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Moment bounds for some fractional stochastic heat equations on the ball*

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Abstract

In this paper, we obtain upper and lower bounds for the moments of the solution to a class of fractional stochastic heat equations on the ball driven by a Gaussian noise which is white in time and has a spatial correlation in space of Riesz kernel type. We also consider the space-time white noise case on an interval.

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1 Introduction

Consider the fractional stochastic heat equation on the unit ball $D = \{y \in \mathbb{R}^d : |y| < 1\}$, $d \geq 1$, with zero Dirichlet boundary conditions:

$$\begin{cases} \partial_t u_t(x) = -(-\Delta)^{\alpha/2} u_t(x) + \lambda \sigma(u_t(x)) \dot{W}(t, x) & x \in D, \quad t > 0, \\ u_t(x) = 0 \quad x \in D^c, \ \ t > 0, \end{cases} \tag{1.1}$$

and the initial condition is a measurable and bounded function $u_0 : D \to \mathbb{R}_+$. The operator $-(-\Delta)^{\alpha/2}$, where $0 < \alpha \leq 2$, is the L^2-generator of a symmetric α-stable process killed when exiting the ball D. The coefficient $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function. The Gaussian noise $\dot{W}(t, x)$ is white in time and coloured in space, that is,

$$\mathrm{E}\left(\dot{W}(t,x)\dot{W}(s,y)\right) = \delta_0(t-s) f(x-y), \tag{1.2}$$

where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a nonnegative definite (generalized) function whose Fourier transform $\hat{f} = \mu$ is a tempered measure. Finally, the parameter $\lambda > 0$ measures the level of the noise.

Following Walsh [21], we define the mild solution to equation (1.1) as the adapted and jointly measurable random field $u = \{u_t(x)\}_{t>0, x\in D}$ satisfying

$$u_t(x) = \int_D u_0(y) p_D(t,x,y)\, dy + \lambda \int_D \int_0^t p_D(t-s,\, x,\, y) \sigma(u_s(y)) W(ds, dy), \tag{1.3}$$

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where $p_D(t,x,y)$ denotes the Dirichlet fractional heat kernel on D and the stochastic integral is understood in an extended Itô sense.

Following Dalang [7], it is well-known (see also [19, Appendix] and [8]), that if the spectral measure μ satisfies that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^\alpha} < \infty, \tag{1.4}$$

then there exists a unique random field solution u to equation (1.3). Moreover, for all $p \geq 2$ and $T > 0$,

$$\sup_{t\in[0,T],x\in D} \mathrm{E}|u_t(x)|^p < \infty.$$

Examples of spatial correlations satisfying (1.4) are:

1. The Riesz kernel $f(x) = |x|^{-\beta}$, $0 < \beta < d$. In this case, $\mu(d\xi) = c|\xi|^{-(d-\beta)}d\xi$ and it is easy to check that condition (1.4) holds whenever $\beta < \alpha$.
2. The fractional kernel $f(x) = \prod_{i=1}^d |x_i|^{2H_i-2}$, where $H_i \in (\frac{1}{2}, 1)$ for $i = 1, ..., d$. In this case, $\mu(d\xi) = c\prod_{i=1}^d |\xi_i|^{1-2H_i}d\xi$ and condition (1.4) holds whenever $\sum_{i=1}^d H_i > d - \frac{\alpha}{2}$.
3. The Bessel kernel $f(x) = \int_0^\infty y^{\frac{\eta-d}{2}} e^{-y} e^{-\frac{|x|^2}{4y}} dy$. In this case, $\mu(d\xi) = c(1+|\xi|^2)^{-\eta/2}d\xi$ and condition (1.4) holds whenever $\eta > d - \alpha$.
4. The space-time white noise case $f = \delta_0$. In this case, $\mu(d\xi) = d\xi$, and (1.4) is only satisfied when $\alpha > d$, that is, $d = 1$ and $1 < \alpha \leq 2$.

Recall that the fractional heat kernel $p_D(t,x,y)$ has spectral decomposition

$$p_D(t,x,y) = \sum_{n=1}^\infty e^{-\mu_n t}\Phi_n(x)\Phi_n(y) \quad \text{for all} \quad x,\, y \in D, \quad t > 0,$$

where $\{\Phi_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(D)$ and $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$ is a sequence of positive numbers such that, for every $n \geq 1$,

$$\begin{cases} -(-\Delta)^{\alpha/2}\Phi_n(x) = -\mu_n\Phi_n(x) & x \in D, \\ \Phi_n(x) = 0 & x \in D^c. \end{cases} \tag{1.5}$$

Some properties of these families of eigenvalues and eigenfunctions are the following. From [3, Theorem 2.3] there exist positive constants c_1 and c_2 such that, for every $n \geq 1$,

$$c_1 n^{\alpha/d} \leq \mu_n \leq c_2 n^{\alpha/d}.$$

Moreover, from [5, Theorem 4.2] there exists $c > 1$ such that, for all $x \in D$,

$$c^{-1}(1-|x|)^{\alpha/2} \leq \Phi_1(x) \leq c(1-|x|)^{\alpha/2}, \tag{1.6}$$

In the case that $d = 1$, $\alpha = 2$, and $D = (-1,1)$, we have $\Phi_n(x) = \sin(\frac{n\pi x}{2})$ and $\mu_n = (\frac{n\pi}{2})^2$.

The aim of this paper is to obtain upper and lower bounds in terms of $t > 0$ and $\lambda > 0$ for the moments of the solution to equation (1.3). For this, we need some further assumptions. We consider the following class of covariances that generalizes the Riesz kernel.

