

## Squared Bessel processes of positive and negative dimension embedded in Brownian local times\*

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### Abstract

The Ray–Knight theorems show that the local time processes of various path fragments derived from a one-dimensional Brownian motion  $B$  are squared Bessel processes of dimensions 0, 2, and 4. It is also known that for various singular perturbations  $X = |B| + \mu \ell$  of a reflecting Brownian motion  $|B|$  by a multiple  $\mu$  of its local time process  $\ell$  at 0, corresponding local time processes of  $X$  are squared Bessel with other real dimension parameters, both positive and negative. Here, we embed squared Bessel processes of all real dimensions directly in the local time process of  $B$ . This is done by decomposing the path of  $B$  into its excursions above and below a family of continuous random levels determined by the Harrison–Shepp construction of skew Brownian motion as the strong solution of an SDE driven by  $B$ . This embedding connects to Brownian local times a framework of point processes of squared Bessel excursions of negative dimension and associated stable processes, recently introduced by Forman, Pal, Rizzolo and Winkel to set up interval partition evolutions that arise in their approach to the Aldous diffusion on a space of continuum trees.

**Keywords:** Brownian motion; local times; excursions; squared Bessel processes.

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## 1 Introduction and statement of main results

Squared Bessel processes are a family of one-dimensional diffusions on  $[0, \infty)$ , defined by continuous solutions  $Y = (Y(x), 0 \leq x \leq \zeta)$  of the stochastic differential equation

$$dY(x) = \delta dx + 2\sqrt{Y(x)}dB(x), \quad Y(0) = y \geq 0, \quad 0 < x < \zeta \quad (1.1)$$

where  $\delta$  is a real parameter,  $B = (B(x), x \geq 0)$  is standard Brownian motion, and  $\zeta$  is the *lifetime* of  $Y$ , defined by

$$\zeta := \begin{cases} \infty & \text{if } \delta > 0 \\ T_0 := \inf\{x \geq 0: Y(x) = 0\} & \text{if } \delta \leq 0. \end{cases} \quad (1.2)$$

It is known [24, Chapter XI], [11] that this SDE has a unique strong solution, with  $\zeta < \infty$  and  $Y(\zeta) = 0$  almost surely if  $\delta \leq 0$ , when we make the boundary state 0 absorbing by setting  $Y(x) = 0$  for  $x \geq \zeta$ . The distribution on the path space  $C[0, \infty)$  of the process  $Y$

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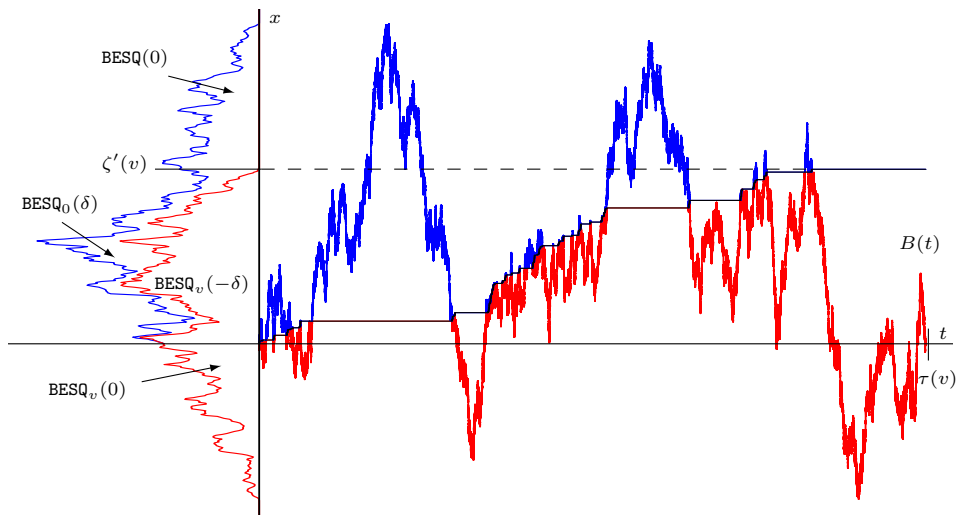


Figure 1: Simulation of Brownian motion  $(B(t), 0 \leq t \leq \tau(v))$  split along an increasing path depending on  $\delta > 0$ . The red excursions below the path have total local time process  $\text{BESQ}_v(-\delta)$  on  $[0, \infty)$ . The blue excursions above the path have total local time process on  $[0, \infty)$  which is  $\text{BESQ}_0(\delta)$  up to level  $\zeta'(v)$ , then continue as  $\text{BESQ}(0)$  above level  $\zeta'(v)$ .

so defined, for each starting state  $y \geq 0$  and each real  $\delta$ , is denoted  $\text{BESQ}_y(\delta)$ . For each real  $\delta$ , the collection of laws  $(\text{BESQ}_y(\delta), y \geq 0)$  defines a Markovian diffusion process on  $[0, \infty)$ , the *squared Bessel process of dimension  $\delta$* , denoted  $\text{BESQ}(\delta)$ .

Several other constructions and interpretations of  $\text{BESQ}(\delta)$  are known. In particular,

- $\text{BESQ}_y(\delta)$  can be represented as  $(yY_1(x/y), x \geq 0)$  for  $Y_1$  a  $\text{BESQ}_1(\delta)$ ; indeed, up to linear time-change and, for  $\delta \in (0, 2)$ , up to the extension after first hitting 0,  $\text{BESQ}(\delta)$ ,  $\delta \in \mathbb{R}$ , are the only diffusions on  $[0, \infty)$  that have this *scaling property* [15];
- $\text{BESQ}(\delta)$  for  $\delta = 1, 2, \dots$  is the squared norm of standard Brownian motion in  $\mathbb{R}^d$ ;
- $\text{BESQ}(\delta)$  may be understood for all real  $\delta$  as a continuous-state branching process, with an immigration rate  $\delta$  if  $\delta > 0$ , emigration rate  $|\delta|$  if  $\delta < 0$ , and lifetime  $\zeta$  at which the population dies out.

The case of immigration has been well-studied [13, 26, 25, 24, 14, 19]. The literature on the case of emigration is rather sparse [22, 9], but scaling limit results for discrete branching processes with emigration [27, 28] with  $\text{BESQ}(\delta)$  limits for  $\delta < 0$  can be obtained from [1, 2, 22].

