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Shifted critical threshold in the loop $O(n)$ model at arbitrarily small n*

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Abstract

In the loop $O(n)$ model a collection of mutually-disjoint self-avoiding loops is drawn at random on a finite domain of a lattice with probability proportional to

$$\lambda^{\# \text{ edges}} n^{\# \text{ loops}},$$

where $\lambda, n \in [0, \infty)$. Let μ be the connective constant of the lattice and, for any $n \in [0, \infty)$, let $\lambda_c(n)$ be the largest value of λ such that the loop length admits uniformly bounded exponential moments. It is not difficult to prove that $\lambda_c(n) = 1/\mu$ when $n = 0$ (in this case the model corresponds to the self-avoiding walk) and that for any $n \geq 0$, $\lambda_c(n) \geq 1/\mu$. In this note we prove that,

$$\lambda_c(n) > 1/\mu \quad \forall n > 0,$$
$$\lambda_c(n) \geq 1/\mu + c_0\, n + O(n^2) \quad \text{as } n \to 0,$$

on $\mathbb{Z}^d$, with $d \geq 2$, and on the hexagonal lattice, where $c_0 > 0$. This means that, when n is positive (even arbitrarily small), as a consequence of the mutual repulsion between the loops, a phase transition can only occur at a strictly larger critical threshold than in the self-avoiding walk.

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1 Introduction

The loop $O(n)$ model is defined as follows. Consider an infinite undirected graph $\mathcal{G} = (V, E)$ of bounded degree. For any finite sub-graph $G = (V_G, E_G) \subset \mathcal{G}$, let Ω_G be the set of spanning sub-graphs of G such that every vertex has degree either zero or two. It follows from this definition that every connected component of the graph $\kappa \in \Omega_G$ is either an isolated vertex or a loop. For any κ, let $o_G(\kappa)$ be the total number of edges of κ and let $L_G(\kappa)$ be the total number of loops of κ. Let $n, \lambda \in [0, \infty)$ be two parameters. The measure of the loop $O(n)$ model is a probability measure on Ω_G which assigns weights,

$$\mathbb{P}_{G,\lambda,n}(\kappa) := \frac{\lambda^{o_G(\kappa)}\, n^{L_G(\kappa)}}{Z_{\lambda,n}(G)}, \quad \kappa \in \Omega_G, \tag{1.1}$$

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where $Z_{\lambda,n}(G)$ is a normalizing constant, to which we will refer as *partition function* (we adopt the convention that $0^0 = 1$).

The loop $O(n)$ model was introduced on the hexagonal lattice as a graphical representation of the spin $O(n)$ model [4]. The central question concerning this model is describing the structure and the size of the loops in the limit of large graphs. This model presents a mathematically interesting and rich behaviour, which depends on the value of the parameters and on the structure of the underlying graph. It can be viewed as a model for random polymers interacting with a random environment through a 'rigid' potential. The study of random polymers in random environment is of great physical and mathematical interest (see for example [14] for a review). Another reason to consider this model is that it interpolates between several paradigmatic statistical mechanics models, to which it reduces for specific values of n, and, thus, allows to compare them. More precisely, the model reduces to self-avoiding walk when $n = 0$, the Ising model when $n = 1$, critical percolation when $n = \lambda = 1$, the dimer model when $n = 1$ and $\lambda = \infty$, proper 4-coloring when $n = 2$ and $\lambda = \infty$, integer-valued ($n = 2$) and tree-valued (integer $n \geq 3$) Lipschitz functions and the hard hexagon model ($n = \infty$) on the hexagonal lattice. We refer to [20] for an extensive discussion. Some of these relations are also valid on $\mathbb{Z}^d$ for a variant of this model where the loops are allowed to overlap and the number of overlaps receives a weight which depends on n [3]. Furthermore, when $n = 2$, the loop $O(n)$ model is related to nearest-neighbour random lattice permutations [1, 2, 10], whose study stems from physics, where they are related to the theory of Bose-Einstein condensation [9], and when $n = 2$ and $\lambda = \infty$, it is related to the double-dimer model (the only difference is that in random permutations and in the double-dimer model also 'loops' of length two are allowed).