Hypothesis 1.1. *There exist positive constants c_1, c_2 and $0 < \beta < \alpha \wedge d$ such that, for all $x \in \mathbb{R}^d$,*

$$c_1|x|^{-\beta} \leq f(x) \leq c_2|x|^{-\beta}.$$

Since we are interested in upper and lower bounds for the moments, we also need the following assumption on the coefficient σ.

Hypothesis 1.2. *There exist positive constants l_σ and L_σ such that, for all $x \in \mathbb{R}$,*

$$l_\sigma |x| \leq |\sigma(x)| \leq L_\sigma |x|.$$

Finally, for the lower bounds, we need the following additional assumption on the initial data.

Hypothesis 1.3. *There exists $\epsilon \in (0, \frac{1}{2})$ such that*

$$\inf_{x \in D_\epsilon} u_0(x) > 0,$$

where $D_\epsilon = \{y \in \mathbb{R}^d : |y| \leq 1 - \epsilon\}$.

Essentially Hypothesis 1.3 says that there exists a large enough closed set of positive measure inside D where the initial condition stays positive. The condition $\epsilon < \frac{1}{2}$ ensures that $\min(\epsilon, 1-\epsilon) = \epsilon$ which implies that D_ϵ contains the closed ball of radius ϵ. This fact will be used in the proof of Proposition 3.1 below, which is a key step to obtain the lower bounds. Hypotheses 1.2 and 1.3 are usual when studying intermittency properties of SPDEs.

We are now ready to state the main result of this paper.

Theorem 1.4. *Assume Hypothesis 1.3.*

a) *If f satisfies Hypothesis 1.1 and $\sigma(x) = x$, then for all $p \geq 2$ and $\delta > 0$, there exist positive constants c_1, $\bar{c}_1$, $c_2(\epsilon)$, $\bar{c}_2(\epsilon)$ such that for all $\lambda > 0$,*

$$\bar{c}_2^p e^{pt\left(c_2 \lambda^{\frac{2\alpha}{\alpha-\beta}} - \mu_1\right)} \leq \inf_{x \in D_\epsilon} \mathrm{E}|u_t(x)|^p \leq \sup_{x \in D} \mathrm{E}|u_t(x)|^p \leq \bar{c}_1^p e^{pt\left(c_1 p^{\frac{\alpha}{\alpha-\beta}} \lambda^{\frac{2\alpha}{\alpha-\beta}} - (1-\delta)\mu_1\right)}.$$

b) *If $f = \delta_0$ and σ satisfies Hypothesis 1.2, then for all $p \geq 2$ and $\delta > 0$, there exist positive constants $c_1(L_\sigma)$, $\bar{c}_1(\delta)$, $c_2(\epsilon, \ell_\sigma)$, $\bar{c}_2(\epsilon)$ such that for all $\lambda > 0$,*

$$\bar{c}_2^p e^{pt\left(c_2 \lambda^{\frac{2\alpha}{\alpha-1}} - \mu_1\right)} \leq \inf_{x \in D_\epsilon} \mathrm{E}|u_t(x)|^p \leq \sup_{x \in D} \mathrm{E}|u_t(x)|^p \leq \bar{c}_1^p e^{pt\left(c_1 z_p^{\frac{2\alpha}{\alpha-1}} \lambda^{\frac{2\alpha}{\alpha-1}} - (1-\delta)\mu_1\right)},$$

where z_p is the optimal constant in Burkholder-Davis-Gundy's inequality (see [10]).

Both upper bounds hold for all $t > 0$ while both lower bounds hold for all $t > c(\alpha)\lambda^{-\frac{2\alpha}{\alpha-1}}$. When $\alpha = 2$, both lower bounds hold for all $t > 0$.

Several remarks are in order. Observe that the bounds are not sharp in p because the proof of the lower bound is based on a second moment argument. However, as explained below, we are mainly interested in the dependence on λ, t and μ_1 of the moment bounds. Observe also that in the multidimensional case, we only consider the case $\sigma(x) = x$ known as parabolic Anderson model, since the method used in the space-time white noise case does not seem to apply in the multidimensional space setting. Instead, have used the Wiener-chaos expansion of the solution which is very suitable when $\sigma(x) = x$.

The lower bound of Theorem 1.4b) when $\alpha = 2$ was already obtained in the recent paper [22]. Thus, Theorem 1.4 extends this lower bound to the fractional Laplacian and higher space dimensions, and provides an upper bound of a similar type. Remark that Theorem 1.4 can be easily extended to the ball of radius $R > 0$. However, the extension to a bounded domain is not straightforward, because of the argument used in the proof of Proposition 3.1.

