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Martingale approximations for random fields

Peligrad Magda* Na Zhang†

Abstract

In this paper we provide necessary and sufficient conditions for the mean square approximation of a random field by an ortho-martingale. The conditions are formulated in terms of projective criteria. Applications are given to linear and nonlinear random fields with independent innovations.

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1 Introduction

A random field consists of multi-indexed random variables $(X_u)_{u\in Z^d}$. An important class of random fields are ortho-martingales which were introduced by Cairoli (1969) and have resurfaced in many recent works. The central limit theorem for stationary ortho-martingales was recently investigated by Volný (2015). It is remarkable that Volný (2015) imposed the ergodicity condition to only one direction of the stationary random field. In order to exploit the richness of the martingale techniques, in this paper we obtain necessary and sufficient conditions for an ortho-martingale approximation in mean square. These approximations extend to random fields the corresponding results obtained for sequences of random variables by Dedecker et al. (2007), Zhao and Woodroofe (2008) and Peligrad (2010). The tools for proving these results consist of projection decomposition. We present applications of our results to linear and nonlinear random fields.

We would like to mention several remarkable recent contributions, which provide interesting sufficient conditions for ortho-martingale approximations, by Gordin (2009), El Machkouri et al. (2013), Volný and Wang (2014), Cuny et al. (2015), Peligrad and Zhang (2017), and Giraudo (2017). A special type of ortho-martingale approximation, so called co-boundary decomposition, was studied by El Machkouri and Giraudo (2017) and Volný (2017). Other recent results involve interesting mixingale-type conditions in Wang and Woodroofe (2013), and mixing conditions in Bradley and Tone (2017).

Our results could also be formulated in the language of dynamical systems, leading to new results in this field.

*Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, OH 45221-0025, USA.
E-mail: peligrm@ucmail.uc.edu
E-mail: zhangn4@mail.uc.edu

2 Results

For the sake of clarity, especially due to the complicated notation, we shall explain first the results for double indexed random fields and, at the end, we shall formulate the results for general random fields. No technical difficulties arise when the double indexed random field is replaced by a multiple indexed one.

We shall introduce a stationary random field adapted to a stationary filtration. In order to construct a flexible filtration it is customary to start with a stationary real valued random field $(\xi_{n,m})_{n,m\in Z}$ defined on a probability space $(\Omega,\mathcal{K},P)$ and to introduce another stationary random field $(X_{n,m})_{n,m\in Z}$ defined by

$$X_{n,m}=f(\xi_{i,j},i\leq n,j\leq m), \tag{2.1}$$

where f is a measurable function defined on R^{Z^2}. Note that $X_{n,m}$ is adapted to the filtration $\mathcal{F}_{n,m}=\sigma(\xi_{i,j},i\leq n,j\leq m)$. Without restricting the generality we shall define $(\xi_{\mathbf{u}})_{\mathbf{u}\in Z^2}$ in a canonical way on the probability space $\Omega=R^{Z^2}$, endowed with the $\sigma-$field, $\mathcal{B}$, generated by cylinders. Then, if $\omega=(x_{\mathbf{v}})_{\mathbf{v}\in Z^2}$, we define $\xi'_{\mathbf{u}}(\omega)=x_{\mathbf{u}}$. We construct a probability measure P' on $\mathcal{B}$ such that for all $B\in\mathcal{B}$ and any m and $\mathbf{u}_1,...,\mathbf{u}_m$ we have

$$P'((x_{\mathbf{u}_1},...,x_{\mathbf{u}_m})\in B)=P((\xi_{\mathbf{u}_1},...,\xi_{\mathbf{u}_m})\in B).$$