We now briefly review the rigorous results on the loop $O(n)$ model. It was proved in [6] that, when $\mathcal{G}$ is the hexagonal lattice, $\mathbb{H}$, and n is large enough, the loops are *exponentially small* for any value of $\lambda \in (0, \infty)$ and that at least two distinct regimes exist: a disordered phase in which each vertex is unlikely to be surrounded by any loops (when $n\lambda^6$ is small), and an ordered phase which is a small perturbation of one of the three ground states (when $n\lambda^6$ is large). It was proved in [7] that, when $\mathcal{G} = \mathbb{H}$, $n \in [1, 2]$, and $\lambda = 1/\sqrt{2 + \sqrt{2 - n}}$ (the so called *Nienhuis' critical point*), the loop $O(n)$ model exhibits macroscopic loops. When $n = 0$, the loop $O(n)$ model corresponds to the single non-interacting random self-avoiding polygon (a self-avoiding walk which returns to the starting vertex). To see this formally, one could slightly modify the definition (1.1) and let $L_G(\kappa)$ be the number of loops in $\kappa \setminus \mathcal{P}_o(\kappa)$, with $\mathcal{P}_o(\kappa)$ being the connected component of κ containing the origin, o. This way, when $n = 0$, only the loop containing the origin can be observed and it gets a weight proportional to $\lambda^{|\mathcal{P}_o|}$. It is well known that in this case the length of $\mathcal{P}_o$ admits uniformly bounded exponential moments when $\lambda \in (0, 1/\mu)$, with $\mu = \mu(\mathcal{G})$ being the so-called *connective constant* of $\mathcal{G}$ (see (2.1) for a definition). The exact value of this constant is known on the hexagonal lattice [8], $\mu(\mathbb{H}) = 1/\sqrt{2 + \sqrt{2}}$. Moreover, it was proved in [5] (in a slightly different setting) that $\mathcal{P}_0$ is *weakly space-filling* when $\lambda \in (1/\mu, \infty)$. A variant of this model, (1.1), where the loops are allowed to intersect and the number of overlaps is weighted throught some vertex-factors which depend on n has been considered in [3]. There, it was proved that, on the torus of $\mathbb{Z}^d$, for any $d \geq 2$, if n is a large enough integer, the loops are exponentially small for any value of $\lambda \in (0, \infty)$, and that, when $d = 2$, for any positive integer n a break of translational symmetry occurs at a non-trivial value of λ. However, such results do not apply to the model under consideration in this paper, since they require that the vertex-factors are bounded from below and from above by positive constants uniformly in n.

Thus, only part of the conjectured phase diagram of the loop $O(n)$ model has been rigorously proved. This note proves a new fact concerning the phase diagram and the loop structure of the loop $O(n)$ model in $\mathbb{H}$ and in $\mathbb{Z}^d$, $d \geq 2$. Let $\lambda_c(n)$ be the supremum among all values of λ such that the loops are exponentially small (see (1.2) for a formal definition). In this paper we prove that, whenever $n > 0$, $\lambda_c(n) > \lambda_c(0) = 1/\mu(\mathcal{G})$. This means that, as a result of the mutual repulsion between the loops, which is present only when $n > 0$, it is more difficult for the loops to be long and, thus, the regime of macroscopic loops (if it exists) can only occur above a critical threshold which is strictly larger than in the case of no interaction. This is in accordance with the conjecture which was formulated by Nienhuis [17, 18, 19], namely that on the hexagonal lattice the critical threshold is strictly increasing with n when n is in $[0, 2]$ and, more precisely, it equals $1/\sqrt{2 + \sqrt{2 - n}}$. A similar fact was proved in [1], where it was proved that the critical threshold of random lattice permutations is strictly larger than $1/\mu(\mathcal{G})$, but the proof presented there is not valid for the model under consideration in this paper, since it essentially requires the existence of 'loops' of length two. Moreover, we provide a bound on the speed of convergence of $\lambda_c(n)$ to $1/\mu$ as n goes to zero, $\lambda_c(n) \geq 1/\mu + c_0 n + O(n^2)$, where $c_0 > 0$, corroborating another qualitative feature of the predicted phase diagram.