A direct consequence of Theorem 1.4 are the following bounds for the moment-type Lyapunov upper and lower exponents, in terms of $\lambda > 0$. In case a), we obtain that for all $\lambda > 0$,

$$\begin{aligned} p\left(c_2\lambda^{\frac{2\alpha}{\alpha-\beta}} - \mu_1\right) &\leq \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in D_\epsilon} \mathrm{E}|u_t(x)|^p \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} \mathrm{E}|u_t(x)|^p \leq p\left(\tilde{c}_1(p)\lambda^{\frac{2\alpha}{\alpha-\beta}} - (1-\delta)\mu_1\right), \end{aligned} \tag{1.7}$$

where $\tilde{c}_1(p) = c_1 p^{\frac{\alpha}{\alpha-\beta}}$. Similar bounds hold for case b). Recall from [10] that u is said to be weakly intermittent if for all $x \in D$

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}|u_t(x)|^2 > 0$$

and for all $p \geq 2$ and $x \in D$

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}|u_t(x)|^p < \infty.$$

Heuristically, this phenomenon says that the solution u will be concentrated into a few very high peaks when t is large. See [10] and the references therein for a more detailed explanation of this phenomenon. The bounds in (1.7) show that in case a), if $\lambda \leq ((1-\delta)\mu_1/\tilde{c}_1(2))^{\frac{\alpha-\beta}{2\alpha}}$, then the solution to equation (1.1) is not weakly intermittent, while if $\lambda \geq (\mu_1/c_2)^{\frac{\alpha-\beta}{2\alpha}}$, then the solution is weakly intermittent. A similar result holds for case b). However, using the definition of $\tilde{c}_1(p)$ we observe that for any fixed $\lambda > 0$ we can choose p_0 large enough such that $\lambda > ((1-\delta)\mu_1/\tilde{c}_1(p_0))^{\frac{\alpha-\beta}{2\alpha}}$. In this case, we cannot guarantee that there is no intermittency and we would need to have a sharp lower bound in p in order to check what happens in those cases. The intermittency would be of a different type that the one stated above since it would only hold for sufficiently large moments ($p > p_0$). But this would be sufficient in order to see the large peaks phenomenon. We leave this question open for future research.

Intermittency for equations of the type (1.1) but in all $\mathbb{R}^d$ have been largely studied in the literature, see e.g. [11, 16, 1, 14]. However, much less is known in the case of bounded domains. In the recent paper [12] (see also [13] for the extension to the fractional Laplacian), it is shown the existence of $\lambda_0(\mu_1) > 0$ such that for all $\lambda < \lambda_0$,

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} \mathrm{E}|u_t(x)|^p < 0,$$

and for all $\epsilon > 0$, the existence of a $\lambda_1(\mu_1, \epsilon) > 0$ such that for all $\lambda > \lambda_1$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in D_\epsilon} \mathrm{E}|u_t(x)|^p < \infty.$$

Here u is the solution to equation (1.1) with a general spatial covariance function f, and σ satisfying Hypothesis 1.2. However, λ_0 and λ_1 are not explicit in those papers. Therefore, Theorem 1.4 provides an extension of these results with an explicit dependence of λ_0 and λ_1 in terms of μ_1. Observe that the results in [12, 22, 13] already imply a dicotomy on the intermittency of the solution depending on large and small values of λ. In this paper, precise bounds for the moments for t fixed are proved, which in particular imply more accurate estimates on λ to deduce intermittency or non-intermittency of the solution. Observe also that this dicotomy phenomenon does not occur if one considers the same equation (1.1) in $\mathbb{R}^d$ or in D but with Neumann boundary conditions. In those cases, the solution is weakly intermittent for all $\lambda > 0$, see e.g. [12, 14].

Theorem 1.4a) also implies that for all $p \geq 2$, $t > 0$ and $x \in D_\epsilon$,

$$\lim_{\lambda\to\infty} \frac{\log\log \mathrm{E}|u_t(x)|^p}{\log \lambda} = \frac{2\alpha}{\alpha-\beta},$$

and similarly for the case b), which is known as the excitation index of the solution introduced by [17]. This result was obtained in [19] for $p = 2$, f the Riesz kernel and σ satisfying Hypothesis 1.2. See also [17, 9] for previous results when $\alpha = 2$ and W is space-time white noise. The results in [9, 19] were the first that used the Gronwall's inequalities stated in Propositions A.1 and A.2 to show these type of results. The proof of Theorem 1.4b) will be also based on those inequalities.

Consider now the deterministic heat equation $\partial_t u = \Delta u + \lambda u$ on a bounded domain $\mathcal{O}$ in $\mathbb{R}^d$, $d \leq 3$, with smooth boundary and Dirichlet boundary condition $u_t(x) = 0, x \in \partial\mathcal{O}, t > 0$, and intial condition $u_0(x) = f(x)$, $f \in L^2(\mathcal{O})$. It is shown in [18] that if k_0 is the smallest integer such that $\langle f, e_{k_0}\rangle \neq 0$, then

$$\limsup_{t\to\infty} \frac{1}{t}\log \|u_t\|_{L^2(\mathcal{O})} = \lambda - \mu_{k_0}.$$

In the same paper, the equation $\partial_t u_t(x) = \Delta u_t(x) + \lambda u_t(x) dW_t$, where W_t is a real-valued Wiener process is also considered. In this case, following similar computations as in that paper, it is easy to show that

$$\limsup_{t\to\infty} \frac{1}{t}\log \sqrt{\mathrm{E}\|u_t\|^2_{L^2(\mathcal{O})}} = \frac{\lambda^2}{2} - \mu_{k_0}.$$

Hence, the dycotomy phenomenon is also present in the deterministic case and the space independent white noise case. Observe that in those cases we have precise expressions for the Lyapunov exponents. For our equation (1.1), even in the space-time white noise case and parabolic Anderson model, obtaining an explicit expression for the upper second moment type Lyapunov exponent remains an open problem. Theorem 1.4 gives a first hint of the general form of this expression.