It was shown by Shiga and Watanabe [26] that the distribution of  $\text{BESQ}_y(\delta)$  for all real  $y \geq 0$  and  $\delta \geq 0$  is uniquely determined by the prescription that  $\text{BESQ}_y(1)$  is the distribution of  $(\sqrt{y} + B)^2$ , and the following additivity property: for  $y, y' \geq 0$  and  $\delta, \delta' \geq 0$ , and two independent processes  $Y$  and  $Y'$ ,

$$\text{if } Y \text{ is a } \text{BESQ}_y(\delta) \text{ and } Y' \text{ is a } \text{BESQ}_{y'}(\delta') \text{ then } Y + Y' \text{ is a } \text{BESQ}_{y+y'}(\delta + \delta'). \quad (1.3)$$

Pitman and Yor [23] used the additivity property to construct a  $\text{BESQ}_y(\delta)$  process  $Y_y^{(\delta)}$  for  $y, \delta \geq 0$  as a sum of points in a  $C[0, \infty)$ -valued Poisson point process, whose intensity measure involves the local time profile induced by Itô's law of Brownian excursions. The  $C[0, \infty)$ -valued process  $(Y_y^{(\delta)}, y \geq 0, \delta \geq 0)$  then has stationary independent increments in both  $y \geq 0$  and  $\delta \geq 0$ . This construction explained the multiple appearances of  $\text{BESQ}(\delta)$  processes and their bridges for  $\delta = 0, 2$  and  $4$  in the Ray-Knight descriptions of Brownian local time processes.

This model of Brownian local times and BESQ processes, driven by a Poisson point process of local time pulses from Brownian excursions, led to a number of further developments. In particular, as recalled later in Lemmas 2.1 and 2.2, if a reflecting Brownian motion  $|B|$  is perturbed by adding a multiple  $\mu$  of its local time process  $\ell$  at 0, to form  $X := |B| + \mu\ell$ , where  $\mu$  might be of either sign, then the resulting *perturbed Brownian motion*  $X$  has a local time process from which it is possible, by varying  $\mu$ , and sampling at suitable random times, to construct  $\text{BESQ}(\delta)$  processes for all real  $\delta$ . The more recent notion of a *Poisson loop soup* [17] greatly generalizes this construction of local time fields from one-dimensional Brownian motion to one-dimensional diffusions [20] and much more general Markov processes.

Despite these constructions of  $\text{BESQ}(\delta)$  for all real  $\delta$  in the local time processes of perturbed Brownian motions, and the general importance of additivity properties in the construction of local time fields [21] [17], it is known [24, Exercise XI.(1.33)] and [11, top of p.332] that the additivity property (1.3) of BESQ processes fails without the assumption that both  $\delta \geq 0$  and  $\delta' \geq 0$ . Our starting point here is a weaker form of additivity of BESQ processes, involving both positive and negative dimensions:

**Proposition 1.1.** *For arbitrary real  $\delta, \delta'$  and  $y, y' \geq 0$ , let  $Y, Y'$  and  $Y_1$  be three independent processes, with*

- $Y$  a  $\text{BESQ}_y(\delta)$  with lifetime  $\zeta$ ;
- $Y'$  a  $\text{BESQ}_{y'}(\delta')$  with lifetime  $\zeta'$ ;
- $Y_1$  a  $\text{BESQ}_1(\delta + \delta')$ .

Let  $T$  be a stopping time relative to the filtration generated by the pair of processes  $(Y, Y')$ , with  $T \leq \zeta \wedge \zeta'$ , and let  $Z$  be the process

$$Z(x) := \begin{cases} Y(x) + Y'(x), & \text{if } 0 \leq x \leq T, \\ Z(T)Y_1((x - T)/Z(T)), & \text{if } T < x < \infty. \end{cases} \tag{1.4}$$

Then  $Z$  is a  $\text{BESQ}_{y+y'}(\delta + \delta')$ .

Note that (1.4) sets  $Z := Y + Y'$  on  $[0, T]$  and, by the scaling property, makes  $Z$  evolve as a  $\text{BESQ}(\delta + \delta')$  on  $[T, \infty)$ . This proposition is a straightforward generalization of the case with  $\delta = -1, \delta' = 0$  and  $T = \zeta \wedge \zeta'$ , which was established as [9, Lemma 25]. If both  $\delta, \delta' \geq 0$ , the conclusion Proposition 1.1 holds even without the assumption  $T \leq \zeta \wedge \zeta'$ , by combining the simpler additivity property (1.3) with the strong Markov property of  $\text{BESQ}(\delta + \delta')$ . We are particularly interested in the instance of Proposition 1.1 with  $y = 0, y' = v, \delta' = -\delta < 0$  and  $T = \zeta'$ , which may be paraphrased as follows:

**Corollary 1.2.** *Let  $\delta > 0$  and  $v \geq 0$ . Let  $Y' := (Y'_v(x), x \geq 0)$  be  $\text{BESQ}_v(-\delta)$  absorbed at  $\zeta'(v) := \inf\{x \geq 0 : Y'_v(x) = 0\}$ . Conditionally given  $Y'$  with  $\zeta'(v) = a$ , let  $Y := (Y_{0,v}^{(\delta)}(x), x \geq 0)$  be a time-inhomogeneous Markov process that is  $\text{BESQ}_0(\delta)$  on the time interval  $[0, a]$  and then continues on  $[a, \infty)$  as  $\text{BESQ}(0)$ . Then  $Y + Y'$  is a  $\text{BESQ}_v(0)$ .*

The subtlety here is that we create dependence between  $Y$  and  $Y'$  by specifying that  $Y$  only follows  $\text{BESQ}_0(\delta)$  independently of  $Y'$  until time  $\zeta'(v)$ , when  $Y'$  hits zero, and then  $Y$  continues as needed for the additivity to hold. In [8], the authors encountered the case  $\delta = 1$  of Corollary 1.2 in a more elaborate context which we review in Section 4.

Let  $L = (L(x, t), x \in \mathbb{R}, t \geq 0)$  be the jointly continuous space-time local time process of Brownian motion  $B = (B(t), t \geq 0)$  so that  $\ell = L(0, \cdot)$  is the local time of  $B$  at 0, with inverse  $\tau(v) = \inf\{t \geq 0 : \ell(t) > v\}$ . According to one of the Ray-Knight theorems, the process  $(L(x, \tau(v)), x \geq 0)$  is a  $\text{BESQ}_v(0)$ . This raises the following question:

Can we find the pair  $(Y, Y')$  of Corollary 1.2 embedded in the local times of  $B$ ?

The following theorem provides a positive answer to this question. See Figure 1 for an illustration of the embedding. Before we state the theorem, consider the excursions away from level 0 of reflected Brownian motion and let  $\gamma \in [-1, 1]$ . Independently multiply each excursion by  $-1$  with probability  $\frac{1}{2}(1-\gamma)$ . The resulting process  $X_\gamma = (X_\gamma(t), t \geq 0)$  is known as *skew Brownian motion*. See [18] for a recent survey of constructions of this process, including this *excursion construction* and the *SDE construction* of  $X_\gamma$  by Harrison and Shepp [12] as the unique strong solution to the equation

$$X_\gamma(t) = B(t) - \gamma \ell_\gamma(t), \quad t \geq 0, \tag{1.5}$$

where  $B$  is Brownian motion and  $\ell_\gamma$  is the local time process at 0 of  $X_\gamma$ , that is

$$\ell_\gamma(t) = \lim_{h \downarrow 0} \frac{1}{2h} \int_0^t \mathbb{1}\{-h < X_\gamma(s) < h\} ds.$$

where the limit exists simultaneously for all  $t \geq 0$  almost surely. It follows from the excursion construction of skew Brownian motion that this choice of local time at 0 is so that  $\ell_\gamma \stackrel{d}{=} \ell$  for all  $\gamma$ , where  $\ell = \ell_0 = L(0, \cdot)$  is the usual local time of  $|B|$  at 0, so that  $|B| - \ell \stackrel{d}{=} B$ . In this framework of skew Brownian motion, we now state our main result.