The new sequence $(\xi'_{\mathbf{u}})_{\mathbf{u}\in Z^2}$ is distributed as $(\xi_{\mathbf{u}})_{\mathbf{u}\in Z^2}$ and re-denoted by $(\xi_{\mathbf{u}})_{\mathbf{u}\in Z^2}$. We shall also re-denote P' as P. Now on R^{Z^2} we introduce the operators

$$T^{\mathbf{u}}((x_{\mathbf{v}})_{\mathbf{v}\in Z^2})=(x_{\mathbf{v}+\mathbf{u}})_{\mathbf{v}\in Z^2}.$$

Two of them will play an important role in our paper, namely when $\mathbf{u}=(1,0)$ and when $\mathbf{u}=(0,1)$. By interpreting the indexes as notations for the lines and columns of a matrix, we shall call

$$T((x_{u,v})_{(u,v)\in Z^2})=(x_{u+1,v})_{(u,v)\in Z^2}$$

the vertical shift and

$$S((x_{u,v})_{(u,v)\in Z^2})=(x_{u,v+1})_{(u,v)\in Z^2}$$

the horizontal shift. Then define

$$X_{j,k}=f(T^jS^k(\xi_{a,b})_{a\leq 0,b\leq 0}). \tag{2.2}$$

We assume that $X_{0,0}$ is centered and square integrable. We notice that the variables are adapted to the filtration $(\mathcal{F}_{n,m})_{n,m\in Z}$. To compensate for the fact that, in the context of random fields, the future and the past do not have a unique interpretation, we shall consider commuting filtrations, i.e.

$$E(E(X|\mathcal{F}_{a,b})|\mathcal{F}_{u,v})=E(X|\mathcal{F}_{a\wedge u,b\wedge v}).$$

This type of filtration is induced, for instance, by an initial random field $(\xi_{n,m})_{n,m\in Z}$ of independent random variables, or, more generally can be induced by stationary random fields $(\xi_{n,m})_{n,m\in Z}$ where only the columns are independent, i.e. $\bar{\eta}_m=(\xi_{n,m})_{n\in Z}$ are independent. This model often appears in statistical applications when one deals with repeated realizations of a stationary sequence.

It is interesting to point out that commuting filtrations can be described by the equivalent formulation: for $a\geq u$ we have

$$E(E(X|\mathcal{F}_{a,b})|\mathcal{F}_{u,v})=E(X|\mathcal{F}_{u,b\wedge v}), \tag{2.3}$$

where, as usual, $a \wedge b$ stands for the minimum of a and b. This follows from this Markovian-type property (see for instance Problem 34.11 in Billingsley, 1995).

Below we use the notations

$$S_{k,j} = \sum_{u,v=1}^{k,j} X_{u,v},\ E(X|\mathcal{F}_{a,b}) = E_{a,b}(X).$$

For an integrable random variable X and $(u,v) \in Z^2$, we introduce the projection operators defined by

$$P_{\tilde{u},v}(X) = (E_{u,v} - E_{u-1,v})(X),$$

$$P_{u,\tilde{v}}(X) = (E_{u,v} - E_{u,v-1})(X).$$

Note that, by (2.3), we have

$$\mathcal{P}_{u,v}(X) := P_{\tilde{u},v} \circ P_{u,\tilde{v}}(X) = P_{u,\tilde{v}} \circ P_{\tilde{u},v}(X)$$

and by an easy computation we have that

$$\mathcal{P}_{u,v}(X) = E_{u,v}(X) - E_{u,v-1}(X) - E_{u-1,v}(X) + E_{u-1,v-1}(X). \tag{2.4}$$

We shall introduce the definition of an ortho-martingale, which will be referred to as a martingale with multiple indexes or simply martingale.

Definition 2.1. *Let d be a function and define*

$$D_{n,m} = d(\xi_{i,j}, i \le n, j \le m). \tag{2.5}$$

Assume integrability. We say that $(D_{n,m})_{n,m\in Z}$ is a martingale differences field if $E_{a,b}(D_{n,m}) = 0$ for either $a < n$ or $b < m$.