The rest of the paper is organized as follows. Section 2 is devoted to define rigorously the Gaussian noise W, and the Wiener-chaos expansion of square integrable random variables. In Section 3 we prove several heat kernel estimates that are needed for the proof of Theorem 1.4, and are also interesting in their own right. Section 4 is devoted to the proof of Theorem 1.4. Finally, in the Appendix we recall some heat kernel estimates and fractional Gronwall's inequalities used in the paper.

2 The Gaussian noise W

Let $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ be the space of real-valued infinitely differentiable functions with compact support. Following [7] and [8], on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$, we consider a centered Gaussian family of random variables $\{W(\varphi), \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with covariance

$$\mathrm{E}\left[W(\varphi)W(\psi)\right] = \int_{\mathbb{R}_+\times\mathbb{R}^{2d}} \varphi(t,x)\varphi(t,y)f(x-y)dxdydt,$$

where f is as in (1.2). Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}_+\times\mathbb{R}^{2d}} \varphi(t,x)\varphi(t,y)f(x-y)dxdydt.$$

The mapping $\varphi \mapsto W(\varphi)$ defined in $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W. We will denote the isometry by

$$W(\phi) = \int_{\mathbb{R}_+\times\mathbb{R}^d} \phi(t,x)W(dt,dx), \qquad \phi \in \mathcal{H}.$$

Notice that if $\varphi, \psi \in \mathcal{H}$, then $\mathrm{E}\left[W(\varphi)W(\psi)\right] = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Moreover, $\mathcal{H}$ contains the space of measurable functions ϕ on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} |\phi(t,x)\phi(t,y)| f(x-y) dx dy dt < \infty.$$

When handling equation (1.3) with $\sigma(x) = x$, we will make use of its chaos expansion. For any integer $n \geq 1$, we denote by $\mathbf{H}_n$ the nth Wiener chaos of W. Recall that $\mathbf{H}_0$ is simply $\mathbb{R}$ and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables

$$\{H_n(W(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\},$$

where H_n is the nth Hermite polynomial. For any $n \geq 1$, we denote by $\mathcal{H}^{\otimes n}$ (resp. $\mathcal{H}^{n}$) the nth tensor product (resp. the nth symmetric tensor product) of $\mathcal{H}$. Then, the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ can be extended to a linear isometry bewteen $\mathcal{H}^{\otimes n}$ (equipped with the modified norm $\sqrt{n!}\|\cdot\|_{\mathcal{H}^{\otimes n}}$) and $\mathbf{H}_n$.

Let $\mathcal{F}^W$ the σ-field generated by W. Then, any $\mathcal{F}^W$-measurable random variable F in $L^2(\Omega)$ can be expressed as

$$F = \mathrm{E}(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \mathcal{H}^{\otimes n}$ are determined by F. This identity is called the Wiener-chaos expansion of F.

3 Heat kernel estimates

As a consequence of Theorems A.3 and A.4 in the Appendix, the following upper and lower bounds for the fractional heat kernel on D hold.

Proposition 3.1. *For any $\epsilon \in (0, \frac{1}{2})$, there exist positive constants $c_1(\epsilon)$, $c_2(\epsilon)$ and $c_3(\epsilon)$ such that for all $x \in D_\epsilon = \{y \in \mathbb{R}^d : |y| \leq 1 - \epsilon\}$ and $t > 0$,*

$$\int_{D_\epsilon} p_D(t,x,y) dy \geq c_1 e^{-\mu_1 t}, \tag{3.1}$$

for all $x \in D_\epsilon$ and $t > 0$,

$$\int_{D_\epsilon} p_D^2(t,x,y) dy \geq c_2 e^{-2\mu_1 t} t^{-d/\alpha}, \tag{3.2}$$

and if f satisfies Hypothesis (1.1)*, then for all $x, w \in D_\epsilon$ and $t > 0$ such that $|x - w| \leq t^\alpha$,*

$$\int_{D_\epsilon \times D_\epsilon} p_D(t,x,y) p_D(t,w,z) f(y-z) dy dz \geq c_3 e^{-2\mu_1 t} t^{-\beta/\alpha}. \tag{3.3}$$

Proof. We start assuming $\alpha = 2$. From Theorem A.3 and (1.6), for all $x \in D_\epsilon$ and $t > 0$,

$$\begin{aligned}\int_{D_\epsilon} p_D(t,x,y) dy &\geq c \int_{D_\epsilon} \min\left(1, \frac{\epsilon^2}{1 \wedge t}\right) e^{-\mu_1 t} \frac{e^{-c\frac{|x-y|^2}{t}}}{1 \wedge t^{d/2}} dy \\ &\geq c e^{-\mu_1 t} \left(\int_{D_\epsilon} \min\left(1, \frac{\epsilon^2}{t}\right) \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \mathbf{1}_{\{t<1\}} dy + \int_{D_\epsilon} e^{-c\frac{|x-y|^2}{t}} \mathbf{1}_{\{t \geq 1\}} dy \right).\end{aligned}$$

The second integral in the last display is lower bounded by $c(\epsilon)\mathbf{1}_{\{t\geq 1\}}$. The first one equals

$$\int_{D_\epsilon} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \mathbf{1}_{\{t<\epsilon^2\}} dy + \epsilon^2 \int_{D_\epsilon} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{1+\frac{d}{2}}} \mathbf{1}_{\{\epsilon^2 \leq t < 1\}} dy.$$

The second integral in the last display is lower bounded by $c(\epsilon)\mathbf{1}_{\{\epsilon^2\le t<1\}}$, while the first one is lower bounded by

$$\int_{D_\epsilon} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\mathbf{1}_{\{|x-y|^2\le t<\epsilon^2\}}dy.$$

