

As an analogue to this conjecture one can consider the gradient model in one dimension (i.e. the random walk on $\{-N, -N+1, \ldots, N\}$ with i.i.d. Gaussian increments conditioned to be zero at its endpoints). It is well-known that this model, suitably rescaled, converges weakly in $C^{0,\alpha}([-1,1])$ for $\alpha < \frac{1}{2}$ to a Brownian bridge. Moreover, if one conditions the walk to be non-negative it converges weakly in $C^0([-1,1])$ to a Brownian excursion (see [4] and the references therein). Similar results (in particular a local limit theorem for the conditioned field) have also been shown for the membrane model in one dimension (at least if one only considers zero boundary data on one end of the interval), see [8].

1.3 Notation

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. We use the discrete forward derivative $\nabla_i u(x) = u(x + e_i) - u(x)$ and the discrete backward derivative $\nabla_{-i} u(x) = u(x) - u(x - e_i)$. Then $\Delta u(x) = \sum_{i=}^n \nabla_{-i} \nabla_i u(x)$ denotes the discrete Laplacian, and $\nabla u(x) = (\nabla_i u(x))_{i=1}^n$ is the discrete (forward) gradient.

We denote the L^2-norm of u by $\|u\|_{L^2}^2 = \sum_{x \in \mathbb{Z}^n} u(x)^2$, and the L^2-scalar product by $(u,v)_{L^2} = \sum_{x \in \mathbb{Z}^n} u(x)v(x)$.

For $x \in \mathbb{Z}^n$ let $d_N(x) = \text{dist}_\infty(x, \mathbb{Z}^n \setminus V_N)$ be the distance to the boundary of V_N.

For a set A we denote by $|A|$ its cardinality.

In the following c, C and C' denote constants that may change from line to line, but are always independent of N and L.

2 Preliminaries

Let us recall the relevant results that will be used in the proof of the main theorems. Let G_N be the Green's function of Δ^2 on V_N with 0 boundary data outside V_N, i.e. $G_N(\cdot, y) = 0$ if $y \notin V_N$ and

$$\begin{aligned} \Delta^2 G_N(\cdot, y) &= \delta_y \quad \text{in } V_N \\ G_N(\cdot, y) &= 0 \quad \text{outside } V_N \end{aligned} \tag{2.1}$$

if $y \in V_N$. The Green's function G_N agrees with the covariance matrix of ψ, i.e. we have that $\text{Cov}_N(\psi_x, \psi_y) = G_N(x, y)$, see also [15]. Our proofs are based on the estimates for the Green's function G_N recently found in [20].

Theorem 2.1. *Let $n = 2$ or $n = 3$. Then we have for any $x, y \in V_N$*

$$c d_N(x)^{4-n} \le G_N(x,x) \le C d_N(x)^{4-n}, \tag{2.2}$$

$$|\nabla_x G_N(x,y)| \le C d_N(x)^{3-n}, \tag{2.3}$$

$$|G_N(x,x) - G_N(x,y)| \le C d_N(x)^{3-n} |x-y|_\infty, \tag{2.4}$$

$$|G_N(x,y)| \le C \frac{d_N(x)^2 d_N(y)^2}{(|x-y|_\infty + 1)^n}, \tag{2.5}$$

where ∇_x denotes the discrete gradient with respect to x.

Proof. The estimates (2.2), (2.3) and (2.5) are from [20, Theorem 1.1], while (2.4) follows from (2.3) by discrete integration along a path from x to y. □

The lower bound relies on Dudley's inequality proved in [11]. To state the inequality we introduce the following two notions. For a Gaussian process $(X_t)_{t \in T}$ we define the pseudometric d_X by

$$d_X(s,t) = \sqrt{\mathbb{E}(|X_s - X_t|^2)}. \tag{2.6}$$

The entropy number $\mathcal{N}(T, d_X, r)$ is the minimal number of open balls of radius r in the d_X metric that are needed to cover T.

Theorem 2.2. *Let $(X_t)_{t \in T}$ be a centred Gaussian process. Then*

$$\mathbb{E}\left(\sup_{t \in T} X_t\right) \leq 24 \int_0^\infty \sqrt{\ln \mathcal{N}(T, d_X, r)} \, dr. \tag{2.7}$$

Remark 2.3. The theorem is true for arbitrary sets T if one defines the supremum appropriately, see e.g. [26]. Since we only apply it to finite index sets we do not discuss this issue here any further.

We also use the Gaussian correlation inequality due to Royen [21] (see also [16]).

Theorem 2.4. *Let ν be a centred Gaussian measure on $\mathbb{R}^m$ and $K, L \subset \mathbb{R}^m$ be closed, symmetric and convex. Then*

$$\nu(K \cap L) \geq \nu(K)\nu(L). \tag{2.8}$$

Finally, we recall a correlation inequality due to Li and Shao [18, Lemma 5.1] that will be used in the proof of the upper bound.

Lemma 2.5. *Let $m \in \mathbb{N}$, and $X = (X_1, \ldots X_m)$, $Y = (Y_1, \ldots Y_m)$ be Gaussian random vectors with mean 0 and positive definite covariance matrices Σ_X, Σ_Y, and let $\mathbb{P}$ denote their joint measure. If $\Sigma_Y \geq \Sigma_X$ (in the sense of symmetric matrices, i.e., $\Sigma_Y - \Sigma_X$ is positive semidefinite) then for every Borel set $F \subset \mathbb{R}^m$*

$$\mathbb{P}(Y \in F) \geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} P(X \in F). \tag{2.9}$$

For the convenience of the reader we repeat the short proof.