Set

$$M_{k,j} = \sum_{u,v=1}^{k,j} D_{u,v}.$$

In the sequel we shall denote by $||\cdot||$ the norm in L^2. By $\Rightarrow$ we denote the convergence in distribution.

Definition 2.2. *We say that a random field $(X_{n,m})_{n,m\in Z}$ defined by (2.1) admits a martingale approximation if there is a sequence of martingale differences $(D_{n,m})_{n,m\in Z}$ defined by (2.5) such that*

$$\lim_{n\wedge m\to\infty} \frac{1}{nm}||S_{n,m} - M_{n,m}||^2 = 0. \tag{2.6}$$

Theorem 2.3. *Assume that (2.3) holds. The random field $(X_{n,m})_{n,m\in Z}$ defined by (2.1) admits a martingale approximation if and only if*

$$\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} ||\mathcal{P}_{1,1}(S_{j,k}) - D_{1,1}||^2 \to 0 \text{ when } n\wedge m \to \infty, \tag{2.7}$$

and both

$$\frac{1}{nm}||E_{0,m}(S_{n,m})||^2 \to 0 \text{ and } \frac{1}{nm}||E_{n,0}(S_{n,m})||^2 \to 0 \text{ when } n\wedge m\to\infty. \tag{2.8}$$

Remark 2.4. Condition (2.8) in Theorem 2.3 can be replaced by

$$\frac{1}{nm}||S_{n,m}||^2 \to ||D_{1,1}||^2. \tag{2.9}$$

Theorem 2.5. *Assume that (2.3) holds. The random field* $(X_{n,m})_{n,m\in Z}$ *defined by (2.1) admits a martingale approximation if and only if*

$$\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m}\mathcal{P}_{1,1}(S_{j,k}) \text{ converges in } L^2 \text{ to } D_{1,1} \text{ when } n\wedge m\to\infty \tag{2.10}$$

and the condition (2.9) holds.

Corollary 2.6. *Assume that the vertical shift* T *(or horizontal shift* S*) is ergodic and either the conditions of Theorem 2.3 or Theorem 2.5 hold. Then*

$$\frac{1}{\sqrt{n_1 n_2}}S_{n_1,n_2}\Rightarrow N(0,c^2) \text{ when } n_1\wedge n_2\to\infty, \tag{2.11}$$

where $c^2=||D_{0,0}||^2$.

3 Proofs

Proof of Theorem 2.3. We start from the following orthogonal representation

$$S_{n,m}=\sum_{i=1}^{n}\sum_{j=1}^{m}\mathcal{P}_{i,j}(S_{n,m})+R_{n,m}, \tag{3.1}$$

with

$$R_{n,m}=E_{n,0}(S_{n,m})+E_{0,m}(S_{n,m})-E_{0,0}(S_{n,m}).$$

Note that for all $1\le a\le i-1$, $1\le b\le j-1$ we have $\mathcal{P}_{i,j}(X_{a,b})=0$; for all $a\ge i$, $1\le b\le j-1$ we have $\mathcal{P}_{i,j}(X_{a,b})=0$ and for all $1\le a\le i-1$, $b\ge j$, $\mathcal{P}_{i,j}(X_{a,b})=0$. Whence,

$$\mathcal{P}_{i,j}(S_{n,m})=\mathcal{P}_{i,j}(\sum_{u=i}^{n}\sum_{v=j}^{m}X_{u,v}).$$

This shows that for any martingale differences sequence defined by (2.5), by orthogonality, we obtain

$$\begin{aligned}||S_{n,m}-M_{n,m}||^2&=\sum_{i=1}^{n}\sum_{j=1}^{m}||\mathcal{P}_{i,j}(\sum_{a=i}^{n}\sum_{b=j}^{m}X_{a,b})-D_{i,j}||^2+||R_{n,m}||^2 \\ &=\sum_{i=1}^{n}\sum_{j=1}^{m}||\mathcal{P}_{1,1}(\sum_{a=1}^{n-i+1}\sum_{b=1}^{m-j+1}X_{a,b})-D_{1,1}||^2+||R_{n,m}||^2 \\ &=\sum_{i=1}^{n}\sum_{j=1}^{m}||\mathcal{P}_{1,1}(S_{i,j})-D_{1,1}||^2+||R_{n,m}||^2.\end{aligned} \tag{3.2}$$

A first observation is that we have a martingale approximation if and only if both (2.7) is satisfied and $||R_{n,m}||^2/nm\to 0$ as $n\wedge m\to\infty$.