For any $\kappa \in \Omega_G$, and $x \in V_G$, let $\mathcal{P}_x(\kappa)$ be the subgraph of κ corresponding to the connected component containing x. Let $|\mathcal{P}_x(\kappa)|$ be the number of edges of $\mathcal{P}_x(\kappa)$. If no edge of κ has x as end-point, then the graph $\mathcal{P}_x(\kappa)$ contains only the vertex x and $|\mathcal{P}_x(\kappa)| = 0$. We will not deal with arbitrary graphs $G \subset \mathcal{G}$, but with *domains*. A graph $G = (V_G, E_G) \subset \mathcal{G} = (V, E)$ is a said to be a domain if its edge set is $E_G = \{\{x, y\} \in E : x, y \in V_G\}$. For any $\delta > 0$, $n \in (0, \infty)$ and $\lambda \in [0, \infty)$, define

$$\mathcal{L}(\delta, \lambda, n) := \sup_{\substack{G \subset \mathcal{G}: \\ G \text{ finite domain}}} \sup_{x \in V(G)} \mathbb{E}_{G,\lambda,n}\big(e^{\delta |\mathcal{P}_x|}\big),$$

where $\mathbb{E}_{G,\lambda,n}$ denotes the expectation with respect to $\mathbb{P}_{G,\lambda,n}$. If for some $\delta > 0$ the previous quantity is finite, the loop length admits uniformly bounded exponential moments. For any $n \in (0, \infty)$, we define the critical threshold,

$$\lambda_c(n) := \sup \Big\{ \lambda \in [0, \infty) \, : \, \mathcal{L}(\delta, \lambda, n) < \infty \text{ for some positive } \delta \Big\}. \tag{1.2}$$

Theorem 1.1. *Let $\mathcal{G}$ be $\mathbb{Z}^d$, with $d \geq 2$, or the hexagonal lattice, $\mathbb{H}$, and let $\mu = \mu(\mathcal{G})$ be the connective constant. We have that,*

$$\lambda_c(n) > 1/\mu, \qquad \forall n \in (0, \infty), \tag{1.3}$$

$$\lambda_c(n) \geq 1/\mu + c_0\, n + O(n^2) \quad \text{as } n \to 0, \tag{1.4}$$

where $c_0 = c_0(\mathcal{G}) \in (0, \infty)$ is a constant which depends only on $\mathcal{G}$.

Our proof is very simple and uses two ingredients. The first ingredient is the celebrated Kesten's pattern theorem, Theorem 2.3 below, which is used to prove that the "typical" loop presents a huge number of many little 'open loops' (a self-avoiding walk with one missing edge to make it a closed loop). The second ingredient is a multi-valued map principle to show that it is expensive for the system not to close these many 'open loops'. This leads to the upper bound $\mathbb{P}_{G,\lambda,n}(\mathcal{P}_x = \tilde{\mathcal{P}}) \leq \lambda^{|\tilde{\mathcal{P}}|} c^{|\tilde{\mathcal{P}}|}$ for some $c = (\lambda, n) \in (0, 1)$, which holds uniformly in $\tilde{\mathcal{P}}$, in G and $x \in V_G$. The enhancement $\lambda_c(n) > 1/\mu$ follows from the fact that $c < 1$.

Our result leads to the following natural questions. This paper proves that $\lambda_c(n) > \lambda_c(0)$ when $n > 0$. Is $\lambda_c(n)$ a strictly increasing function of n? The critical threshold of the loop $O(n)$ model on the hexagonal lattice has been conjectured to satisfy such a strict monotonicity property and it seems likely that the same is true also on $\mathbb{Z}^d$, $d \geq 2$. Furthermore, can one prove that $\lambda_c(n) < \infty$ on $\mathbb{Z}^d$, $d \geq 3$, for some values of $n \in (0, \infty)$? This should be the case, at least for small values of n.