Proof. Let f_X, f_Y be the densities of X and Y. The assumption $\Sigma_Y \geq \Sigma_X$ implies that $\Sigma_X^{-1} \geq \Sigma_Y^{-1}$ and hence $(x, \Sigma_X^{-1}x) \geq (x, \Sigma_Y^{-1}x)$ for all $x \in \mathbb{R}^m$. Therefore:

$$\begin{aligned}
f_Y(x) &= \frac{1}{(2\pi)^{\frac{m}{2}}(\det \Sigma_Y)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x, \Sigma_Y^{-1}x)\right) \\
&\geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \frac{1}{(2\pi)^{\frac{m}{2}}(\det \Sigma_X)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x, \Sigma_X^{-1}x)\right) \\
&= \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} f_X(x).
\end{aligned} \tag{2.10}$$

Then

$$\mathbb{P}(Y \in F) = \int_F f_Y(x)\,dx \geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \int_F f_X(x)\,dx = \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}(X \in F). \tag{2.11}$$

□

3 Lower bounds

Let

$$\Omega_{V_{N-L},\infty} := \left\{ \psi \colon |\psi_x| \leq d_N(x)^{\frac{4-n}{2}} \ \forall x \in V_{N-L} \right\} \tag{3.1}$$

be the event that ψ is uniformly small on V_{N-L}.

If ψ was $C^{0,\frac{4-n}{2}}$-Hölder continuous with Hölder constant ≤ 1 with probability bounded below uniformly in N, this event would have a positive probability uniformly in N and L. Now ψ is only $C^{0,\frac{4-n}{2}-\varepsilon}$-Hölder continuous (see [20], [6]), so we cannot expect a lower bound independent of N. Instead, we prove in Subsection 3.2 that the probability of $\Omega_{V_{N-L},\infty}$ is bounded below by $e^{-c\frac{N^{n-1}}{(L+1)^{n-1}}}$. Then, using a change of measure argument, we show in Subsection 3.3 that, given $f \colon V_N \to \mathbb{R}$, we have

$$\mathbb{P}_N(f + \Omega_{V_{N-L},\infty}) \geq e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2} \mathbb{P}_N(\Omega_{V_{N-L},\infty}). \tag{3.2}$$

Suppose now that we can find a function f such that $f(x) \geq d_N(x)^{\frac{4-n}{2}}$ for $x \in V_{N-L}$ and such that $\|\Delta f\|_{L^2}^2 \leq C\frac{N^{n-1}}{(L+1)^{n-1}}$. Then $\Omega_{V_{N-L},+} \supset f + \Omega_{V_{N-L},\infty}$ and thus (3.2) will imply that

$$\begin{aligned} \mathbb{P}_N(\Omega_{V_{N-L},+}) &\geq \mathbb{P}_N(f + \Omega_{V_{N-L},\infty}) \\ &\geq e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2} \mathbb{P}_N(\Omega_{V_{N-L},\infty}) \\ &\geq e^{-C\frac{N^{n-1}}{(L+1)^{n-1}}} \mathbb{P}_N(\Omega_{V_{N-L},\infty}). \end{aligned} \tag{3.3}$$

Combined with a lower bound on $\mathbb{P}_N(\Omega_{V_{N-L},\infty})$ this implies the lower bound in Theorem 1.1. In Lemma 3.4 we construct an f with the desired properties.

3.1 Local smallness of the field

We first prove that locally the field is small with a positive probability. For $x_0 \in V_N$ and $\gamma > 0$ we define the set

$$A_{x_0,\gamma} := \{x \in V_N \colon |x - x_0|_\infty \leq \gamma d_N(x_0)\}. \tag{3.4}$$

Lemma 3.1. *Let $n = 2$ or $n = 3$. There is a pair of constants $\gamma, \delta > 0$ with the following property: For all $x_0 \in V_N$ the following estimate holds*

$$\mathbb{P}_N\left(\psi \colon |\psi_x| \leq d_N(x)^{\frac{4-n}{2}} \ \forall x \in A_{x_0,\gamma}\right) \geq \delta. \tag{3.5}$$

Proof. We apply Theorem 2.2 to the Gaussian process ψ distributed according to $\mathbb{P}_N$. We assume $\gamma < \frac{1}{2}$ so that $x \in A_{x_0,\gamma}$ implies

$$\frac{d_N(x_0)}{2} \leq d_N(x) \leq \frac{3d_N(x_0)}{2}. \tag{3.6}$$

Therefore we will always estimate distances to the boundary for $x \in A_{x_0,\gamma}$ by $d_N(x_0)$ in the following. The bound (2.4) implies

$$\mathbb{E}_N(\psi_x - \psi_y)^2 \leq |G_N(x,x) - G_N(x,y)| + |G_N(y,y) - G_N(y,x)| \leq \Theta d_N(x_0)^{3-n}|x - y|_\infty \tag{3.7}$$

for $x, y \in A_{x_0,\gamma}$ and some $\Theta > 0$. Therefore we estimate the Gaussian pseudometric defined in (2.6) by