Computation, involving the fact that the filtration is commuting, shows that

$$||R_{n,m}||^2=||E_{n,0}(S_{n,m})||^2+||E_{0,m}(S_{n,m})||^2-||E_{0,0}(S_{n,m})||^2, \tag{3.3}$$

and since $||E_{0,0}(S_{n,m})||\le||E_{0,m}(S_{n,m})||$ we have that $||R_{n,m}||^2/nm\to 0$ as $n\wedge m\to\infty$ if and only if (2.8) holds. □

Proof of Theorem 2.5. Let us first note that $D_{1,1}$ defined by (2.10) is a martingale difference. By using the translation operators we then define the sequence of martingale differences $(D_{u,v})_{u,v\in Z}$ and the sum of martingale differences $(M_{u,v})_{u,v\in Z}$. This time we evaluate

$$||S_{n,m}-M_{n,m}||^2=E(S_{n,m}^2)+E(M_{n,m}^2)-2E(S_{n,m}M_{n,m}).$$

By using the martingale property, stationarity and simple algebra we obtain

$$E(S_{n,m}M_{n,m})=\sum_{u=1}^{n}\sum_{v=1}^{m}\sum_{i\geq u}^{n}\sum_{j\geq v}^{m}E(D_{u,v}X_{i,j})=\sum_{u=1}^{n}\sum_{v=1}^{m}E(D_{1,1}S_{u,v}).$$

A simple computation involving the properties of conditional expectation and the martingale property shows that

$$E(D_{1,1}S_{u,v})=E(D_{1,1}\mathcal{P}_{1,1}(S_{u,v})).$$

By (2.10) this identity gives that

$$\lim_{n\wedge m\to\infty}\frac{1}{nm}E(S_{n,m}M_{n,m})=E(D_{1,1}^2).$$

From the above considerations

$$\lim_{n\wedge m\to\infty}\frac{1}{nm}||S_{n,m}-M_{n,m}||^2=\lim_{n\wedge m\to\infty}\frac{1}{nm}E(S_{n,m}^2)-E(D_{1,1}^2),$$

whence the martingale approximation holds by (2.9).

Let us assume now that we have a martingale approximation. According to Theorem 2.3 condition (2.7) is satisfied. In order to show that (2.7) implies (2.10) we apply the Cauchy-Schwarz inequality twice:

$$\begin{aligned}||\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}(\mathcal{P}_{1,1}(S_{i,j})-D_{1,1})||^2&\leq\frac{1}{nm^2}\sum_{i=1}^{n}||\sum_{j=1}^{m}(\mathcal{P}_{1,1}(S_{i,j})-D_{1,1})||^2\\&\leq\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}||\mathcal{P}_{1,1}(S_{i,j})-D_{1,1})||^2.\end{aligned}$$

Also, by the triangular inequality

$$|\frac{1}{\sqrt{nm}}||S_{n,m}||-||D_{1,1}||\,|\leq\frac{1}{\sqrt{nm}}||S_{n,m}-M_{n,m}||\to 0 \text{ as } n\wedge m\to\infty,$$

and (2.9) follows. □

Proof of Remark 2.4. If we have a martingale decomposition, then by Theorem 2.3 we have (2.7) and by Theorem 2.5 we have (2.9). Now, in the opposite direction, just note that (2.7) implies (2.10) and then apply Theorem 2.5. □