**Theorem 1.3.** *Let  $\delta > 0$  and  $\gamma := 1/(1 + \delta)$ . Let  $X_\gamma$  be skew Brownian motion driven by  $B$  as in (1.5), with local time  $\ell_\gamma$  at zero. Let  $S_\delta(x) = \inf\{t \geq 0: \gamma \ell_\gamma(t) > x\}$ ,  $x \geq 0$ . Then the following two families of random variables are independent*

- $Y_0^{(\delta)} := (L(x, S_\delta(x)), x \geq 0) \stackrel{d}{=} \text{BESQ}_0(\delta)$ ;
- $Y'_v := (L(x, \tau(v)) - L(x, S_\delta(x) \wedge \tau(v)), x \geq 0) \stackrel{d}{=} \text{BESQ}_v(-\delta)$  for all  $v \geq 0$ .

For each  $v \geq 0$ , the random level  $\zeta'(v) := \inf\{x \geq 0: S_\delta(x) > \tau(v)\}$  is almost surely finite, and coincides with the absorption time of  $Y'_v$ . Conditionally given  $\zeta'(v) = a$ ,

- the process  $Y_{0,v}^{(\delta)} := (L(x, S_\delta(x) \wedge \tau(v)), x \geq 0)$  is independent of  $Y'_v$  and a time-inhomogeneous Markov process that is  $\text{BESQ}_0(\delta)$  on the time interval  $[0, a]$  and then continues as  $\text{BESQ}(0)$ .

Note that, with  $\gamma = 1/(1 + \delta)$ , we have  $B(S_\delta(x)) = X_\gamma(S_\delta(x)) + \gamma \ell_\gamma(S_\delta(x)) = 0 + x = x$ , since  $S_\delta(x)$  is an inverse local time of  $X_\gamma$ . Hence,  $S_\delta$  is a right inverse of  $B$ . Right inverses of Lévy processes were studied by Evans [6], also [30], to construct stationary local time processes. The main focus has been on the minimal right inverse, which for  $B$  is the first passage process. Theorem 1.3 involves a family of non-minimal right inverses.

Thinking of  $\text{BESQ}(\delta)$  as a branching processes with immigration or emigration, according to the sign of  $\delta$ , Theorem 1.3 provides a frontier  $S_\delta$  varying with  $x$ , across which the emigration of  $\text{BESQ}(-\delta)$  is the immigration of  $\text{BESQ}(\delta)$ .

**Corollary 1.4.** *In the setting of Theorem 1.3, the  $C[0, \infty)$ -valued process  $(Y_0^{(\delta)}, \delta \geq 0)$ , with  $Y_0^{(0)} \equiv 0$ , has stationary and independent increments in  $\delta \geq 0$ .*

The weaker form of additivity in Theorem 1.3 raises further questions. Here are some:

1. Is the (right-continuous increasing) process  $(S_\delta(x), x \geq 0)$  of stopping times uniquely identified by the distribution of  $Y_0^{(\delta)}$  specified in the first bullet point of Theorem 1.3?
2. In the context of Corollary 1.4, the decomposition of  $Y_0^\delta$  into local time pulses established in [23] yields that for values of  $\delta$  for which there is a pulse at  $\delta$ , the increasing path  $t \mapsto \inf\{x \geq 0: S_\delta(x) > t\}$  in Figure 1 is below  $t \mapsto \inf\{x \geq 0: S_{\delta-}(x) > t\}$  in a way affecting local times discontinuously. As such pulses capture local time of  $B$  between increasing paths, what is the structure of the point process of the corresponding “excursions” of  $B$ ?

3. In Corollary 1.2, what is the conditional distribution of  $Y'$  or of  $\zeta'(v)$  given  $Y + Y'$ ?
4. Suppose a non-negative process  $Y'$  absorbed at 0 at time  $\zeta'$  is such that  $Y + Y'$  is  $\text{BESQ}_v(0)$  for  $Y$  conditionally given  $Y'$  as in Corollary 1.2. Is  $Y'$  a  $\text{BESQ}_v(-\delta)$  process?

The rest of this article is organized as follows: Section 2 presents the proofs of Theorem 1.3 and Corollary 1.4. In Section 3, we explore the implications of Proposition 1.1 by checking the laws of some marginals and functionals. We conclude in Section 4 by pointing out some related developments.

## 2 Proofs of the main results

The proof of Proposition 1.1 is a straightforward generalisation of the proof of [9, Lemma 25]. The special case as formulated in Corollary 1.2 is central to this paper, so we prove it here and leave the generalisation to a full proof of the proposition to the reader.

*Proof of Corollary 1.2.* Without loss of generality, we may assume that the processes in the proposition are defined on a probability space that supports three independent Brownian motions  $B'$ ,  $B_0$  and  $B_1$ . Specifically, we may assume that

$$Y'(0) = v, \quad dY'(x) = -\delta dx + 2\sqrt{Y'(x)}dB'(x), \quad 0 \leq t \leq \zeta' = \inf\{x \geq 0: Y'(x) = 0\}$$

and  $Y'(x) = 0, x \geq \zeta'$ , so that  $Y'$  is a  $\text{BESQ}_v(-\delta)$ . We may also assume that

$$Y_0(0) = 0, \quad dY_0(x) = \delta dx + 2\sqrt{Y_0(x)}dB_0(x),$$

$$\text{and } Y_1(0) = 1, \quad dY_1(x) = 2\sqrt{Y_1(x)}dB_1(x).$$

Then we define  $Y$  to follow  $Y_0$  on  $[0, \zeta']$ , and we scale  $Y_1$  to continue  $Y$  on  $[\zeta', \infty)$ :

$$Y(x) = Y_0(x), \quad 0 \leq x \leq \zeta', \quad Y(\zeta' + x) = Y_0(\zeta')Y_1(x/Y_0(\zeta')), \quad x \geq 0,$$

so that given  $Y'$  with  $\zeta' = a$ , the process  $Y$  is an inhomogeneous Markov process that is  $\text{BESQ}_0(\delta)$  on the time interval  $[0, a]$  and then  $Y$  continues as  $\text{BESQ}(0)$ . Elementary stochastic calculus shows that

$$Y(0) + Y'(0) = v, \quad d(Y(x) + Y'(x)) = 2\sqrt{Y(x) + Y'(x)}dW_1(x), \quad 0 \leq x \leq \zeta',$$

for some Brownian motion  $W_1$  constructed from  $B'$  and  $B_0$ . Furthermore,

$$W(x) = W_1(x), \quad 0 \leq x \leq \zeta', \quad W(\zeta' + x) = W_1(\zeta') + \sqrt{Y(\zeta')}B_1(x/Y(\zeta')), \quad x \geq 0,$$

is a Brownian motion, by Lévy's characterisation. Furthermore,  $U = Y + Y'$  solves

$$U(0) = v, \quad dU(x) = 2\sqrt{U(x)}dW(x).$$

This is the stochastic differential equation defining  $\text{BESQ}_v(0)$ . That it has a unique strong solution can be read from [24, Chapter XI].  $\square$

Our proof of Theorem 1.3 exploits the following known variants of the Ray–Knight theorems for perturbed Brownian motions  $R_\mu^\pm := |B| \pm \mu\ell$ , where  $\ell = L(0, \cdot)$  is the local time of  $B$  at 0 normalized so that  $|B| - \ell \stackrel{d}{=} B$ , and we assume  $\mu > 0$ .