$$d_\psi(x,y) \leq \sqrt{\Theta d_N(x_0)^{3-n}|x - y|_\infty}. \tag{3.8}$$

This implies that for $r > 0$ and $x, y \in A_{x_0,\gamma}$ such that $|x - y|_\infty \leq \frac{r^2}{\Theta d_N(x_0)^{3-n}}$ we have

$$d_\psi(x, y) \leq r. \tag{3.9}$$

In particular $B_\infty \left(x, \frac{r^2}{\Theta d_N(x_0)^{3-n}} \right) \subset B_{d_\psi}(x, r)$ and therefore

$$\mathcal{N}(A_{x_0,\gamma}, d_\psi, r) \leq \left\lceil \frac{\gamma d_N(x_0)}{\frac{r^2}{\Theta d_N(x_0)^{3-n}}} \right\rceil^n \leq 1 \vee \left(\frac{2\gamma\Theta d_N(x_0)^{4-n}}{r^2} \right)^n. \tag{3.10}$$

Then Theorem 2.2 implies

$$\mathbb{E}_N \left(\sup_{x \in A_{x_0,\gamma}} \psi_x \right) \leq 24 \int_0^{\sqrt{2\gamma\Theta d_N(x_0)^{4-n}}} \sqrt{\ln \left(\frac{2\gamma\Theta d_N(x_0)^{4-n}}{r^2} \right)^n} \, \mathrm{d}r$$

$$\leq 24 d_N(x_0)^{\frac{4-n}{2}} \sqrt{2\gamma\Theta n} \int_0^1 \sqrt{-2\ln r} \, \mathrm{d}r \leq \Lambda\sqrt{\gamma} d_N(x_0)^{\frac{4-n}{2}} \tag{3.11}$$

where Λ only depends on n. If we take $\gamma = (16\Lambda)^{-2}$ we obtain

$$\mathbb{E}_N \left(\sup_{x \in A_{x_0,\gamma}} \psi_x \right) \leq \frac{1}{16} d_N(x_0)^{\frac{4-n}{2}}. \tag{3.12}$$

Define the oscillation of a function f on a set T as usual by

$$\operatorname{osc}_T f = \sup_T f - \inf_T f. \tag{3.13}$$

Since ψ_x is a centred process (3.12) implies

$$\mathbb{E}_N \left(\operatorname{osc}_{A_{x_0,\gamma}} \psi_x \right) \leq \frac{1}{8} d_N(x_0)^{\frac{4-n}{2}}. \tag{3.14}$$

This implies that

$$\mathbb{P}_N \left(\operatorname{osc}_{A_{x_0,\gamma}} \psi_x \leq \tfrac{1}{4} d_N(x_0)^{\frac{4-n}{2}} \right) \geq \frac{1}{2}. \tag{3.15}$$

Note that we have the inclusions

$$\left\{ \psi : |\psi_x| \leq d_N(x)^{\frac{4-n}{2}} \ \forall x \in A_{x_0,\gamma} \right\} \supset \left\{ \psi : |\psi_x| \leq \tfrac{1}{2} d_N(x_0)^{\frac{4-n}{2}} \ \forall x \in A_{x_0,\gamma} \right\}$$

$$\supset \left\{ \psi : \operatorname{osc}_{A_{x_0,\gamma}} \psi_x \leq \tfrac{1}{4} d_N(x_0)^{\frac{4-n}{2}} \right\} \cap \left\{ \psi : |\psi_{x_0}| \leq \tfrac{1}{4} d_N(x_0)^{\frac{4-n}{2}} \right\}. \tag{3.16}$$

Now the Gaussian correlation inequality (2.8) together with (2.2) imply that

$$\mathbb{P}_N \left(\left\{ \psi : |\psi_x| \leq d_N(x)^{\frac{4-n}{2}} \ \forall x \in A_{x_0,\gamma} \right\} \right) \geq \frac{1}{2} \mathbb{P}_N \left(|\psi_{x_0}| \leq \tfrac{1}{4} d_N(x_0)^{\frac{4-n}{2}} \right) \geq \delta \tag{3.17}$$

for some fixed $\delta > 0$. $\qquad\square$

Remark 3.2. The use of the Gaussian correlation inequality could be avoided here: from (3.12) and (2.2) one easily obtains

$$\mathbb{E}_N \left(\sup_{x \in A_{x_0,\gamma}} |\psi_x| \right) \leq \mathbb{E}_N \left(\sup_{x \in A_{x_0,\gamma}} |\psi_x - \psi_{x_0}| \right) + \mathbb{E}_N(|\psi_{x_0}|) \leq \Xi d_N(x_0)^{\frac{4-n}{2}} \tag{3.18}$$

for some $\Xi > 0$ and therefore

$$\mathbb{P}_N \left(\psi : |\psi_x| \leq 4\Xi d_N(x)^{\frac{4-n}{2}} \ \forall x \in A_{x_0,\gamma} \right) \geq \mathbb{P}_N \left(\psi : |\psi_x| \leq 2\Xi d_N(x_0)^{\frac{4-n}{2}} \ \forall x \in A_{x_0,\gamma} \right) \geq \frac{1}{2}. \tag{3.19}$$

We could work with this estimate instead of (3.5) by using

$$\tilde{\Omega}_{V_{N-L},\infty} := \left\{ \psi : |\psi_x| \leq 4\Xi d_N(x)^{\frac{4-n}{2}} \ \forall x \in V_{N-L} \right\} \tag{3.20}$$

instead of $\Omega_{V_{N-L},\infty}$ in the following.