Proof of Corollary 2.6. This Corollary follows as a combination of Theorem 2.3 (or Theorem 2.5) with the main result in Volný (2015) via Theorem 25.4 in Billingsley (1995). □

4 Multidimensional index sets

The extensions to random fields indexed by Z^d, for $d>2$, are straightforward following the same lines of proofs as for a double indexed random field. By $\mathbf{u}\leq\mathbf{n}$ we understand $\mathbf{u}=(u_1,...,u_d)$, $\mathbf{n}=(n_1,...,n_d)$ and $1\leq u_1\leq n_1$,..., $1\leq u_d\leq n_d$. We shall start with a strictly stationary real valued random field $\xi=(\xi_{\mathbf{u}})_{\mathbf{u}\in Z^d}$, defined on the canonical probability space R^{Z^d} and define the filtrations $\mathcal{F}_{\mathbf{u}}=\sigma(\xi_{\mathbf{j}}:\mathbf{j}\leq\mathbf{u})$. We shall assume that the filtration is commuting if $E_{\mathbf{u}}E_{\mathbf{a}}(X) = E_{\mathbf{u}\wedge\mathbf{a}}(X)$, where the minimum is taken coordinate-wise. We define

$$X_{\mathbf{m}} = f((\xi_{\mathbf{j}})_{\mathbf{j}\le\mathbf{m}}) \text{ and set } S_{\mathbf{k}} = \sum\nolimits_{\mathbf{u}=\mathbf{1}}^{\mathbf{k}} X_{\mathbf{u}}.$$

We also define T_i the coordinate-wise translations and then

$$X_{\mathbf{k}} = f(T_1^{k_1} \circ ... \circ T_d^{k_d}(\xi_{\mathbf{u}})_{\mathbf{u}\le\mathbf{0}}).$$

Let d be a function and define

$$D_{\mathbf{m}} = d((\xi_{\mathbf{j}})_{\mathbf{j}\le\mathbf{m}}) \text{ and set } M_{\mathbf{k}} = \sum\nolimits_{\mathbf{u}=\mathbf{1}}^{\mathbf{k}} D_{\mathbf{u}}. \tag{4.1}$$

Assume integrability. We say that $(D_{\mathbf{m}})_{\mathbf{m}\in Z^d}$ is a martingale differences field if $E_{\mathbf{a}}(D_{\mathbf{m}}) = 0$ is at least one coordinate of $\mathbf{a}$ is strictly smaller than the corresponding coordinate of $\mathbf{m}$. We have to introduce the d-dimensional projection operators. By using the fact that the filtration is commuting, it is convenient to define

$$\mathcal{P}_{\mathbf{u}}(X) = P_{\mathbf{u},1} \circ P_{\mathbf{u},2} \circ ... \circ P_{\mathbf{u},d}(X),$$

where

$$P_{\mathbf{u},j}(Y) = E(Y|\mathcal{F}_{\mathbf{u}}) - E(Y|\mathcal{F}_{\mathbf{u}}^{(j)}).$$

Above, we used the notation: $\mathcal{F}_{\mathbf{u}}^{(j)} = \mathcal{F}_{\mathbf{u}'}$ where $\mathbf{u}'$ has all the coordinates of $\mathbf{u}$ with the exception of the j-th coordinate, which is $u_j - 1$. For instance when $d = 3$, $P_{\mathbf{u},2}(Y) = E(Y|\mathcal{F}_{u_1,u_2,u_2}) - E(Y|\mathcal{F}_{u_1,u_2-1,u_3})$.

We say that a random field $(X_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ admits a martingale approximation if there is a sequence of martingale differences $(D_{\mathbf{m}})_{\mathbf{m}\in Z^d}$ such that

$$\frac{1}{|\mathbf{n}|}||S_{\mathbf{n}} - M_{\mathbf{n}}||^2 \to 0 \text{ when } \min_{1\le i\le d} n_i \to \infty, \tag{4.2}$$

where $|\mathbf{n}| = n_1...n_d$.