We now observe that if $t<\epsilon^2$ and $\epsilon<\frac{1}{2}$, then $\min(\sqrt{t},1-\epsilon)=\sqrt{t}$. Thus, for all $x\in D_\epsilon$,

$$\mathrm{Vol}(y\in D_\epsilon : |x-y|^2\le t)=\mathrm{Vol}(B_x(\sqrt{t})\cap B_0(1-\epsilon))\ge \frac{1}{2}\mathrm{Vol}(B_0(\sqrt{t}))=c_d t^{d/2},$$

where $B_x(r)=\{y\in\mathbb{R}^d : |y-x|\le r\}$, for $x\in\mathbb{R}^d$ and $r>0$, and Vol denotes the d-dimensional volume. This shows that

$$\int_{D_\epsilon} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\mathbf{1}_{\{|x-y|^2\le t<\epsilon^2\}}dy \ge c_d\mathbf{1}_{\{t<\epsilon^2\}},$$

which proves (3.1) for $\alpha=2$.

Following along the same lines, we get that

$$\begin{aligned}\int_{D_\epsilon} p_D^2(t,x,y)dy &\ge ce^{-2\mu_1 t}\left\{\int_{D_\epsilon}\frac{e^{-c\frac{|x-y|^2}{t}}}{t^d}\mathbf{1}_{\{|x-y|^2\le t<\epsilon^2\}}dy+\mathbf{1}_{\{t\ge\epsilon^2\}}\right\}\\ &\ge ce^{-2\mu_1 t}\left\{t^{-d/2}\mathbf{1}_{\{t<\epsilon^2\}}+\mathbf{1}_{\{t\ge\epsilon^2\}}\right\}\\ &\ge ce^{-2\mu_1 t}t^{-d/2},\end{aligned}$$

which shows (3.2) when $\alpha=2$.

We next show (3.3) for $\alpha=2$. We assume that f satisfies Hypothesis 1.1. From Theorem A.3 and (1.6), for all $x,w\in D_\epsilon$ and $t>0$,

$$\begin{aligned}&\int_{D_\epsilon\times D_\epsilon} p_D(t,x,y)p_D(t,w,z)f(y-z)dydz\\ &\ge ce^{-2\mu_1 t}\left\{\int_{D_\epsilon\times D_\epsilon}\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\mathbf{1}_{\{|x-y|^2\le t<\epsilon^2\}}\frac{e^{-c\frac{|w-z|^2}{t}}}{t^{d/2}}\mathbf{1}_{\{|w-z|^2\le t\}}|y-z|^{-\beta}dydz+\mathbf{1}_{\{t\ge\epsilon^2\}}\right\}.\end{aligned}$$

Next observe that since $|x-w|<\sqrt{t}$, $|x-y|^2\le t$, and $|w-z|^2\le t$, we get that $|y-z|^{-\beta}\ge t^{-\beta/2}$. Therefore, proceeding as above, we conclude that if $x,w\in D_\epsilon$ and $t>0$ are such that $|x-w|<\sqrt{t}$, then

$$\begin{aligned}\int_{D_\epsilon\times D_\epsilon} p_D(t,x,y)p_D(t,w,z)f(y-z)dydz &\ge ce^{-2\mu_1 t}\left\{t^{-\beta/2}\mathbf{1}_{\{t<\epsilon^2\}}+\mathbf{1}_{\{t\ge\epsilon^2\}}\right\}\\ &\ge ce^{-2\mu_1 t}t^{-\beta/2},\end{aligned}$$

which concludes the proof of (3.2) when $\alpha=2$.

We now assume $\alpha\in(1,2)$. Similarly as above, appealing to Theorem A.4 and (1.6), for all $x\in D_\epsilon$ and $t>0$,

$$\begin{aligned}&\int_{D_\epsilon} p_D(t,x,y)dy\\ &\ge ce^{-\mu_1 t}\left\{\int_{D_\epsilon}\min\left(1,\frac{\epsilon^\alpha}{t}\right)\min\left(t^{-1/\alpha},\frac{t}{|x-y|^{\alpha+d}}\right)\mathbf{1}_{\{t<1\}}dy+\mathbf{1}_{\{t\ge 1\}}\right\}.\end{aligned}$$

The integral in the last display equals

$$\int_{D_\epsilon}\min\left(t^{-1/\alpha},\frac{t}{|x-y|^{\alpha+d}}\right)\mathbf{1}_{\{t<\epsilon^\alpha\}}dy+\epsilon^\alpha\int_{D_\epsilon}\min\left(t^{-1/\alpha-1},\frac{1}{|x-y|^{\alpha+d}}\right)\mathbf{1}_{\{\epsilon^\alpha\le t<1\}}dy.$$

The second integral in the last display is lower bounded by $c(\epsilon)\mathbf{1}_{\{\epsilon^\alpha \le t<1\}}$, while the first one is lower bounded by

$$\int_{D_\epsilon} t^{-d/\alpha} \min\left(t^{(d-1)/\alpha}, \left(\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}\right)\mathbf{1}_{\{|x-y|^\alpha<t<\epsilon^\alpha\}}dy.$$