This note is organized as follows. In Section 2 we present all the definitions and state Kesten's pattern theorem. In Section 3 we present the proof of Theorem 1.1.

2 Kesten's pattern theorem

In this section we introduce the definitions which are necessary to present the proof of Theorem 1.1 and we state Kesten's pattern theorem. All definitions and statements refer to $\mathbb{Z}^d$, with $d \geq 2$. Their generalization to the hexagonal lattice, $\mathbb{H}$, is simple.

A *self-avoiding walk* ω on $\mathbb{Z}^d$ beginning at the site $x \in \mathbb{Z}^d$ is defined as a sequence of sites $(\omega(0), \omega(1), \ldots \omega(N))$ with $\omega(0) = x$, satisfying $|\omega(j+1) - \omega(j)|_2 = 1$, where $|\cdot|_2$ denotes the L_2 norm, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. We write $|\omega| = N$ to denote the *length* of ω. We let $SAW_x(N)$ be the total number of self-avoiding walks of length N beginning at the site $x \in \mathbb{Z}^d$. The limit

$$\mu := \lim_{N\to\infty} \left(|SAW_x(N)| \right)^{\frac{1}{N}}, \tag{2.1}$$

exists [11], it is known as *connective constant*, and it satisfies $\mu = \mu(\mathbb{Z}^d) \in [d, 2d-1]$.

A pattern is a short self-avoiding walk occurring in a longer self-avoiding walk.

Definition 2.1. *A pattern $P = (p(0), \ldots, p(n))$ is said to occur at the j-th step of the self-avoiding walk $\omega = (\omega(0), \ldots, \omega(N))$ if there exists a vector $v \in \mathbb{Z}^d$ such that $\omega(j+k) = p(k) + v$ for every $k = 0, \ldots, n$.*

Kesten's pattern theorem does not apply to general patterns, but to *proper internal patterns*.

Definition 2.2. *A pattern P is a proper internal pattern if for every $k \in \mathbb{N}$ there exists a self-avoiding walk on which P occurs at k or more different steps.*

We are ready to state Kesten's pattern theorem, which was proved in [15] (see also [16][Chapter 7]). For a pattern P, an integer N, a vertex $x \in \mathbb{Z}^d$, and a real number w, let $SAW_x[N, w, P] \subset SAW_x(N)$ be the set of N-steps self-avoiding walks presenting the pattern P at less than w steps.

Theorem 2.3 (Kesten, 1963)**.** *Recall that $\mu = \mu(\mathbb{Z}^d)$ is the connective constant. For any proper internal pattern P, there exists an $a > 0$ small enough such that*

$$\limsup_{N\to\infty} \left(|SAW_x[N, aN, P]|\right)^{\frac{1}{N}} < \mu. \tag{2.2}$$

Before presenting the proof of the main theorem, we will provide a rigorous definition of self-avoiding polygon and state one important property. For $N \geq 4$, an N-step *self-avoiding polygon* $\mathcal{P}$ is an undirected graph $\mathcal{P} \subset G$ consisting of N nearest-neighbour sites and edges connecting them with the following property: there exists a corresponding $(N-1)$-step self-avoiding walk ω having $|\omega(N-1) - \omega(0)|_2 = 1$ such that the vertex set of $\mathcal{P}$ contains all the elements of ω and the edge set of $\mathcal{P}$ contains the edge joining $\omega(N-1)$ to $\omega(0)$ and the $N-1$ edges joining $\omega(i-1)$ to $\omega(i)$ $(i = 1, \ldots, N-1)$. Let $SAP_x(N)$ be the set of N-step self-avoiding polygons $\mathcal{P}$ such that one vertex of $\mathcal{P}$ is x. We also define the set $SAP_x(1)$, which includes only one graph, the (degenerate) 1-step self-avoiding polygon $\mathcal{P}$, which contains only the vertex x and no edges, and $SAP_x(N)$ is empty for $N = 2$ or N odd.