3.2 Global smallness of the field

From the previous we know that on small boxes the field is small with probability bounded away from zero. We can cover V_{N-L} with these small boxes, and then use the Gaussian correlation inequality to obtain a bound on the probability that the field is globally small.

Lemma 3.3. *Let $n = 2$ or $n = 3$, let $\Omega_{V_{N-L}, \infty}$ be as before. Then we have*

$$\mathbb{P}_N(\Omega_{V_{N-L}, \infty}) \geq e^{-C \frac{N^{n-1}}{(L+1)^{n-1}}}. \tag{3.21}$$

Proof. Recall the definition of $A_{x, \gamma}$ in (3.4). Fix γ such that the conclusion of Lemma 3.1 holds and use the shorter notation $A_x := A_{x, \gamma}$.

We want to construct a subset B_N of V_N such that $|B_N| \leq C \frac{N^{n-1}}{(L+1)^{n-1}}$ and such that

$$V_{N-L} \subset \bigcup_{x \in B_N} A_x. \tag{3.22}$$

If we have found such a set, then the Gaussian correlation inequality (Theorem 2.4) and Lemma 3.1 imply that

$$
\begin{aligned}
\mathbb{P}_N(\Omega_{V_{N-L}, \infty}) &\geq \mathbb{P}_N \left(\bigcap_{x \in B_N} \{\psi \colon |\psi_y| < d_N(y)^{\frac{4-n}{2}} \ \forall y \in A_x\} \right) \\
&\geq \prod_{x \in B_N} \mathbb{P}_N \left(\psi \colon |\psi_y| < d_N(y)^{\frac{4-n}{2}} \ \forall y \in A_x \right) \\
&\geq \prod_{x \in B_N} \delta = \delta^{|B_N|} \geq e^{-C \frac{N^{n-1}}{(L+1)^{n-1}}}.
\end{aligned}
\tag{3.23}
$$

It remains to prove the existence of B_N. The size of the boxes A_x depends on the distance to the boundary, so in order to construct B_N it is convenient to split V_N into the dyadic annuli $W_{N,k} = \{x \in V_N \colon 2^k \leq d_N(x) < 2^{k+1}\}$ for $k = 0, 1, \dots, \lfloor \log_2 N \rfloor$. For $x \in W_{N,k}$ the cube A_x has diameter $2\gamma d_N(x) \geq \gamma 2^{k+1}$. Because $W_{N,k}$ has outer sidelength $2(N - 2^k) \leq 2N$ and thickness 2^k, we can cover it by at most

$$2n \left(2 \frac{2N}{\gamma 2^{k+1}} \right)^{n-1} 2 \frac{2^k}{\gamma 2^{k+1}} \leq C \frac{N^{n-1}}{2^{k(n-1)}} \tag{3.24}$$

cubes A_x, i.e. we find a set $B_{N,k}$ of at most $C \frac{N^{n-1}}{2^{k(n-1)}}$ points in V_N such that

$$W_{N,k} \subset \bigcup_{x \in B_{N,k}} A_x. \tag{3.25}$$

Let $k_0 = \lfloor \log_2(L+1) \rfloor$ which implies that $V_{N-L} \subset \bigcup_{k \geq k_0} W_{N,k}$.

Consider $B_N = \bigcup_{k=k_0}^{\log_2 N} B_{N,k}$. Then $V_{N-L} \subset \bigcup_{x \in B_N} A_x$, and we have

$$|B_N| \leq \sum_{k=k_0}^{\lfloor \log_2 N \rfloor} |B_{N,k}| \leq C \sum_{k=k_0}^{\infty} \frac{N^{n-1}}{2^{k(n-1)}} \leq C \frac{N^{n-1}}{2^{k_0(n-1)}} \leq C \frac{N^{n-1}}{(L+1)^{n-1}}. \tag{3.26}$$

$\square$

3.3 Change of measure

We can now prove the lower bound in Theorem 1.1. The idea is simple: We use an explicit calculation with densities to prove that the probability of the event $\mathbb{P}_N(f + \Omega_{V_{N-L}, \infty})$ is bounded below by $e^{-\|\Delta f\|_{L^2}^2} \mathbb{P}_N(\Omega_{V_{N-L}, \infty})$. Then it remains to make a good choice of f.

Proof of Theorem 1.1, lower bound. Let $f : V_N \to \mathbb{R}$ be a function to be specified later, and extend it by 0 to all of $\mathbb{Z}^n$. We want to estimate the probability of the event $f + \Omega_{V_{N-L},\infty} = \{f + \psi : \psi \in \Omega_{V_{N-L},\infty}\}$. To do so, we calculate

$$
\begin{aligned}
\mathbb{P}_N(f + \Omega_{V_{N-L},\infty}) &= \int_{f+\Omega_{V_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta\psi\|_{L^2}^2\right) \mathrm{d}\psi \\
&= \int_{\Omega_{V_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta(f+\psi)\|_{L^2}^2\right) \mathrm{d}\psi \\
&= \int_{\Omega_{V_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta f\|_{L^2}^2 - \frac{1}{2}\|\Delta\psi\|_{L^2}^2 - (\Delta f, \Delta\psi)_{L^2}\right) \mathrm{d}\psi.
\end{aligned}
\tag{3.27}
$$