Using the same argument as before, if $t<\epsilon^\alpha$ and $\epsilon<\frac{1}{2}$, then for all $x\in D_\epsilon$

$$\mathrm{Vol}(y\in D_\epsilon : |x-y|^\alpha \le t) = \mathrm{Vol}(B_x(t^{1/\alpha})\cap B_0(1-\epsilon)) \ge \frac{1}{2}\mathrm{Vol}(B_0(t^{1/\alpha})) = c_d t^{d/\alpha},$$

which concludes the proof of (3.1) for $\alpha\in(1,2)$. Similarly,

$$\begin{aligned}
&\int_{D_\epsilon} p_D^2(t,x,y)dy \\
&\ge ce^{-2\mu_1 t}\left\{\int_{D_\epsilon} \min\left(1,\frac{\epsilon^{2\alpha}}{t^2}\right)\min\left(t^{-2/\alpha}, \frac{t^2}{|x-y|^{2(\alpha+d)}}\right)\mathbf{1}_{\{t<1\}}dy + \mathbf{1}_{\{t\ge 1\}}\right\} \\
&\ge ce^{-2\mu_1 t}\left\{t^{-d/\alpha}\mathbf{1}_{\{t<\epsilon^\alpha\}} + \mathbf{1}_{\{t\ge\epsilon^\alpha\}}\right\} \\
&\ge ce^{-2\mu_1 t}t^{-d/\alpha},
\end{aligned}$$

which shows (3.1) for $\alpha\in(1,2)$.

Finally, for all $x,w\in D_\epsilon$ and $t>0$ such that $|x-w|<t^{1/\alpha}$,

$$\begin{aligned}
\int_{D_\epsilon\times D_\epsilon} p_D(t,x,y)p_D(t,w,z)f(y-z)dydz &\ge ce^{-2\mu_1 t}\left\{t^{-\beta/\alpha}\mathbf{1}_{\{t<\epsilon^\alpha\}} + \mathbf{1}_{\{t\ge\epsilon^\alpha\}}\right\} \\
&\ge ce^{-2\mu_1 t}t^{-\beta/\alpha},
\end{aligned}$$

which proves (3.3) when $\alpha\in(1,2)$. □

Proposition 3.2. *For all $\delta>0$, there exist $c_1, c_2(\delta)>0$ such that for all $x,w\in D$ and $t>0$,*

$$\int_D p_D(t,x,y)dy \le c_1 e^{-\mu_1 t}, \tag{3.4}$$

and

$$\int_{D\times D} p_D(t,x,y)p_D(t,w,z)f(y-z)dydz \le c_2 e^{-(2-\delta)\mu_1 t}t^{-a/\alpha}, \tag{3.5}$$

where

$$a = \begin{cases} d, & \textit{if} \quad f=\delta_0, \\ \beta, & \textit{if} \quad f \textit{ satisfies Hypothesis } 1.1. \end{cases}$$

Proof. We first assume $\alpha=2$. By Theorem A.3, for all $x\in D$ and $t>0$,

$$\int_D p_D(t,x,y)dy \le ce^{-\mu_1 t}\int_D \frac{e^{-c\frac{|x-y|^2}{t}}}{1\wedge t^{d/2}}dy \le ce^{-\mu_1 t},$$

which shows (3.4). Let $f=\delta_0$. By the semigroup property and Theorem A.3, for all $\delta>0$,

$$\int_D p_D^2(t,x,y)dy = p_D(2t,x,x) \le ce^{-2\mu_1 t}\frac{1}{1\wedge t^{d/2}} \le c(\delta)e^{-(2-\delta)\mu_1 t}t^{-d/2}.$$

Finally, by Theorem A.3, when f satisfies Hypothesis 1.1, we get that

$$\begin{aligned}
\int_{D\times D} p_D(t,x,y)p_D(t,w,z)f(y-z)dydz &\le ce^{-2\mu_1 t}\int_{D\times D}\frac{e^{-c\frac{|x-y|^2}{t}}}{1\wedge t^{d/2}}\frac{e^{-c\frac{|w-z|^2}{t}}}{1\wedge t^{d/2}}|y-z|^{-\beta}dydz \\
&\le c(\delta)e^{-(2-\delta)\mu_1 t}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\frac{e^{-c\frac{|w-z|^2}{t}}}{t^{d/2}}|y-z|^{-\beta}dydz \\
&\le c(\delta)e^{-(2-\delta)\mu_1 t}t^{-a/\alpha},
\end{aligned}$$

where we have used [19, Lemma 4.1] in the last inequality. □

4 Proof of Theorem 1.4

4.1 Proof of the lower bound of Theorem 1.4b)

By Jensen's inequality, for any $p \geq 2$,

$$\mathrm{E}|u_t(x)|^p \geq \left(\mathrm{E}|u_t(x)|^2\right)^{p/2}. \tag{4.1}$$

Therefore, it suffices to prove the lower bound for $p = 2$. Taking the second moment to the mild formulation (1.3) we obtain that, for all $x \in D$ and $t > 0$,

$$\mathrm{E}|u_t(x)|^2 = \left(\int_D u_0(y) p_D(t,x,y) dy\right)^2 + \lambda^2 \int_0^t \int_D p_D^2(t-s,x,y) \mathrm{E}|\sigma(u_s(y))|^2 dy ds.$$

Hypothesis 1.3 and the heat kernel estimate (3.1) yield to

$$\int_D u_0(y) p_D(t,x,y) dy \geq \inf_{y \in D_\epsilon} u_0(y) \int_{D_\epsilon} p_D(t,x,y) dy \geq c e^{-\mu_1 t}.$$