Hammersley proved in [12] the remarkable fact that the connective constant of the self-avoiding polygons exists and is the same as the connective constant of self-avoiding walks,

$$\mu(\mathbb{Z}^d) = \lim_{N\to\infty} \left(|SAP_x(N)| \right)^{\frac{1}{N}}. \tag{2.3}$$

From the super-multiplicativity property of self-avoiding polygons it also follows that

$$|SAP_x(N)| \leq \frac{(d-1)}{d} N\mu^N. \tag{2.4}$$

(see for example [16][Equations (3.2.1) and (3.2.5)]).

3 Proof of Theorem 1.1

Fix a dimension $d \geq 2$. We want to assign an orientation to self-avoiding polygons in order define pattern occurrences. For any vertex $x \in \mathbb{Z}^d$, any integer $N > 1$, and any self-avoiding polygon $\mathcal{P} \in SAP_x(N)$, one can identify precisely two $N-1$ steps self-avoiding walks, $\omega^1 = (\omega^1(0), \ldots, \omega^1(N-1))$, and $\omega^2 = (\omega^2(0), \ldots, \omega^2(N-1)) \in SAW_x(N-1)$, such that, for any $k \in \{1,2\}$ and $i \in [0, N-2]$, $\{\omega^k(i), \omega^k(i+1)\}$ is an edge of $\mathcal{P}$ and $\{\omega(N-1), \omega(0)\}$ is an edge of $\mathcal{P}$. Since the map which assigns to any self-avoiding polygon $\mathcal{P} \in SAP_x(N)$ the corresponding pair of self-avoiding walks $\{\omega^1, \omega^2\}$ is a bijection, we can define a new bijection $f : SAP_x(N) \mapsto SAW_x(N-1)$ which assigns to any self-avoiding polygon $\mathcal{P} \in SAP_x(N)$ a unique self-avoiding walk $f(\mathcal{P}) \in \{\omega^1, \omega^2\}$ in some arbitrary manner (for example, f might depend on some features $\mathcal{P}$). The function f is fixed in the whole proof and its definition will never be made explicit. We say that a pattern P occurs at the step $j \in [0, N-1]$ of a self-avoiding polygon $\mathcal{P} \in SAP_x(N)$ if it occurs at the step $j \in [0, N-1]$ of the self-avoiding walk $f(\mathcal{P}) \in SAW_x(N)$. We let $SAP_x(N, w, P) \subset SAP_x(N)$ be the set of self-avoiding polygons of length N such that the pattern P is present at less than w steps.

Consider a finite sub-graph $G = (V_G, E_G) \subset \mathbb{Z}^d$. Let also

$$Z_{\lambda,n}(G) = \sum_{\kappa \in \Omega_G} \lambda^{o_G(\kappa)} n^{L_G(\kappa)}, \tag{3.1}$$

be the partition function, which depends on the graph G.

We now define one specific pattern. Let P' be the pattern corresponding to the sequence of vertices $(o, \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1)$, with $o \in \mathbb{Z}^d$ being the origin and $\boldsymbol{e_i}$ the Cartesian unit vectors (see Figure 1). It is not difficult to see that such a pattern is proper internal. We start with an auxiliary lemma, which involves the self-avoiding polygons presenting such a pattern at many steps. Given two graphs $G_1 = (V_{G_1}, E_{G_1}) \subset G_2 = (V_{G_2}, E_{G_2})$, we let $G_2 \setminus G_1$ be the graph whose vertex set is $V_{G_2} \setminus V_{G_1}$ and whose edge set is $\{\{x,y\} \in E_{G_2} \ : \ x, y \in V_{G_2} \setminus V_{G_1}\}$.