Let us introduce the following regularity condition

$$\frac{1}{|\mathbf{n}|}||S_{\mathbf{n}}||^2 \to E(D_{\mathbf{1}}^2) \text{ when } \min_{1\le i\le d} n_i \to \infty. \tag{4.3}$$

Theorem 4.1. *Assume that the filtration is commuting. The following statements are equivalent:*
(a) The random field $(X_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ admits a martingale approximation.
(b) The random field satisfies (4.3) and

$$\frac{1}{|\mathbf{n}|}\sum\nolimits_{\mathbf{j}\ge\mathbf{1}}^{\mathbf{n}} ||\mathcal{P}_{\mathbf{1}}(S_{\mathbf{j}}) - D_{\mathbf{1}}||^2 \to 0 \textit{ when } \min_{1\le i\le d} n_i \to \infty. \tag{4.4}$$

(c) The random field satisfies (4.4) and for all j, $1 \le j \le d$, we have

$$\frac{1}{|\mathbf{n}|}||E_{\mathbf{n}_j}(S_{\mathbf{n}})||^2 \to 0 \textit{ when } \min_{1\le i\le d} n_i \to \infty,$$

where and $\mathbf{n}_j \in Z^d$ has the $j-th$ coordinate 0 and the other coordinates equal to the coordinates of $\mathbf{n}$.
(d) The random field satisfies (4.3) and

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} P_{\mathbf{1}}(S_{\mathbf{j}}) \textit{ converges in } L^2 \textit{ to } D_{\mathbf{1}} \textit{ when } \min_{1\le i\le d} n_i \to \infty. \tag{4.5}$$

Corollary 4.2. *Assume that one of the shifts $(T_i)_{1\le i\le d}$ is ergodic and either one of the conditions of Theorem 4.1 holds. Then*

$$\frac{1}{\sqrt{|\mathbf{n}|}}S_{\mathbf{n}} \Rightarrow N(0,c^2) \text{ when } \min_{1\le i\le d} n_i \to \infty,$$

where $c^2 = ||D_{\mathbf{0}}||^2$.

5 Examples

Let us apply these results to linear and nonlinear random fields with independent innovations.

Example 5.1. (Linear field) Let $(\xi_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ be a random field of independent, identically distributed random variables which are centered and have finite second moment, $\sigma^2 = E(\xi_{\mathbf{0}}^2)$. For $\mathbf{k}\ge\mathbf{0}$ define

$$X_{\mathbf{k}} = \sum_{\mathbf{j}\ge\mathbf{0}} a_{\mathbf{j}}\xi_{\mathbf{k}-\mathbf{j}}.$$

Assume that $\sum_{\mathbf{j}\ge\mathbf{0}} a_{\mathbf{j}}^2 < \infty$ and denote $b_{\mathbf{j}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}} a_{\mathbf{k}}$. Also assume that

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} b_{\mathbf{j}} \to c \text{ when } \min_{1\le i\le d} n_i \to \infty \tag{5.1}$$

and

$$\frac{E(S_{\mathbf{n}}^2)}{|\mathbf{n}|} \to c^2\sigma^2 \text{ when } \min_{1\le i\le d} n_i \to \infty.$$

Then the martingale approximation holds.

Proof of Example 5.1. The result follows by simple computations and by applying Theorem 4.1 (d). □