On the other hand, from Hypothesis 1.2 and the heat kernel estimate (3.2), we get that

$$\begin{aligned}\int_0^t \int_D p_D^2(t-s,x,y) \mathrm{E}|\sigma(u_s(y))|^2 dy ds &\geq \ell_\sigma^2 \int_0^t \int_{D_\epsilon} p_D^2(t-s,x,y) \mathrm{E}|u_s(y)|^2 dy ds \\ &\geq \ell_\sigma^2 \int_0^t h_\epsilon(s) \int_{D_\epsilon} p_D^2(t-s,x,y) dy ds \\ &\geq c \int_0^t h_\epsilon(s) e^{-2\mu_1(t-s)} (t-s)^{-1/\alpha} ds,\end{aligned}$$

where $h_\epsilon(s) := \inf_{y \in D_\epsilon} \mathrm{E}|u_s(y)|^2$. Now, set $g_\epsilon(t) = e^{2\mu_1 t} h_\epsilon(t)$. The estimates above show that for all $t > 0$,

$$g_\epsilon(t) \geq c \left(1 + \lambda^2 \int_0^t (t-s)^{-1/\alpha} g_\epsilon(s) ds\right).$$

Finally, Proposition A.1 with $\rho = 1 - \frac{1}{\alpha}$ and Proposition A.2 conclude the desired lower bound.

4.2 Proof of the upper bound of Theorem 1.4b)

Taking the pth moment to the mild formulation (1.3) and appealing to Burkholder-Davis-Gundy's and Minkowski's inequalities, it holds that for all $x \in D$ and $t > 0$,

$$\begin{aligned}\mathrm{E}|u_t(x)|^p \leq 2^{p-1} \Bigg\{ &\left(\int_D u_0(y) p_D(t,x,y) dy\right)^p \\ &+ \lambda^p z_p^p \left(\int_0^t \int_D p_D^2(t-s,x,y) (\mathrm{E}|\sigma(u_s(y))|^p)^{2/p} dy ds\right)^{p/2} \Bigg\},\end{aligned}$$

where z_p is as in the statement of Theorem 1.4b). Since u_0 is bounded, and using the heat kernel estimate (3.4), we get that

$$\int_D u_0(y) p_D(t,x,y) dy \leq c_1 e^{-\mu_1 t}.$$

Using Hypothesis 1.2 and the heat kernel estimate (3.5), we obtain

$$\begin{aligned}\int_0^t \int_D p_D^2(t-s,x,y)(\mathrm{E}|\sigma(u_s(y))|^p)^{2/p}dyds &\le L_\sigma^2 \int_0^t \int_D p_D^2(t-s,x,y)(\mathrm{E}|u_s(y)|^p)^{2/p}dyds \\ &\le L_\sigma^2 \int_0^t h(s)\left(\int_D p_D^2(t-s,x,y)dy\right)ds \\ &\le c\int_0^t h(s)e^{-(2-\delta)\mu_1(t-s)}(t-s)^{-1/\alpha}ds,\end{aligned}$$

where $h(s)=\sup_{y\in D}(\mathrm{E}|u_s(y)|^p)^{2/p}$. The estimates above show that, for all $t>0$,

$$g(t)\le c\left(1+\lambda^2 z_p^2\int_0^t \frac{g(s)}{(t-s)^{1/\alpha}}ds\right),$$

where $g(t)=e^{(2-\delta)\mu_1 t}h(t)$. Finally, Proposition A.1 with $\rho=1-\frac{1}{\alpha}$ concludes.

4.3 Proof of Theorem 1.4a)

In this case, following [1], the solution to (1.1) has the following Wiener-chaos expansion in $L^2(\Omega)$

$$u_t(x)=h_0(t,x)+\sum_{n\ge1}\lambda^n I_n(h_n(\cdot,t,x)), \tag{4.2}$$

where $h_0(t,x)=\int_D u_0(y)p_D(t,x,y)dy$, and for $n\ge1$, I_n denotes the multiple Wiener integral with respect to W in $\mathbb{R}_+^n\times D^n$, and for any $(t_1,...,t_n)\in\mathbb{R}_+^n$ and $x_1,...,x_n\in D$,

$$\begin{aligned}h_n(t_1,x_1,...,t_n,x_n,t,x)=p_D(t-t_n,x,x_n)p_D(t_n-t_{n-1},x_n,x_{n-1})\\ \cdots p_D(t_2-t_1,x_2,x_1)h_0(t_1,x_1)\mathbf{1}_{\{0<t_1<\cdots<t_n<t\}}.\end{aligned}$$

Therefore,

$$\mathrm{E}|u_t(x)|^2=|h_0(t,x)|^2+\sum_{n\ge1}\lambda^{2n}n!\|\tilde h_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes2}}^2,$$

where $\tilde h_n$ denotes the symmetrization of h_n. That is,

$$\begin{aligned}n!\|\tilde h_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes2}}^2&=\int_{0<t_1<\cdots<t_n<t}\int_{D^{2n}}p_D(t-t_n,x,x_n)p_D(t-t_n,x,y_n)f(x_n-y_n)\\ &\times p_D(t_n-t_{n-1},x_n,x_{n-1})p_D(t_n-t_{n-1},x_n,y_{n-1})f(x_{n-1}-y_{n-1})\cdots p_D(t_2-t_1,x_2,x_1)\\ &\times p_D(t_2-t_1,x_2,y_1)f(x_1-y_1)|h_0(t_1,x_1)|^2dx_1\cdots dx_ndy_1\cdots dy_ndt_1\cdots dt_n.\end{aligned}$$