Lemma 3.1. *For any $a \in (0,1)$ and $N \in \mathbb{N}$, let $G = (V_G, E_G) \subset \mathbb{Z}^d$ be an arbitrary finite domain, let $x \in V_G$ be an arbitrary vertex, let $\mathcal{P} \in SAP_x(N)$ be such that $\mathcal{P} \subset G$ and such that $\mathcal{P} \notin SAP_x(N, aN, P')$. Then,*

$$\frac{Z_{\lambda,n}(G \setminus \mathcal{P})}{Z_{\lambda,n}(G)} \leq \frac{1}{\left(1 + \lambda^4 n\right)^{aN}}.$$

Proof. Given a self-avoiding polygon $\mathcal{P} \subset G$ (which was defined as a graph), we let $\mathcal{U}(\mathcal{P})$ be the graph whose vertex set is $V_{\mathcal{P}}$ and whose edge set is $\{\{x,y\} \in E_G \ : \ x, y \in V_{\mathcal{P}}\}$. Note that $\mathcal{P}$ does not necessarily equal $\mathcal{U}(\mathcal{P})$, but it is always contained in $\mathcal{U}(\mathcal{P})$. The following relation holds,

$$Z_{\lambda,n}(G) \geq Z_{\lambda,n}(G \setminus \mathcal{P}) Z_{\lambda,n}(\mathcal{U}(\mathcal{P})). \tag{3.2}$$

Indeed, in the right-hand side we have the weight of configurations $\kappa \in \Omega_G$ such that no loop contains one vertex in $V_{\mathcal{P}}$ and one vertex in $V_G \setminus V_{\mathcal{P}}$ at the same time, while in the left-hand side we have the weight of *all* configurations $\kappa \in \Omega_G$.

For a self-avoiding polygon $\mathcal{P} \in SAP_x(N)$ satisfying the assumptions of the lemma, let x_1, x_2, $\ldots x_{aN}$ be the sequence of the first aN sites of $f(\mathcal{P})$ where the pattern P' occurs, ordered in order of appearance along $f(\mathcal{P})$, writing aN in place of $\lceil aN \rceil$. For any $i \in [1, aN]$, let now Q_i be the (unique) self-avoiding polygon of length four containing the vertices $\{x_i, x_i + \boldsymbol{e_2}, x_i + \boldsymbol{e_1} + \boldsymbol{e_2}, x_i + \boldsymbol{e_1}\}$ and the edges connecting them (see Figure 1). Let $\cup_{i=1}^{aN} Q_i$ be the graph corresponding to the union of the vertex sets and of the edge

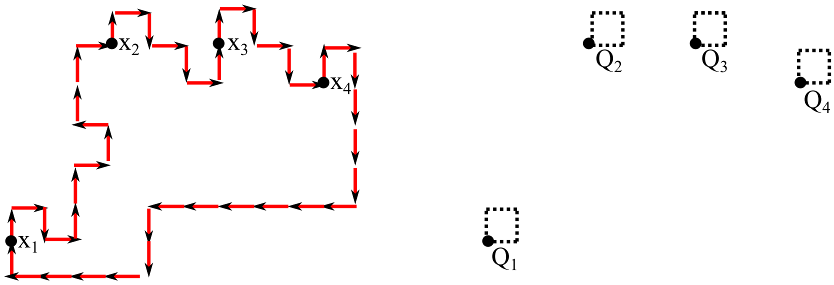

Figure 1: *Left:* A self-avoiding polygon $\mathcal{P}$ presenting the pattern P' at the vertices x_i. *Right:* Self-avoiding polygons of length four, Q_i, for the self-avoiding polygon $\mathcal{P}$ represented on the left.

sets of the self-avoiding polygons Q_i, $i \in [1, aN]$. Since $\cup_{i=1}^{aN} Q_i \subset \mathcal{U}(\mathcal{P})$ (here we use the fact that G is a domain), we deduce that,

$$Z_{\lambda,n}(\mathcal{U}(\mathcal{P})) \geq Z_{\lambda,n}(\cup_{i=1}^{aN} Q_i). \tag{3.3}$$

We deduce from (3.2) and (3.3) that,

$$\frac{Z_{\lambda,n}(G \setminus \mathcal{P})}{Z_{\lambda,n}(G)} \leq \frac{Z_{\lambda,n}(G \setminus \mathcal{P})}{Z_{\lambda,n}(G \setminus \mathcal{P}) Z_{\lambda,n}(\mathcal{U}(\mathcal{P}))} \leq \frac{1}{Z_{\lambda,n}\big(\cup_{i=1}^{aN} Q_i\big)}. \tag{3.4}$$

We now claim that

$$Z_{\lambda,n}\big(\cup_{i=1}^{aN} Q_i\big) = (1 + \lambda^4 n)^{aN}, \tag{3.5}$$

which concludes the proof of the lemma when replaced in the previous expression.