Because $\Omega_{V_{N-L},\infty}$ is symmetric around the origin, we can replace ψ by $-\psi$ and obtain that

$$
\mathbb{P}_N(f + \Omega_{V_{N-L},\infty}) = \int_{\Omega_{V_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta f\|_{L^2}^2 - \frac{1}{2}\|\Delta\psi\|_{L^2}^2 + (\Delta f, \Delta\psi)_{L^2}\right) \mathrm{d}\psi.
\tag{3.28}
$$

If we add (3.27) and (3.28) and use the estimate $e^t + e^{-t} \geq 2$, we conclude

$$
\begin{aligned}
\mathbb{P}_N(f + \Omega_{V_{N-L},\infty}) &= \frac{1}{2} \int_{\Omega_{V_{N-L},\infty}} \frac{e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2 - \frac{1}{2}\|\Delta\psi\|_{L^2}^2} \left(e^{(\Delta f, \Delta\psi)_{L^2}} + e^{-(\Delta f, \Delta\psi)_{L^2}}\right)}{Z_N} \mathrm{d}\psi \\
&\geq e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2} \int_{\Omega_{V_{N-L},\infty}} \frac{e^{-\frac{1}{2}\|\Delta\psi\|_{L^2}^2}}{Z_N} \mathrm{d}\psi \\
&= e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2} \mathbb{P}_N(\Omega_{V_{N-L},\infty}).
\end{aligned}
\tag{3.29}
$$

Note that the conclusion in (3.29) could also be derived from (3.27) using Jensen's inequality.

We now choose f as in Lemma 3.4 below. Then

$$
\|\Delta f\|_{L^2}^2 \leq C \frac{N^{n-1}}{(L+1)^{n-1}}.
\tag{3.30}
$$

Moreover this choice of f ensures that $\Omega_{V_{N-L},+} \supset f + \Omega_{V_{N-L},\infty}$, and so (3.29), (3.30) and Lemma 3.3 imply that

$$
\begin{aligned}
\mathbb{P}_N(\Omega_{V_{N-L},+}) &\geq \mathbb{P}_N(f + \Omega_{V_{N-L},\infty}) \geq e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2} \mathbb{P}_N(\Omega_{V_{N-L},\infty}) \\
&\geq e^{-C\frac{N^{n-1}}{(L+1)^{n-1}}} e^{-C\frac{N^{n-1}}{(L+1)^{n-1}}} = e^{-C'\frac{N^{n-1}}{(L+1)^{n-1}}}.
\end{aligned}
\tag{3.31}
$$

$\square$

Lemma 3.4. *There is a constant $C > 0$ such that for every N and $0 \leq L \leq N$ there is a function $f : \mathbb{Z}^n \to \mathbb{R}$ such that* $\mathrm{supp}\, f \subset V_N$, $f(x) \geq d_N(x)^{\frac{4-n}{2}}$ *for all $x \in V_{N-L}$ and*

$$
\sum_{x \in \mathbb{Z}^n} |\Delta f(x)|^2 \leq C \frac{N^{n-1}}{(L+1)^{n-1}}.
\tag{3.32}
$$

Proof. We again use a dyadic construction. Recall $W_{N,k} = \{x \in V_N : 2^k \leq d_N(x) < 2^{k+1}\}$ for $k = 0, 1, \ldots, \lfloor \log_2 N \rfloor$. Let in addition $W_{N,-1} = \mathbb{Z}^n \setminus V_N$.
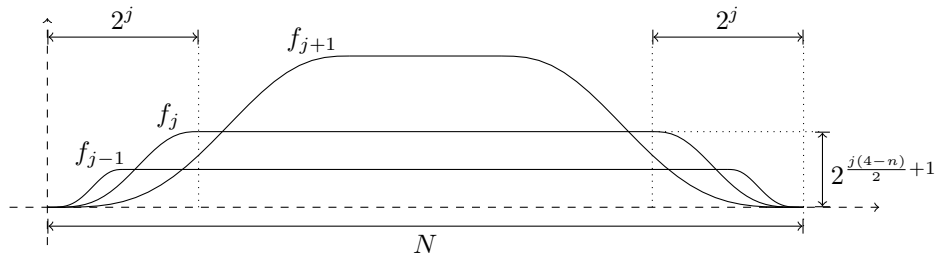
Figure 1: The functions f_j.

Fix a smooth function $\eta\colon \mathbb{R} \to \mathbb{R}$ such that $\eta \geq 0$, $\eta = 1$ on $[1, \infty)$ and $\eta = 0$ on $(-\infty, 0]$. For $i \in \{1, 2, \ldots, n\}$ and $x \in \mathbb{Z}^n$ we introduce the distance $d_i(x) = \mathrm{dist}(x, \mathbb{Z}^n \setminus (\mathbb{Z}^{i-1} \times \{-N, \ldots, N\} \times \mathbb{Z}^{n-i}))$ of x to the boundary in direction x_i.