**Lemma 2.1** (Théorème 2 of Le Gall and Yor [16], Theorems 3.3–3.4 of [31]). *The space-time local time process  $(L_\mu^+(x, t), x \in \mathbb{R}, t \geq 0)$  of  $R_\mu^+$  is such that for each  $a \in [0, \infty]$*

$$(L_\mu^+(x, \tau(a/\mu)), x \geq 0) \text{ is } \text{BESQ}_0(2/\mu) \text{ on } [0, a] \text{ continued on } [a, \infty) \text{ as } \text{BESQ}(0).$$

**Lemma 2.2** (Theorem 3.3 of Carmona, Petit and Yor [5]). *For each fixed  $v \geq 0$  the local time process  $(L_\mu^-(x, t), x \in \mathbb{R}, t \geq 0)$  of  $R_\mu^-$  evaluated at  $\tau_\mu^-(v) := \inf\{t \geq 0: L_\mu^-(0, t) > v\}$  yields independent processes*

$$(L_\mu^-(-x, \tau_\mu^-(v)), x \geq 0) \stackrel{d}{=} \text{BESQ}_v(2 - 2/\mu) \quad \text{and} \quad (L_\mu^-(x, \tau_\mu^-(v)), x \geq 0) \stackrel{d}{=} \text{BESQ}_v(0).$$

The following lemma exhibits independent copies of  $R_\mu^\pm$  in the setting of skew Brownian motion.

**Lemma 2.3.** *For  $\gamma \in (0, 1)$ , let  $X_\gamma$  be the skew Brownian motion driven by  $B$  as in (1.5). Let  $I^+ := (0, \infty)$  and  $I^- := (-\infty, 0)$ , and consider time changes  $\kappa_\gamma^\pm(s) = \inf\{t \geq 0: A_\gamma^\pm(t) > s\}$  where  $A_\gamma^\pm(t) := \int_0^t \mathbf{1}\{X_\gamma(r) \in I^\pm\} dr$ . Then*

- $W_\gamma^\pm := \pm B \circ \kappa_\gamma^\pm \stackrel{d}{=} |B| \pm \mu_\gamma^\pm \ell$ , where  $\mu_\gamma^\pm = 2\gamma/(1 \pm \gamma) > 0$ ,
- $W_\gamma^+$  and  $W_\gamma^-$  are independent.

*Proof.* Denote by  $n_{\text{ex}}$  the excursion intensity measure of reflecting Brownian motion relative to increments of  $\ell$ . See e.g. [24, Chapter XII]. Recall the excursion construction of skew Brownian motion from a Poisson point process with intensity measure  $n_{\text{ex}}$ . By standard thinning properties of Poisson point processes, the absolute values of excursions of  $X_\gamma$  away from 0 into  $I^\pm$  form independent Poisson point processes with intensity measures  $\frac{1}{2}(1 \pm \gamma)n_{\text{ex}}(d\omega)ds$ . By [18, Remark 6 and Equation (39)],  $\ell_\gamma(t)$  splits naturally into the contributions from these positive and negative excursions:

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_0^t \mathbf{1}\{X_\gamma(s) \in I^\pm \cap [-h, h]\} ds = \frac{1}{2}(1 \pm \gamma)\ell_\gamma(t) \tag{2.1}$$

By Knight’s theorem [24, Theorem V.(1.9)], the time changes  $\kappa_\gamma^\pm$  give rise to two independent reflecting Brownian motions  $X_\gamma^\pm = \pm X_\gamma \circ \kappa_\gamma^\pm$ . This argument is detailed in [24, page 242] for the case  $\gamma = 0$ , and extends easily to general  $|\gamma| < 1$ . See also the discussion after [18, Proposition 11].

By these time changes, the local times of  $X_\gamma^\pm$  at 0 are the time changes of the limits (2.1), namely  $\ell_\gamma^\pm(s) = \frac{1}{2}(1 \pm \gamma)\ell_\gamma(\kappa_\gamma^\pm(s))$ . We read (1.5) as a decomposition of  $B(t) = X_\gamma(t) + \gamma\ell_\gamma(t)$  into excursions away from the increasing process  $(\gamma\ell_\gamma(t), t \geq 0)$ . Then

$$W_\gamma^\pm(s) = \pm B(\kappa_\gamma^\pm(s)) = X_\gamma^\pm(s) \pm \gamma\ell_\gamma(\kappa_\gamma^\pm(s)) = X_\gamma^\pm(s) \pm \frac{2\gamma}{1 \pm \gamma}\ell_\gamma^\pm(s),$$

so that  $W_\gamma^\pm \stackrel{d}{=} R_\mu^\pm := |B| \pm \mu^\pm \ell$  as in Lemmas 2.1 and 2.2 for  $\mu^\pm := 2\gamma/(1 \pm \gamma)$ . □

*Proof of Theorem 1.3.* The definition of  $S_\delta(x)$  is such that the increasing path in Figure 1 is a multiple of the local time of the skew Brownian motion  $X_\gamma$ . We write (1.5) as  $B(t) = X_\gamma(t) + \gamma\ell_\gamma(t)$ . The meaning of this expression is that the positive excursions of  $X_\gamma$  are found in  $B$  as excursions above  $\gamma\ell_\gamma(t)$ , while the negative excursions of  $X_\gamma$  are found in  $B$  as excursions below  $\gamma\ell_\gamma(t)$ . Recall that  $W_\gamma^+$  and  $W_\gamma^-$  comprise excursions of  $B$  above and below  $\gamma\ell_\gamma$ , respectively. The theorem identifies the distributions of these local times by application of Lemmas 2.1 and 2.2, as will now be detailed.

In the setting of Lemma 2.3, we can apply Lemma 2.2 to  $W_\gamma^-$  to see that for all  $v > 0$ , the process  $W_\gamma^-$  up to the inverse  $\tau_\gamma^-(v) = \inf\{s \geq 0: L_\gamma^-(0, s) > v\}$  of the local time  $(L_\gamma^-(0, t), t \geq 0)$  of  $W_\gamma^-$  at zero has two independent local time processes  $(L_\gamma^-(-x, \tau_\gamma^-(v)), x \geq 0) \stackrel{d}{=} \text{BESQ}_v(0)$  and  $(L_\gamma^-(x, \tau_\gamma^-(v)), x \geq 0) \stackrel{d}{=} \text{BESQ}_v(2 - 2/\mu_\gamma^-)$ , where  $\mu_\gamma^- = 2\gamma/(1 + \gamma)$ , i.e.  $2 - 2/\mu_\gamma^- = 1 - 1/\gamma = -\delta < 0$ , since  $\gamma = 1/(1 + \delta) \in (0, 1)$ .