Thus, for a subset $B \subset \{1, 2, \dots aN\}$ (which might be $B = \emptyset$), let $\kappa_B \in \Omega_{\cup_{i=1}^{aN} Q_i}$ be the configuration such that, for all $i \in B$, $\mathcal{P}_{x_i} = Q_i$, and for all $i \in \{1, 2, \dots aN\} \setminus B$, $\mathcal{P}_{x_i}$ is a degenerate self-avoiding polygon containing only the vertex x_i. We have that, $L_G(\kappa_B) = |B|$ and that $o(\kappa_B) = \lambda^{4|B|}$. Thus,

$$Z_{\lambda,n}(\cup_{i=1}^{aN} Q_i) = \sum_{B \subset \{1,2,\dots aN\}} n^{|B|} \lambda^{4|B|} = \sum_{j=0}^{aN} \binom{aN}{j} n^j \lambda^{4j} = \big(1 + \lambda^4 n\big)^{aN}.$$

This concludes the proof of (3.5) and thus the proof of the lemma. □

We now present the proof of Theorem 1.1. The starting point of the proof is the observation that, if $\mathcal{P} \in SAP_x[N]$ with $N > 1$, then

$$\mathbb{P}_{G,\lambda,n}(\mathcal{P}_x = \mathcal{P}) = n\, \lambda^{|\mathcal{P}|} \frac{Z_{\lambda,n}(G \setminus \mathcal{P})}{Z_{\lambda,n}(G)} \leq n \lambda^{|\mathcal{P}|}. \tag{3.6}$$

We have that, for an arbitrary real $a \in (0, 1)$, and $\ell \in \mathbb{N}$,

$$\mathbb{P}_{G,\lambda,n}(|\mathcal{P}_x| > \ell) = \sum_{N=\ell+1}^{\infty} \sum_{\substack{\mathcal{P} \in SAP_x(N): \\ \mathcal{P} \subset G}} \mathbb{P}_{G,\lambda,n}(\mathcal{P}_x = \mathcal{P}) \tag{3.7}$$

$$= \sum_{N=\ell+1}^{\infty} \Big(\sum_{\substack{\mathcal{P} \in SAP_x(N, aN, P'): \\ \mathcal{P} \subset G}} \mathbb{P}_{G,\lambda,n}(\mathcal{P}_x = \mathcal{P}) + \sum_{\substack{\mathcal{P} \in SAP_x(N): \\ \mathcal{P} \notin SAP_x(N, aN, P'), \mathcal{P} \subset G}} \mathbb{P}_{G,\lambda,n}(\mathcal{P}_x = \mathcal{P}) \Big). \tag{3.8}$$

We will now provide an upper bound for the two terms above. For the first term, we apply Kesten's pattern theorem, Theorem 2.3. Thus, fix $a' > 0$ small enough such that

$$\mu' := \limsup_{N\to\infty} |SAP_x[N, a'N, P']|^{\frac{1}{N}} \leq \limsup_{N\to\infty} |SAW_x[N, a'N, P']|^{\frac{1}{N}} < \mu. \tag{3.9}$$