For $j = 0, 1, \ldots \lfloor \log_2 N \rfloor - 1$ consider the function

$$f_j(x) = 2^{\frac{j(4-n)}{2}+1} \prod_{i=1}^{n} \eta\left(\frac{d_i(x)}{2^j}\right) \tag{3.33}$$

(cf. Figure 1). Note that

$$f_j(x) = 2^{\frac{j(4-n)}{2}+1} \tag{3.34}$$

for all $x \in V_N$ such that $d_N(x) \geq 2^j$. Moreover

$$|\Delta f_j(x)| \leq C 2^{\frac{j(4-n)}{2}+1} \|\eta''\|_{L^\infty} \frac{1}{2^{2j}} \leq \frac{C \|\eta''\|_{L^\infty}}{2^{\frac{jn}{2}}}. \tag{3.35}$$

In fact $\Delta f_j(x) = 0$ if $d_N(x) > 2^j$ because f_k is constant on V_{N-2^j}. We define the function

$$f = \sum_{j=\lfloor \log_2(L+1) \rfloor}^{\lfloor \log_2 N \rfloor} f_j. \tag{3.36}$$

For $x \in V_{N-L}$ let now k be such that $x \in W_{N,k}$, and observe that $\lfloor \log_2(L+1) \rfloor \leq k \leq \lfloor \log_2 N \rfloor$. The estimate (3.34) implies

$$f(x) \geq f_k(x) \geq 2^{\frac{k(4-n)}{2}+1} \geq \left(2 \cdot 2^k\right)^{\frac{4-n}{2}} \geq d_N(x)^{\frac{4-n}{2}}. \tag{3.37}$$

For an arbitrary $x \in \mathbb{Z}^n$ let again $k \in \{-1, 0, 1, \ldots\}$ be such that $x \in W_{N,k}$. Then (3.35) implies that

$$|\Delta f(x)| \leq \sum_{j=k \vee \lfloor \log_2(L+1) \rfloor}^{\lfloor \log_2 N \rfloor} |\Delta f_j| \leq \sum_{j=k \vee \lfloor \log_2(L+1) \rfloor}^{\infty} \frac{C \|\eta''\|_{L^\infty}}{2^{\frac{jn}{2}}} \leq \frac{C'}{2^{\frac{(k \vee \lfloor \log_2(L+1) \rfloor)n}{2}}}. \tag{3.38}$$

Using that $|W_{N,k}| \leq C 2^k N^{n-1}$ for $k \geq 0$ and that $\Delta f(x)$ is zero on $W_{N,-1}$ except possibly on the set $V_{N+1} \setminus V_N$ of cardinality $C N^{n-1} \leq C' 2^{-1} N^{n-1}$, the previous estimate implies that

$$\sum_{x \in \mathbb{Z}^n} |\Delta f(x)|^2 \leq \sum_{k=-1}^{\lfloor \log_2 N \rfloor} \sum_{x \in W_{N,k}} |\Delta f(x)|^2 \leq \sum_{k=-1}^{\infty} \frac{C2^k N^{n-1}}{2^{(k \vee \lfloor \log_2(L+1) \rfloor)n}}$$

$$\leq \sum_{k=-1}^{\lfloor \log_2(L+1) \rfloor} \frac{C2^k N^{n-1}}{2^{\lfloor \log_2(L+1) \rfloor n}} + \sum_{k=\lfloor \log_2(L+1) \rfloor + 1}^{\infty} \frac{C2^k N^{n-1}}{2^{kn}} \qquad (3.39)$$

$$\leq C \frac{N^{n-1}}{(L+1)^{n-1}} + C \frac{N^{n-1}}{(L+1)^{n-1}} = C' \frac{N^{n-1}}{(L+1)^{n-1}}.$$

$\square$

4 Upper bounds

In order to prove the upper bound in Theorem 1.1, we will find a suitably sparse set $E_{N,L}$ of points at the boundary such that the random variables $\{\psi_x \colon x \in E_{N,L}\}$ are almost independent in the sense that their covariance matrix is diagonally dominant. We can then use Lemma 2.5 to compare them to actually independent random variables. The following argument is taken from [25, Section 6.2.1].

Proof of Theorem 1.1, upper bound. Note that for $N \geq L > \frac{N}{2}$ the upper bound is trivial. Indeed, V_{N-L} is nonempty and so the symmetry of the field implies $\mathbb{P}_N(\Omega_{V_{N-L},+}) \leq \frac{1}{2}$, while the right hand side of (1.2) exceeds $\frac{1}{2}$ if $L > \frac{N}{2}$ and $c < 2^{-n}$. We assume $L \leq \frac{N}{2}$ in the following. Let $E_{N,L} = V_{N-L} \cap ((\lceil \alpha(L+1) \rceil \mathbb{Z})^{n-1} \times \{N-L\})$ where $\alpha \geq 1$ is a constant to be chosen later. This is a set of points on one face of $[-N+L, N-L]^n$ such that any two points have distance at least αL. Its cardinality satisfies

$$|E_{N,L}| = \left(2 \left\lfloor \frac{N-L}{\lceil \alpha(L+1) \rceil} \right\rfloor + 1\right)^{n-1} \geq \left(\left\lfloor \frac{N - \frac{N}{2}}{\alpha(L+1) + 1} \right\rfloor + 1\right)^{n-1}$$

$$\geq \left(\frac{\frac{N}{2}}{\alpha(L+1) + 1}\right)^{n-1} \geq c \frac{N^{n-1}}{\alpha^{n-1}(L+1)^{n-1}}. \qquad (4.1)$$