Similarly, Lemma 2.1 with  $a = \infty$ , yields that  $W_\gamma^+$  has ultimate local time process  $(L_\gamma^+(x, \infty), x \geq 0) \stackrel{d}{=} \text{BESQ}_0(2/\mu_\gamma^+)$  where  $\mu_\gamma^+ = 2\gamma/(1 - \gamma)$ , i.e.  $2/\mu_\gamma^+ = -1 + 1/\gamma = \delta$ .

Let us rewrite the results of the last two paragraphs in terms of the local times  $L = (L(x, t), x \in \mathbb{R}, t \geq 0)$  of  $B$ . Recall that  $S_\delta(x) = \inf\{t \geq 0: \gamma\ell_\gamma(t) > x\}$ ,  $x \geq 0$ , where  $\gamma = 1/(1 + \delta)$ , and also set  $S_\delta(x) := 0$  for  $x < 0$ . Note that

$$W_\gamma^-(A_\gamma^-(t)) \leq \gamma\ell_\gamma(t) \leq W_\gamma^+(A_\gamma^+(t)) \quad \text{and} \quad W_\gamma^-(A_\gamma^-(t)) \leq B(t) \leq W_\gamma^+(A_\gamma^+(t)),$$

where for each  $t \geq 0$  and in each of the two statements, at least one of the inequalities is an equality. Let

$$\mathcal{R}_\gamma^+ := \{(t, x) \in [0, \infty) \times \mathbb{R}: x \geq \gamma\ell_\gamma(t)\} = \{(t, x) \in [0, \infty) \times \mathbb{R}: S_\delta(x) \geq t\}.$$

Then the occupation measure  $U_\gamma^+$  of  $W_\gamma^+$  can be related to the occupation measure  $U$  of  $B$  by the usual change of variables  $u = \kappa_\gamma^+(r)$ , separately on each excursion interval of  $X_\gamma$  of a positive excursion to give

$$\begin{aligned} U_\gamma^+([0, s] \times [0, x]) &= \int_0^s \mathbf{1}_{[0, x]}(W_\gamma^+(r)) dr = \int_0^{\kappa_\gamma^+(s)} \mathbf{1}_{[0, x]}(B(u)) \mathbf{1}_{\mathcal{R}_\gamma^+}(u, B(u)) du \\ &= U([0, \kappa_\gamma^+(s)] \times [0, x]) \cap \mathcal{R}_\gamma^+. \end{aligned}$$

By the occupation density formula for  $U$  in its general form for time-varying integrands, we obtain

$$U_\gamma^+([0, s] \times [0, x]) = \int_0^x \int_0^{\kappa_\gamma^+(s)} \mathbf{1}_{\mathcal{R}_\gamma^+}(u, y) dy L(y, u) du = \int_0^x L(y, \kappa_\gamma^+(s) \wedge S_\delta(y)) dy.$$

Hence  $(L(y, \kappa_\gamma^+(s) \wedge S_\delta(y)), y \in \mathbb{R}, s \geq 0)$  is a local time for  $W_\gamma^+$  that is right-continuous in  $s$  and in  $y$ . In particular, we deduce from the continuity of  $y \mapsto L_\gamma^+(y, \infty)$  that  $L_\gamma^+(y, \infty) = L(y, S_\delta(y))$  for all  $y \geq 0$  almost surely. Similarly, we use the joint continuity of local times of  $W_\gamma^-$  of [5, Theorem 3.2] to obtain that  $L(y, \kappa_\gamma^-(s) \wedge S_\delta(y)) - L(y, S_\delta(y)) = L_\gamma^-(y, s)$  almost surely. In particular, we have  $L(0, t) = L_\gamma^-(0, A_\gamma^-(t))$ , hence  $\tau_\gamma^-(v) = A_\gamma^-(\tau(v))$  for all  $v \geq 0$ , and hence, for all  $x \geq 0, v \geq 0$  almost surely,

$$L(x, \tau(v)) - L(x, \tau(v) \wedge S_\delta(x)) = L_\gamma^-(x, \tau_\gamma^-(v)).$$

To complete the proof, we consider the random level  $\zeta'(v) = \inf\{x \geq 0: L(x, \tau(v)) = L(x, S_\delta(x) \wedge \tau(v))\}$ . Since  $L$  and  $S_\delta$  are both increasing and  $B(\tau(v)) = 0$  while  $B(S_\delta(x)) = B(S_\delta(x-)) = x$ , we can only have  $L(x, \tau(v)) = L(x, S_\delta(x) \wedge \tau(v))$  if  $S_\delta(x) > \tau(v)$ , and  $\zeta'(v) = \inf\{x \geq 0: S_\delta(x) > \tau(v)\}$ . Note that  $\zeta'(v) = \inf\{x \geq 0: L_\gamma^-(x, \tau_\gamma^-(v)) = 0\}$ , as a function of  $W_\gamma^-$ , is independent of  $W_\gamma^+$ . Conditionally given  $\zeta'(v) = a$ , we can apply Lemma 2.1 to obtain that  $(L_\gamma^+(x, A_\gamma^-(\tau(v))), x \geq 0) = (L(x, S_\delta(x) \wedge \tau(v)), x \geq 0)$  also has the desired distribution.  $\square$

*Proof of Corollary 1.4.* Let  $\delta > \delta' > 0$ . Let  $\Upsilon_{\text{top}} = (L(x, S_{\delta'}(x)), x \geq 0)$ ,  $\Upsilon_{\text{mid}} = (L(x, S_\delta(x)) - L(x, S_{\delta'}(x)), x \geq 0)$  and  $\Upsilon_{\text{rest}} = (L(x, \tau(v)) - L(x, S_\delta(x) \wedge \tau(v)), x \geq 0, v \geq 0)$ . By the proof of Theorem 1.3, we have  $(\Upsilon_{\text{top}}, \Upsilon_{\text{mid}})$  independent of  $\Upsilon_{\text{rest}}$ , and we have  $\Upsilon_{\text{top}}$  independent of  $(\Upsilon_{\text{mid}}, \Upsilon_{\text{rest}})$ . Hence,  $\Upsilon_{\text{mid}}$  is independent of  $\Upsilon_{\text{top}}$ . By additivity,  $\Upsilon_{\text{mid}} \stackrel{d}{=} \text{BESQ}_0(\delta - \delta')$ , independent of  $\Upsilon_{\text{top}} \stackrel{d}{=} \text{BESQ}(\delta)$ . A straightforward induction completes the proof.  $\square$

### 3 Some checks on Proposition 1.1

In this section, we further explore Proposition 1.1. Besides the SDE proof indicated in the previous section, the setting lends itself to a variety of computations, making explicit many instances of this result via beta-gamma algebra and giving rise to interesting integrals. We also indicate a Sturm–Liouville proof of Proposition 1.1.