Then, define $\lambda_1' := \frac{2}{\mu+\mu'}$, which satisfies $\lambda_1' > \frac{1}{\mu}$, and assume that $\lambda \in (0, \lambda_1')$. We deduce from (3.6) and (3.9) that there exists a constant $c_1 \in (0, \infty)$ such that, for any $\ell \in \mathbb{N}$,

$$\begin{aligned}\sum_{N=\ell+1}^{\infty} \sum_{\substack{\mathcal{P}\in SAP_x(N,a'N,P'):\\ \mathcal{P}\subset G}} & \mathbb{P}_{G,\lambda,n}(\mathcal{P}_x = \mathcal{P}) \\ & \leq \; n \sum_{N=\ell+1}^{\infty} |SAP_x(N, a'N, P')|\lambda^N \\ & \leq c_1 \sum_{N=\ell+1}^{\infty} \big(\frac{\mu+\mu'}{2}\big)^N \lambda^N \leq \frac{c_1}{1-\frac{\lambda}{\lambda_1'}} \big(\frac{\lambda}{\lambda_1'}\big)^{(\ell+1)}. \end{aligned} \tag{3.10}$$

We now use the previous lemma to provide an upper bound for the second term in the right-hand side of (3.8). From (2.4), (3.6) and Lemma 3.1, we deduce that, if

$$\lambda < \frac{\big(1+\lambda^4 n\big)^{a'N}}{\mu} \tag{3.11}$$

then there exists $c_2, c_3 \in (0, \infty)$, which depend only on λ and n, such that, for any $\ell \in \mathbb{N}$,

$$\begin{aligned}\sum_{N=\ell+1}^{\infty} \sum_{\substack{\mathcal{P}\in SAP_x(N):\\ \mathcal{P}\notin SAP_x(N,a'N,P),\mathcal{P}\subset G}} & \mathbb{P}_{G,\lambda,n}(\mathcal{P}_x = \mathcal{P}) \\ & = \; n \sum_{N=\ell+1}^{\infty} \sum_{\substack{\mathcal{P}\in SAP_x(N):\\ \mathcal{P}\notin SAP_x(N,a'N,P),\mathcal{P}\subset G}} \lambda^{|\mathcal{P}|}\frac{Z_{\lambda,n}(G\setminus\mathcal{P})}{Z_{\lambda,n}(G)} \\ & \leq \; n \sum_{N=\ell+1}^{\infty} |SAP_x(N)|\lambda^N \big(\frac{1}{1+\lambda^4 n}\big)^{a'N} \\ & \leq \; n\,\frac{(d-1)}{d} \sum_{N=\ell+1}^{\infty} N\,\big(\frac{\lambda\,\mu}{(1+\lambda^4 n)^{a'}}\big)^N = c_2 e^{-c_3\ell}. \end{aligned} \tag{3.12}$$

Let $\lambda_1 = \lambda_1(n)$ be the solution of

$$\lambda\,\mu \;=\; (1+\lambda^4 n)^{a'} \tag{3.13}$$

and note that $\lambda_1(n) > \frac{1}{\mu}$ for any $n > 0$ and that (3.11) and (3.12) hold whenever $\lambda \in (0, \lambda_1)$. Combining (3.10) and (3.12) in (3.8), we deduce that, if

$$\lambda < \min\{\lambda_1', \lambda_1(n)\}, \tag{3.14}$$

we can find $\delta > 0$ such that $\mathcal{L}(\delta, \lambda, n) < \infty$. Thus, we proved that $\lambda_c(n) \geq \min\{\lambda_1', \lambda_1(n)\} > 1/\mu$ and obtained (1.3).

We now prove (1.4). Using the fact that, for any n smaller than a positive value n_0, $\min\{\lambda_1', \lambda_1(n)\} = \lambda_1(n)$, using (3.13) and performing a Taylor expansion, we obtain that, for any $n \in (0, n_0)$,

$$\begin{aligned}\lambda_c(n) - 1/\mu \geq \lambda_1(n) - 1/\mu \\ &= \frac{(1+\lambda_1^4(n)\,n)^{a'} - 1}{\mu} \\ &= \frac{a'}{\mu}\lambda_1^4(0)\ n + O(n^2) \\ &= \frac{a'}{\mu^5}\,n + O(n^2).\end{aligned}$$

This leads to (1.4) and concludes the proof.

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