Clearly $d_N(x) = L+1$ for any $x \in E_{N,L}$. Therefore according to (2.5) for $x \neq y$

$$|G_N(x,y)| \leq C \frac{(L+1)^4}{(|x-y|_\infty + 1)^n} \leq C \frac{(L+1)^4}{|x-y|_\infty^n}. \qquad (4.2)$$

If we combine this with (2.2) we obtain for any $x \in E_{N,L}$

$$\sum_{\substack{y \in E_{N,L} \\ y \neq x}} \frac{|G_N(x,y)|}{\sqrt{G_N(x,x)G_N(y,y)}} \leq C \sum_{\substack{y \in E_{N,L} \\ y \neq x}} \frac{(L+1)^4}{(L+1)^{4-n}|x-y|_\infty^n}$$

$$= C \sum_{j=1}^{\infty} |\{y \in E_{N,L} \colon |y-x|_\infty = j \lceil \alpha(L+1) \rceil\}| \frac{(L+1)^n}{(j \lceil \alpha(L+1) \rceil)^n}$$

$$\leq \frac{C}{\alpha^n} \sum_{j=1}^{\infty} \frac{a_j}{j^n}$$

$$(4.3)$$

where $a_j = 2$ for $n = 2$ and $a_j = 8j$ for $n = 3$. Thus $\sum_{j=1}^{\infty} \frac{a_j}{j^n} < \infty$ and hence

$$\sum_{\substack{y \in E_{N,L} \\ y \neq x}} \frac{|G_N(x,y)|}{\sqrt{G_N(x,x)G_N(y,y)}} \leq \frac{C}{\alpha^n}. \qquad (4.4)$$

We now choose α large enough that the right hand side of (4.4) becomes less than $\frac{1}{4}$.

We define the Gaussian random vector $(X_x)_{x \in E_{N,L}}$ by $X_x = \frac{\psi_x}{\sqrt{G_N(x,x)}}$. Let Σ_X be its covariance matrix. Then $(\Sigma_X)_{x,x} = 1$ for all x and (4.4) implies that

$$\sum_{\substack{y \in E_{N,L} \\ y \neq x}} |(\Sigma_X)_{x,y}| \leq \frac{1}{4}. \tag{4.5}$$

Let $\{Y_x\}_{x \in E_{N,L}}$ be i.i.d. normal variables distributed according to $\mathcal{N}\left(0, \frac{3}{2}\right)$ and let $\Sigma_Y = \frac{3}{2}\mathbb{1}_{E_{N,L}}$ be their joint covariance matrix, where $\mathbb{1}_{E_{N,L}}$ is a unit matrix indexed by $E_{N,L}$.

Because of (4.5) the matrix $\Sigma_Y - \Sigma_X$ then satisfies

$$(\Sigma_Y - \Sigma_X)_{x,x} = \frac{3}{2} - 1 = \frac{1}{2} > \sum_{\substack{y \in E_{N,L} \\ y \neq x}} (\Sigma_X)_{x,y}. \tag{4.6}$$

This means that $\Sigma_Y - \Sigma_X$ is strictly diagonally dominant and hence positive definite. Hence we can apply Lemma 2.5 and obtain

$$\begin{aligned}
\left(\frac{1}{2}\right)^{|E_{N,L}|} &= \mathbb{P}(Y \in (0, \infty)^{E_{N,L}}) \\
&\geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}(X \in (0, \infty)^{E_{N,L}}) \\
&= \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}_N(\psi_x \geq 0 \; \forall x \in E_{N,L}) \\
&\geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}_N(\Omega_{V_{N-L},+}).
\end{aligned} \tag{4.7}$$

It remains to estimate $\frac{\det \Sigma_X}{\det \Sigma_Y}$. Since Σ_Y is diagonal, $\det \Sigma_Y = \left(\frac{3}{2}\right)^{|E_{N,L}|}$. On the other hand, by (4.5) the matrix $\Sigma_X - \frac{3}{4}\mathbb{1}_{E_{N,L}}$ is still diagonally dominant and hence positive semidefinite. Hence all eigenvalues of Σ_X must be at least $\frac{3}{4}$. Therefore $\det \Sigma_X \geq \left(\frac{3}{4}\right)^{|E_{N,L}|}$.

We conclude

$$\mathbb{P}_N(\Omega_{V_{N-L},+}) \leq \left(\frac{1}{2}\right)^{|E_{N,L}|} \left(\frac{\det \Sigma_Y}{\det \Sigma_X}\right)^{\frac{1}{2}} \leq \left(\frac{1}{2}\right)^{|E_{N,L}|} \left(\frac{3/2}{3/4}\right)^{\frac{|E_{N,L}|}{2}} = \left(\frac{1}{\sqrt{2}}\right)^{|E_{N,L}|}. \tag{4.8}$$

Recall that by (4.1) we have $|E_{N,L}| \geq c\frac{N^{n-1}}{\alpha^{n-1}(L+1)^{n-1}}$. Thus we finally obtain

$$\mathbb{P}_N(\Omega_{V_{N-L},+}) \leq \exp\left(-c\frac{N^{n-1}}{(L+1)^{n-1}}\right) \tag{4.9}$$

for $c = \frac{1}{2\alpha^{n-1}} \log 2$. $\qquad\square$

References

[1] Vladimir I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR-1642391

[2] Erwin Bolthausen, Jean-Dominique Deuschel, and Giambattista Giacomin, *Entropic repulsion and the maximum of the two-dimensional harmonic crystal*, Ann. Probab. **29** (2001), no. 4, 1670–1692. MR-1880237