To simplify presentation, for any real  $\delta$  and  $v \geq 0$  we will denote by  $\text{BESQ}_v^{(\delta)} = (\text{BESQ}_v^{(\delta)}(x), x \geq 0)$  a process with law  $\text{BESQ}_v(\delta)$ . For  $r \geq 0$  let  $\gamma(r)$  denote a gamma variable, with  $\gamma(0) = 0$  and

$$\frac{\mathbb{P}(\gamma(r) \in dt)}{dt} = f_r(t) := \frac{1}{\Gamma(r)} t^{r-1} e^{-t} \mathbf{1}(t > 0). \tag{3.1}$$

Fix  $x > 0$ . To check the implication of Corollary 1.2 that  $Y(x) + Y'(x) \stackrel{d}{=} \text{BESQ}_v^{(\delta)}(x)$  for all  $v \geq 0$ , by uniqueness of Laplace transforms it suffices to show for all  $\mu > 0$  that in the modified setting where  $Y' \stackrel{d}{=} \text{BESQ}_{\gamma(1)/\mu}^{(-\delta)}$ , meaning that  $Y(0)$  is assigned the exponential distribution of  $\gamma(1)/\mu$ , that  $Y(x) + Y'(x) \stackrel{d}{=} \text{BESQ}_{\gamma(1)/\mu}^{(0)}(x)$ . To show this, first recall some known facts:

**Lemma 3.1.** *Let  $\delta \geq 0$ . Then*

- (a)  $\text{BESQ}_0^{(\delta)}(x) \stackrel{d}{=} 2x\gamma(\delta/2)$ .
- (b)  $\text{BESQ}_{\gamma(1)/\mu}^{(-\delta)}(x) \stackrel{d}{=} (2\mu x + 1)\gamma(1)I(1/(2\mu x + 1)^{1+\delta/2})$  where  $\gamma(1)$  is independent of the indicator variable  $I(p)$  with Bernoulli ( $p$ ) distribution for  $p = 1/(2\mu x + 1)^{1+\delta/2}$ .

Here (a) is a consequence of the additivity property (1.3), while (b) details for  $-\delta = 2 - 2\alpha \leq 0$  the entrance law for  $\text{BESQ}(2 - 2\alpha)$  killed at  $T_0$ , with  $\alpha > 0$ , which was identified by [23, (3.2) and (3.5)]. The case of (b) for  $\delta = 0$  and  $\mu = 1/(2b)$  is also an easy consequence of the Ray–Knight description of Brownian local times  $L(x, T_{-b}), x \geq 0) \stackrel{d}{=} \text{BESQ}_{2b\gamma(1)}^{(0)}$ . Applying this instance of (b), we find that  $\text{BESQ}_{\gamma(1)/\mu}^{(0)}(x)$  has Laplace transform (in  $\lambda$ )

$$\left(1 - \frac{1}{2\mu x + 1}\right) + \frac{1}{2\mu x + 1} \frac{1}{1 + \lambda(2\mu x + 1)/\mu} = \frac{(2\lambda x + 1)\mu}{(2\lambda x + 1)\mu + \lambda}. \tag{3.2}$$

On the other hand, we obtain the Laplace transform of  $Y(x) + Y'(x)$  by conditioning  $Y(x)$  and  $Y'(x)$  on  $\zeta' = \inf\{x \geq 0: Y'(x) = 0\}$ . Specifically, now using (b) for  $Y'$  and (a) for  $Y$ , we find

$$\mathbb{E}(\exp(-\lambda(Y(x) + Y'(x)))) = \frac{1}{(2\mu x + 1)^{1+\delta/2}} \frac{1}{1 + \lambda(2\mu x + 1)/\mu} \frac{1}{(1 + \lambda 2x)^{\delta/2}} + \int_0^x \frac{(\delta + 2)\mu}{(2\mu m + 1)^{2+\delta/2}} \mathbb{E}\left(e^{-\lambda \text{BESQ}_{2m\gamma(\delta/2)}^{(0)}(x-m)}\right) dm. \tag{3.3}$$

By the additivity property of  $\text{BESQ}(0)$  and then proceeding as for (3.2), we have

$$\mathbb{E}\left(e^{-\lambda \text{BESQ}_{2m\gamma(\delta/2)}^{(0)}(x-m)}\right) = \left(\mathbb{E}\left(e^{-\lambda \text{BESQ}_{2m\gamma(1)}^{(0)}(x-m)}\right)\right)^{\delta/2} = \left(\frac{1 + 2(x-m)\lambda}{1 + 2\lambda x}\right)^{\delta/2}.$$

The change of variables  $m = xu, dm = xdu$  allows the integral in (3.3) to be expressed as

$$\frac{(\delta + 2)\mu x}{(1 + 2\lambda x)^{\delta/2}} \int_0^1 \frac{(1 + 2\lambda x(1 - u))^{\delta/2}}{(1 + 2\mu x u)^{2+\delta/2}} du.$$

Writing  $a = 2\lambda x, b = 2\mu x$  and  $q = 1 + \delta/2$ , this integral is of the form

$$\int_0^1 \frac{(1 + a(1 - u))^{q-1}}{(1 + bu)^{q+1}} du = \frac{(1 + a)^q - (1 + b)^{-q}}{(a + ab + b)q} \tag{3.4}$$

where the integral is evaluated as  $F(1) - F(0)$  for the indefinite integral

$$F(u) = -\frac{(1 + a(1 - u))^q (1 + bu)^{-q}}{(a + ab + b)q}.$$

The identification of (3.2) and (3.3) is now elementary using (3.4).



Consider next the distribution of  $\int_0^\infty (Y + Y')(x)dx$  in the setting of Corollary 1.2. By the corollary, this is the distribution of the corresponding integral of a  $\text{BESQ}_v(0)$ , which according to the Ray–Knight theorem for local times of  $B$  at time  $\tau(v)$  is that of

$$\tau_+(v) := \int_0^{\tau(v)} \mathbb{1}_{\{B_+ > 0\}} dt \stackrel{d}{=} \tau(v/2).$$

The equivalent equality of Laplace transforms at  $\frac{1}{2}\lambda^2$  reads

$$\mathbb{E} \left( \exp \left( -\frac{1}{2}\lambda^2 \int_0^\infty (Y + Y')(x)dx \right) \right) = \exp \left( -\frac{v}{2}\lambda \right). \tag{3.5}$$