Example 5.2. (Volterra field) Let $(\xi_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ be a random field of independent random variables identically distributed, centered and with finite second moment, $\sigma^2 = E(\xi_{\mathbf{0}}^2)$. For $\mathbf{k}\ge\mathbf{1}$, define

$$X_{\mathbf{k}} = \sum_{(\mathbf{u},\mathbf{v})\ge(\mathbf{0},\mathbf{0})} a_{\mathbf{u},\mathbf{v}}\xi_{\mathbf{k}-\mathbf{u}}\xi_{\mathbf{k}-\mathbf{v}},$$

where $a_{\mathbf{u},\mathbf{v}}$ are real coefficients with $a_{\mathbf{u},\mathbf{u}} = 0$ and $\sum_{\mathbf{u},\mathbf{v}\ge\mathbf{0}} a_{\mathbf{u},\mathbf{v}}^2 < \infty$. Denote

$$c_{\mathbf{n},\mathbf{u},\mathbf{v}} = \frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{j}}(a_{\mathbf{k}-\mathbf{u},\mathbf{k}-\mathbf{v}} + a_{\mathbf{k}-\mathbf{v},\mathbf{k}-\mathbf{u}}). \tag{5.2}$$

Denote $A = \{\mathbf{u}\le\mathbf{1}$,there is $1\le i\le d$ with $u_i = 1\}$ and $B = \{\mathbf{u}\le\mathbf{1}\}$ and assume that

$$\lim_{\mathbf{n}>\mathbf{m}\to\infty}\sum_{(\mathbf{u},\mathbf{v})\in(A,B)}(c_{\mathbf{n},\mathbf{u},\mathbf{v}} - c_{\mathbf{m},\mathbf{u},\mathbf{v}})^2 = 0. \tag{5.3}$$

Also assume that

$$\frac{E(S_{\mathbf{n}}^2)}{|\mathbf{n}|} \to \sigma^4c^2 \text{ when } \min_{1\le i\le d} n_i \to \infty,$$

where c^2 is the limit of

$$\frac{1}{|\mathbf{n}|}\sum_{(\mathbf{u},\mathbf{v})\in(A,B)} c_{\mathbf{n},\mathbf{u},\mathbf{v}}^2 \text{ when } \min_{1\le i\le d} n_i \to \infty.$$

Then the martingale approximation holds.

Proof of Example 5.2. We have

$$\mathcal{P}_{\mathbf{1}}(X_{\mathbf{k}}) = \sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{0},\mathbf{0})} a_{\mathbf{u},\mathbf{v}}\mathcal{P}_{\mathbf{1}}(\xi_{\mathbf{k}-\mathbf{u}}\xi_{\mathbf{k}-\mathbf{v}}) = \sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{k},\mathbf{k})} a_{\mathbf{k}-\mathbf{u},\mathbf{k}-\mathbf{v}}\mathcal{P}_{\mathbf{1}}(\xi_{\mathbf{u}}\xi_{\mathbf{v}}).$$

Note that $\mathcal{P}_{\mathbf{1}}(\xi_{\mathbf{u}}\xi_{\mathbf{v}}) \neq 0$ if and only if $\mathbf{u} \in A$ and $\mathbf{v} \in B$ or $\mathbf{v} \in A$ and $\mathbf{u} \in B$. Therefore,

$$\mathcal{P}_{\mathbf{1}}(X_{\mathbf{k}}) = \sum_{(\mathbf{u},\mathbf{v})\in(A,B)} (a_{\mathbf{k}-\mathbf{u},\mathbf{k}-\mathbf{v}} + a_{\mathbf{k}-\mathbf{v},\mathbf{k}-\mathbf{u}})\xi_{\mathbf{u}}\xi_{\mathbf{v}} \ ,$$

and

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}}\mathcal{P}_{\mathbf{1}}(S_{\mathbf{j}}) = \frac{1}{|\mathbf{n}|}\sum_{(\mathbf{u},\mathbf{v})\in(A,B)}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{j}}(a_{\mathbf{k}-\mathbf{u},\mathbf{k}-\mathbf{v}} + a_{\mathbf{k}-\mathbf{v},\mathbf{k}-\mathbf{u}})\xi_{\mathbf{u}}\xi_{\mathbf{v}} \ .$$

By independence, and with the notation (5.2) this convergence happens if (5.3) holds. It remains to apply Theorem 4.1 (d). □

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