[3] Erwin Bolthausen, Jean-Dominique Deuschel, and Ofer Zeitouni, *Entropic repulsion of the lattice free field*, Comm. Math. Phys. **170** (1995), no. 2, 417–443. MR-1334403

[4] Francesco Caravenna and Loïc Chaumont, *An invariance principle for random walk bridges conditioned to stay positive*, Electron. J. Probab. **18** (2013), no. 60, 32. MR-3068391

[5] Francesco Caravenna and Jean-Dominique Deuschel, *Pinning and wetting transition for $(1+1)$-dimensional fields with Laplacian interaction*, Ann. Probab. **36** (2008), no. 6, 2388–2433. MR-2478687

[6] Alessandra Cipriani, Biltu Dan, and Rajat Subhra Hazra, *The scaling limit of the membrane model*, 2018, arXiv:1801.05663.

[7] Amir Dembo, Jian Ding, and Fuchang Gao, *Persistence of iterated partial sums*, Ann. Inst. Henri Poincaré Probab. Stat. **49** (2013), no. 3, 873–884. MR-3112437

[8] Denis Denisov and Vitali Wachtel, *Exit times for integrated random walks*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), no. 1, 167–193. MR-3300967

[9] Jean-Dominique Deuschel, *Entropic repulsion of the lattice free field. II. The 0-boundary case*, Comm. Math. Phys. **181** (1996), no. 3, 647–665. MR-1414304

[10] Jean-Dominique Deuschel and Giambattista Giacomin, *Entropic repulsion for the free field: pathwise characterization in $d \geq 3$*, Comm. Math. Phys. **206** (1999), no. 2, 447–462. MR-1722113

[11] Richard M. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Functional Analysis **1** (1967), 290–330. MR-0220340

[12] Giambattista Giacomin, *Aspects of statistical mechanics of random surfaces.*, Notes of the lectures given at the IHP, 2001.

[13] Christin Hiergeist and Reinhard Lipowsky, *Local contacts of membranes and strings*, Physica A: Statistical Mechanics and its Applications **244** (1997), no. 1, 164 – 175.

[14] Noemi Kurt, *Entropic repulsion for a class of Gaussian interface models in high dimensions*, Stochastic Process. Appl. **117** (2007), no. 1, 23–34. MR-2287101

[15] Noemi Kurt, *Maximum and entropic repulsion for a Gaussian membrane model in the critical dimension*, Ann. Probab. **37** (2009), no. 2, 687–725. MR-2510021

[16] Rafał Latała and Dariusz Matlak, *Royen's proof of the Gaussian correlation inequality*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 2169, Springer, Cham, 2017, pp. 265–275. MR-3645127

[17] Joel L. Lebowitz and Christian Maes, *The effect of an external field on an interface, entropic repulsion*, J. Statist. Phys. **46** (1987), no. 1-2, 39–49. MR-887234

[18] Wenbo V. Li and Qi-Man Shao, *Lower tail probabilities for Gaussian processes*, Ann. Probab. **32** (2004), no. 1A, 216–242. MR-2040781

[19] Reinhard Lipowsky, *Generic interactions of flexible membranes*, Structure and Dynamics of Membranes (R. Lipowsky and E. Sackmann, eds.), Handbook of Biological Physics, vol. 1, North-Holland, 1995, pp. 521–602.

[20] Stefan Müller and Florian Schweiger, *Estimates for the Green's Function of the Discrete Bilaplacian in Dimensions 2 and 3*, Vietnam J. Math. **47** (2019), no. 1, 133–181. MR-3913854

[21] Thomas Royen, *A simple proof of the Gaussian correlation conjecture extended to some multivariate gamma distributions*, Far East J. Theor. Stat. **48** (2014), no. 2, 139–145. MR-3289621

[22] Juan Jesús Ruiz-Lorenzo, Rodolfo Cuerno, Esteban Moro, and Angel Sánchez, *Phase transition in tensionless surfaces*, Biophysical Chemistry **115** (2005), no. 2, 187–193, BIFI 2004 International Conference Biology after the Genoma: A Physical View.

[23] Hironobu Sakagawa, *Entropic repulsion for a Gaussian lattice field with certain finite range interaction*, J. Math. Phys. **44** (2003), no. 7, 2939–2951. MR-1982781

[24] Hironobu Sakagawa, *On the probability that Laplacian interface models stay positive in subcritical dimensions*, Stochastic analysis on large scale interacting systems, RIMS Kôkyûroku Bessatsu, B59, Res. Inst. Math. Sci. (RIMS), Kyoto, 2016, pp. 273–288. MR-3675938

[25] Florian Schweiger, *Discrete Green's functions and statistical physics of membranes*, Master's thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2016.

[26] Michel Talagrand, *Majorizing measures: the generic chaining*, Ann. Probab. **24** (1996), no. 3, 1049–1103. MR-1411488

[27] Yvan Velenik, *Localization and delocalization of random interfaces*, Probab. Surv. **3** (2006), 112–169. MR-2216964