This formula too can be checked from the construction of  $Y$  and  $Y'$  by conditioning on  $\zeta'$ . For the  $\text{BESQ}_0(\delta)$  process  $Y$  on  $[0, m]$  continued as  $\text{BESQ}_{Y(m)}(0)$ , we have

$$\mathbb{E} \left( \exp \left( -\frac{1}{2}\lambda^2 \int_0^\infty Y(x)dx \right) \middle| \zeta' = m \right) = \exp \left( -\frac{1}{2}\delta m \lambda \right), \tag{3.6}$$

by Lemma 2.1 applied with  $a = m$  and  $\mu = 2/\delta$ , since these substitutions make

$$\left( \int_0^\infty Y(x)dx \middle| \zeta' = m \right) \stackrel{d}{=} \int_0^\infty L_\mu^+(x, \tau(a/\mu))dx = \tau(a/\mu)$$

with  $a/\mu = \frac{1}{2}\delta m$ . On the other hand, given  $\zeta' = m$ , an application of the formula of [23, Proposition (5.10)] yields the first passage bridge functional

$$\begin{aligned} & \mathbb{E} \left( \exp \left( -\frac{1}{2}\lambda^2 \int_0^m Y'(x)dx \right) \middle| \zeta' = m \right) \\ &= \left( \frac{\lambda m}{\sinh(\lambda m)} \right)^{(4+\delta)/2} \exp \left( -\frac{v}{2m} (\lambda m \coth(\lambda m) - 1) \right). \end{aligned} \tag{3.7}$$

To complete the calculation of the left side of (3.5) we must integrate the product of expressions in (3.6) and (3.7) with respect to the distribution of  $\zeta'$ , the absorption time of  $\text{BESQ}_v(-\delta)$ , which is the distribution of  $v/(2\gamma(1 + \delta/2))$  with density

$$\frac{\mathbb{P}(\zeta' \in dm)}{dm} = \frac{v}{2m^2} f_{1+\delta/2} \left( \frac{v}{2m} \right)$$

where  $f_r(t)$  is the gamma( $r$ ) density at  $t$  as in (3.1). So at the level of the total integral functional, the identification of the distribution of  $Y + Y'$  as  $\text{BESQ}_v(0)$  implies the identity

$$\exp \left( -\frac{1}{2}\lambda v \right) = \int_0^\infty \frac{v^{1+\delta/2} t^{2+\delta/2}}{2^{\delta/2} \Gamma(1 + \delta/2)} \left( \frac{1}{\sinh(\lambda m)} \right)^{2+\delta/2} \exp \left( -\frac{1}{2}\delta \lambda m - \frac{v\lambda}{2} \coth(\lambda m) \right) dm.$$

Make the change of variables  $x = \lambda m$ ,  $dm = dx/\lambda$ , then set  $t = \lambda v/2$ ,  $\delta = 2p$ , to see that this evaluation is equivalent to the identity for all  $t > 0$  and  $p \geq 0$

$$\int_0^\infty \frac{\exp(-px - t \coth(x))}{(\sinh(x))^{2+p}} dx = \frac{\Gamma(1+p)e^{-t}}{t^{1+p}}. \tag{3.8}$$

This identity is in fact valid for all complex  $p$  with real part greater than  $-1$ . It is just the variation of Euler’s integral representation of the gamma function

$$\int_0^\infty y^p e^{-t(y+1)} dy = \frac{\Gamma(1+p)e^{-t}}{t^{1+p}}$$

disguised by the change of variable  $y + 1 = \coth(x)$ ,  $dy = -dx/\sinh^2(x)$ .

This kind of argument can be extended to a full proof of Proposition 1.1 by the method of computing the Laplace functional

$$\mathbb{E} \left( \exp \left( - \int_0^\infty Z(x) \rho(dx) \right) \right) \tag{3.9}$$

for suitable measures  $\rho$  on  $(0, \infty)$ , and showing that it equals the known Laplace functional of  $\text{BESQ}_{y+y'}(\delta - \delta')$ , found in [23] when  $\delta - \delta' \geq 0$ , for enough such  $\rho$ . Let us briefly sketch this here for the (slightly easier) case when  $\delta + \delta' \geq 0$  and  $y = 0, y' = v$ . Without loss of generality,  $\delta' > 0$ , as otherwise the statement follows from full additivity. The case  $y > 0$  is then also straightforward, while  $\delta - \delta' < 0$  follow similarly. We claim that for any function  $f: [0, \infty) \rightarrow [0, \infty)$  that is Lebesgue integrable on  $[0, z]$  for all  $z \geq 0$ , the Laplace functional (3.9) reduces to the known Laplace functional of  $\text{BESQ}_v(\delta - \delta')$  when  $\rho(dx) = f(x)dx$ . By [24, Theorem XI.(1.7)], the latter is

$$(\phi(\infty))^{(\delta-\delta')/2} \exp \left( \frac{v}{2} \phi'(0) \right) \tag{3.10}$$

where  $\phi$  is the unique positive, non-increasing solution to the Sturm–Liouville equation

$$\phi'' = 2f\phi, \quad \phi(0) = 1. \tag{3.11}$$

This solution is convex and converges at  $\infty$  to some  $\phi(\infty) \in (0, 1]$ . We compute the Laplace functional in the setting of Proposition 1.1 by conditioning first on  $\zeta'(v) = x$  and then on  $Y(x) = y$ . We use notation  $\mathbb{P}_y^{(\gamma)} := \text{BESQ}_y(\gamma)$  and also  $\mathbb{P}_{a,b}^{(\gamma),x}$  for the distribution of a  $\text{BESQ}(\gamma)$  bridge of length  $x$  from  $a \geq 0$  to  $b \geq 0$ . Specifically, for  $\delta' > 0$ , for each  $a > 0$  we define  $\mathbb{P}_{a,0}^{(-\delta'),x}$  for  $x \geq 0$  to be the first passage bridge obtained as the weakly continuous conditional distribution of  $\mathbb{P}_a^{(-\delta')}(\cdot | T_0 = x)$ . By duality, this equals  $\mathbb{P}_{a,0}^{(4+\delta'),x}$ , which is the time reversal of  $\mathbb{P}_{0,a}^{(4+\delta'),x}$ . See [23]. We need several expectations of quantities of the form  $\mathcal{L}(f, w) := \exp(-\int_0^w Y(u)f(u)du)$  and also use notation  $\theta_x(f) = f(x + \cdot)$ . In this notation, we want to compute

$$\int_0^\infty \mathbb{P}_{v,0}^{(-\delta'),x}(\mathcal{L}(f, x)) \left( \int_0^\infty \mathbb{P}_{0,y}^{(\delta),x}(\mathcal{L}(f, x)) \mathbb{P}_y^{(\delta-\delta')}(\mathcal{L}(\theta_x f, \infty)) \mathbb{P}(Y(x) \in dy) \right) \mathbb{P}(\zeta'(v) \in dx). \tag{3.12}$$

The key technical formula is a generalisation of [24, Theorem XI.(3.2)] from unit-length bridges to bridges of length  $x$ , which we express in terms of the solution of  $\phi'' = 2f\phi$  without restricting  $f$  to support in  $[0, x]$ . We obtain for all  $\gamma > 0, a \geq 0, b \geq 0, x > 0$

$$\mathbb{P}_{a,b}^{(\gamma),x}(\mathcal{L}(f, x)) = (\phi(x))^{\gamma/2} \exp \left( \frac{a}{2} \phi'(0) - \frac{b}{2} \frac{\phi'(x)}{\phi(x)} \right) \frac{q_{\sigma^2(x)}^{(\gamma)}(a(\phi(x))^2, b)}{q_x^{(\gamma)}(a, b)}, \tag{3.13}$$

where  $\sigma^2(x) = (\phi(x))^2 \int_0^x (\phi(u))^{-2} du$ , and  $q_u^{(\gamma)}(v, y) = \mathbb{P}_v^{(\gamma)}(Y(u) \in dy)/dy$  is the continuous  $\text{BESQ}(\gamma)$  transition density on  $(0, \infty)$ . Using  $\mathbb{P}_{v,0}^{(-\delta'),x} = \mathbb{P}_{v,0}^{(4+\delta'),x}$  for the first, this yields the two bridge functionals of (3.12) while the remaining functional can be obtained from [24, Theorem XI.(1.7)]. We leave the remaining details to the reader.