Now, appealing to Propositions 3.1 and 3.2, we obtain

$$\begin{aligned}&c_2e^{-2\mu_1 t}\int_{0<t_1<\cdots<t_n<t}(t-t_n)^{-\beta/\alpha}\prod_{i=2}^n(t_i-t_{i-1})^{-\beta/\alpha}dt_1\cdots dt_n\\ &\le n!\|\tilde h_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes2}}^2\le c_1e^{-(2-\delta)\mu_1 t}\int_{0<t_1<\cdots<t_n<t}(t-t_n)^{-\beta/\alpha}\prod_{i=2}^n(t_i-t_{i-1})^{-\beta/\alpha}dt_1\cdots dt_n.\end{aligned}$$

Following similar computations as in [1, 2] it is easy to see that the last display implies that

$$c_2e^{-2\mu_1 t}\sum_{n\ge0}\lambda^{2n}C_2^n(n!)^{\frac{\beta}{\alpha}-1}t^{(1-\frac{\beta}{\alpha})n}\le\mathrm{E}|u_t(x)|^2\le c_1e^{-(2-\delta)\mu_1 t}\sum_{n\ge0}\lambda^{2n}C_1^n(n!)^{\frac{\beta}{\alpha}-1}t^{(1-\frac{\beta}{\alpha})n}.$$

Finally, [1, Lemma A.1] and [2, Lemma 5.2] conclude the proof of Theorem 1.4a) for $p=2$. The lower bound for $p\ge2$ follows using Jensen's inequality as in (4.1). For the upper bound, as in [1, 2], we have that by Minkowski's inequality and the equivalence of norms in a fixed Wiener chaos, for all $p \geq 2$,

$$(\mathrm{E}|u_t(x)|^p)^{1/p} \leq |h_0(t,x)| + \sum_{n\geq 1}(p-1)^{n/2}\lambda^n \left(n!\|\tilde{h}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes 2}}\right)^{1/2},$$

which implies the desired upper bound.

A Appendix

We recall the following fractional Gronwall's inequalities.

Proposition A.1. [15, Lemma 7.1.1], [14] *Let $\rho > 0$ and suppose that $g(t)$ is a locally integrable function satisfying*

$$g(t) \leq c_1 + k\int_0^t (t-s)^{\rho-1} g(s)ds \quad \text{for all} \quad t>0, \tag{A.1}$$

for some positive constants c_1, k. Then there exist positive constants c_2, c_3 such that

$$g(t) \leq c_2 e^{c_3 k^{1/\rho} t} \quad \text{for all} \quad t > 0.$$

If instead of (A.1) *the function is non-negative and satisfies*

$$g(t) \geq c_1 + k\int_0^t (t-s)^{\rho-1} g(s)ds \quad \text{for all} \quad t>0,$$

then

$$g(t) \geq c_2 e^{c_3 k^{1/\rho} t} \quad \text{for all} \quad t > \frac{e}{\rho}(\Gamma(\rho)k)^{-1/\rho}.$$

The next result shows that when $\rho = \frac{1}{2}$, the lower bound can be obtained for all $t > 0$.

Proposition A.2. [9] *Let $g(t)$ be a non-negative locally integrable function satisfying*

$$g(t) \geq c_1 + k\int_0^t \frac{g(s)}{\sqrt{t-s}}ds \quad \text{for all} \quad t>0,$$

for some positive constants c_1, k. Then there exist positive constants c_2, c_3 such that

$$g(t) \geq c_2 e^{c_3 k^2 t} \quad \text{for all} \quad t > 0.$$

We also recall the following estimates of the Dirichlet fractional heat kernel.

Theorem A.3. [20, Theorem 2.2] *Assume $\alpha = 2$. There exist positive constants C, c_1 and c_2 such that, for all $x, y \in D$ and $t > 0$,*

$$\begin{aligned} C^{-1}\min\left(1, \frac{\Phi_1(x)\Phi_1(y)}{1\wedge t}\right) e^{-\mu_1 t}\frac{e^{-c_2\frac{|x-y|^2}{t}}}{1\wedge t^{d/2}} &\leq p_D(t,x,y) \\ &\leq C\min\left(1, \frac{\Phi_1(x)\Phi_1(y)}{1\wedge t}\right) e^{-\mu_1 t}\frac{e^{-c_1\frac{|x-y|^2}{t}}}{1\wedge t^{d/2}}. \end{aligned}$$

Theorem A.4. [6, Theorem 1.1] *Assume $\alpha \in (1,2)$. There exist a positive constant C such that, for all $x, y \in D$ and $t > 0$,*

$$\begin{aligned} &C^{-1}e^{-\mu_1 t}\Big\{\min\left(1, \frac{\Phi_1(x)}{\sqrt{t}}\right)\min\left(1, \frac{\Phi_1(y)}{\sqrt{t}}\right)\min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right)\mathbf{1}_{\{t<1\}} \\ &\qquad + \Phi_1(x)\Phi_1(y)\mathbf{1}_{\{t\geq 1\}}\Big\} \\ &\leq p_D(t,x,y) \\ &\leq Ce^{-\mu_1 t}\Big\{\min\left(1, \frac{\Phi_1(x)}{\sqrt{t}}\right)\min\left(1, \frac{\Phi_1(y)}{\sqrt{t}}\right)\min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right)\mathbf{1}_{\{t<1\}} \\ &\qquad + \Phi_1(x)\Phi_1(y)\mathbf{1}_{\{t\geq 1\}}\Big\}. \end{aligned}$$

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