### 4 Related developments in the literature

We discuss three related developments in the literature. These are interval partition diffusions, an instance of a weaker form of additivity related to sticky Brownian motion, and local time flows generated by skew Brownian motion.

Motivated by Aldous’s conjectured diffusion on a space of continuum trees, the authors of [8] study interval partition diffusions in which interval lengths evolve as

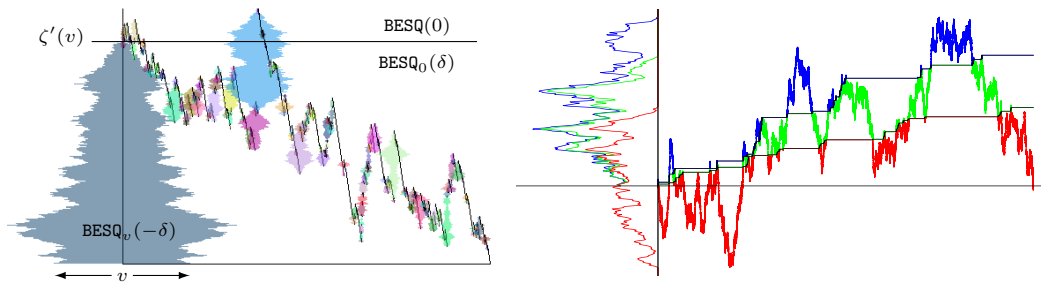


Figure 2: Left: The left-most shaded area is  $Y' \sim \text{BESQ}_v(-\delta)$  for some  $\delta \in (0, 2)$ , represented in the widths of a symmetric “spindle” shape. Other shaded areas form a Poisson point process of  $\text{BESQ}(-\delta)$  excursions placed on a “scaffolding”, an induced  $\text{Stable}(1+\delta/2)$  process whose jump heights are the excursion lifetimes. Simulation, as in [9], due to [7]. Right: Simulation of Brownian motion split along two increasing paths.

independent  $\text{BESQ}(-1)$  processes until absorption at 0, while new intervals are created according to a Poisson point process of  $\text{BESQ}(-1)$  excursions. See also [9] and further references there. Specifically, the construction for one initial interval of length  $v$  is illustrated in Figure 2 (left). Next to a  $\text{BESQ}_v(-1)$  process  $Y'$  with absorption time  $\zeta'(v)$ , the total sums of all other interval lengths form a process  $Y$  that is shown to be  $\text{BESQ}_0(1)$  up to  $\zeta'(v)$  continuing as  $\text{BESQ}(0)$ , as in Corollary 1.2. See [8, Theorem 1.5 and Corollary 5.19]. A generalisation to  $\text{BESQ}(-\delta)$  for  $\delta \in (0, 2)$  is indicated in [10, Section 6.4], to be taken up elsewhere.

Shiga and Watanabe [26] showed that families of one-dimensional diffusions with the additivity property can be parameterised by three real parameters, one of which corresponds to a linear time-change parameter affecting the diffusion coefficient, which we fix here without loss of generality. The family formed by the other two parameters,  $\delta$  and  $\mu$ , are the strong solutions to the stochastic differential equation

$$dY(t) = (\delta - \mu Y(t)) dt + 2\sqrt{Y(t)} dB(t), \quad Y(0) = y, \quad y \geq 0, \quad \delta \geq 0, \mu \in \mathbb{R}. \quad (4.1)$$

We observe that these families may be extended to  $\delta < 0$  with absorption at 0 much as in (1.1), and that the statement and proof of Proposition 1.1 generalizes straightforwardly to this case. Warren [29, Proposition 3] establishes an additivity in the case  $\delta = 0$  that involves the parameter  $\mu$  of (4.1), where the second process  $Y' = (Y'(t), t \geq 0)$  is driven by a Brownian motion  $(B'(t), t \geq 0)$  independent of the first process  $Y$ , but the first process  $(Y(t), t \geq 0)$  appears in the  $dt$ -part of its stochastic differential equation:

$$dY'(t) = \mu Y(t) dt + 2\sqrt{(Y'(t))^+} dB'(t), \quad Y'(0) = 0.$$

Specifically,  $Y + Y' \stackrel{d}{=} \text{BESQ}_y(0)$ . Furthermore, [29, Theorems 10-11 and Proposition 12] demonstrate how to find this decomposition embedded in the local times of a given Brownian motion, using the Brownian motion to drive a stochastic differential equation whose strong solution is sticky Brownian motion of parameter  $\mu \geq 0$ .

Burdzy et al. [3, 4] treat other aspects of what they call the local time flow generated by skew Brownian motion. They study solutions to uncountably many coupled variants of (1.5) jointly. Specifically, [4] focusses on  $(X_\gamma^{s,x}(t), L_\gamma^{s,x}(t)), t \geq s, x \in \mathbb{R}$ , for  $B(t)$  in (1.5) replaced by  $x + B(t) - B(s)$ , while [3] exhibits various one-dimensional families indexed by  $x$  or by  $\gamma$  that form Markov processes in a way reminiscent of Ray–Knight theorems.

Taking  $s = 0$ , one viewpoint is to read these coupled solutions as joint decompositions of  $B(t) = X_\gamma^{s,x}(t) - x + \gamma L_\gamma^{s,x}(t)$  along increasing paths  $-x + \gamma L_\gamma^{s,x}(t)$ . Consider the coupling in  $\gamma \in (0, 1)$  of [3, Theorem 1.3 and 1.4] when  $x = 0$ . They note for  $\gamma_1 < \gamma_2$

that  $\gamma_1 \ell_{\gamma_1}(t) \leq \gamma_2 \ell_{\gamma_2}(t)$  for all  $t \geq 0$ , cf. Figure 2 (right), and they establish a phase transition when  $\gamma_1 = \gamma_2/(1 + 2\gamma_2)$ . By our Corollary 1.4, applied to an increment  $Y_0^{(\delta_1)} - Y_0^{(\delta_2)} \stackrel{d}{=} \text{BESQ}(\delta_1 - \delta_2)$ , we identify the same phase transition with the behaviour around the critical (recurrence-transience threshold [24, Proposition XI.(1.5)]) dimension  $\delta = 2$  of  $\text{BESQ}_0(\delta)$ , since for  $\delta_i = -1 + 1/\gamma_i$ ,  $i = 1, 2$ , we have  $\gamma_1 = \gamma_2/(1 + 2\gamma_2)$  if and only if  $\delta_1 - \delta_2 = 2$ .

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