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Critical radius and supremum of random spherical harmonics (II)

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Abstract

We continue the study, begun in [6], of the critical radius of embeddings, via deterministic spherical harmonics, of fixed dimensional spheres into higher dimensional ones, along with the associated problem of the distribution of the suprema of random spherical harmonics. Whereas [6] concentrated on spherical harmonics of a common degree, here we extend the results to mixed degrees, en passant improving on the lower bounds on critical radii that we found previously.

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1 Introduction

The spherical harmonics of degree $\ell \geq 0$ on the d-dimensional unit sphere S^d are the collection of (real) eigenfunctions $\{\phi_j^{\ell,d}\}_{j=1}^{k_\ell^d}$ of the Laplacian $\Delta_{g_{S^d}}$ on S^d, satisfying

$$\Delta_{g_{S^d}} \phi_j^{\ell,d}(x) \ = -\ell(\ell+d-1)\phi_j^{\ell,d}(x), \tag{1.1}$$

where k_ℓ^d is

$$k_\ell^d \ \triangleq \ \frac{2\ell+d-1}{\ell+d-1}\binom{\ell+d-1}{d-1}. \tag{1.2}$$

In [6], we studied the map

$$\tilde{i}_\ell^d : \ S^d \to S^{k_\ell^d-1}, \quad x \to \sqrt{\frac{s_d}{k_\ell^d}}\left(\phi_1^{\ell,d}(x), \cdots, \phi_{k_\ell^d}^{\ell,d}(x)\right), \tag{1.3}$$

defined by the spherical harmonics of degree ℓ, where, s_d denotes the surface area of the unit sphere S^d. For large enough ℓ, the image $\tilde{i}_\ell^d(S^d)$ is diffeomorphic to S^d if ℓ is odd, and to $\mathbb{R}P^d$ if ℓ is even.

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As shown in [6], and explained again below, the behavior of this *deterministic* map has significant implications for *random* spherical harmonics[1]

$$\tilde{\Phi}^d_\ell(x) \;=\; \sum_{j=1}^{k^d_\ell} a^\ell_j \phi^{\ell,d}_j(x), \tag{1.4}$$

where the a^ℓ_j are either standard Gaussian variables (in which case we talk about the "Gaussian ensemble") or $(a^\ell_1, \ldots, a^\ell_{k^d_\ell})$ is uniform on $S^{k^d_\ell - 1}$ (in which case we talk about the "spherical ensemble"). In particular, one of the most important aspects of the deterministic mapping vis a vis the random process is the critical radius, or reach, of its image in the ambient sphere.

What made the study in [6] most interesting was the fact that the pull back of the standard round metric under $\tilde{i}^d_\ell$ grows with order ℓ^2, indicating that the image $\tilde{i}^d_\ell(S^d)$ in $S^{k^d_\ell - 1}$ becomes more and more 'twisted' as ℓ grows. Intuitively, if a sequence of sets become more twisted in their ambient space, it seems natural that their critical radii, as a measure of smoothness, will tend to zero. (Think of the critical radii of the graphs of $(x, \sin \ell x)$ in $\mathbb{R}^2$, which tend to 0 as $\ell \to \infty$.) Rather surprisingly, the main results of [6] showed that there is a lower bound for the critical radius of the $\tilde{i}^d_\ell(S^d)$ in $S^{k^d_\ell - 1}$, as $\ell \to \infty$. In some sense, this was a result of competition between the 'twistiness' of the image and the 'extra space' available as the ambient spaces $S^{k^d_\ell - 1}$ changed.

As a direct consequence of these deterministic results [6] derived an explicit formula for the distribution of the suprema of the random spherical harmonics $\tilde{\Phi}^d_\ell$ of (1.4) under the spherical ensemble, by exploiting Weyl's tube formula. (For the Gaussian ensemble, see [1, 8, 10] on the connections between spherical and Gaussian ensembles, Weyl's tube formula, suprema of random fields, and the expected Euler characteristic of excursion sets.)

The aim of the present paper is to extend the analysis of [6] to a related, but somewhat different embedding, given by the deterministic map

$$i^d_L:\; S^d \to S^{\pi^d_L - 1}, \qquad x \to \sqrt{\frac{s_d}{\pi^d_L}} \left(\phi^{\ell,d}_j(x)\right)_{\ell=0,\ldots,L,\; j=1,\ldots,k^d_\ell}, \tag{1.5}$$

where

$$\pi^d_L \;\triangleq\; \sum_{\ell=0}^{L} k^d_\ell \;=\; \frac{2L+d}{d}\binom{L+d-1}{d-1}. \tag{1.6}$$

For large enough L (either odd or even) this map is an embedding; viz. $i^d_L(S^d) \cong S^d$ [11]. Following the ideas and proofs in [6], we will prove the existence of a lower bound for the critical radius of $i^d_L(S^d)$ in $S^{\pi^d_L - 1}$. This will allow us to also derive an exact formula for the distribution of the suprema of the family of random spherical harmonics under the spherical ensemble, viz.

$$\Phi^d_L(x) \;\triangleq\; \sum_{\ell=0}^{L} \widetilde{\Phi}^d_\ell(x) \;=\; \sum_{\ell=0}^{L}\sum_{j=1}^{k^d_\ell} a^\ell_j \phi^{\ell,d}_j(x), \tag{1.7}$$

with $(a^\ell_j)_{\ell,j}$ uniform on $S^{\pi^d_L - 1}$.

[1] As is common, we shall allow the term "spherical harmonics" to take two meanings, either one of which will always be clear from the context. The first is the collection of eigenfunctions given by (1.1). The second is the collection of functions satisfying Laplace's equation, and then restricted to a sphere. The first collection provides a basis for the second. See Section 2.1 for a more formal description of this.

The differences between the two random processes Φ_L^d and $\widetilde{\Phi}_\ell^d$ are subtle but important, and best understood in spectral terms. For a fixed ℓ, all the spherical harmonics $\phi_j^{\ell,d}$ are associated with the same eigenvalue, often called 'frequency'. In these terms, $\widetilde{\Phi}_\ell^d$ is a single, or 'pure' frequency random process, whereas the spectrum of Φ_L^d contains (a discrete collection of) frequencies between 0 and $L(L+d-1)$. In terms of the original motivation for studying random spherical harmonics (the 'Berry conjecture' of [4]), mixed spectra processes play a more central role than pure spectra ones. This is one of the main motivations behind the current paper.

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2 Main results

2.1 Spherical harmonics and the deterministic embedding

Let the unit sphere S^d be equipped with the round metric g_{S^d}, and write $\Delta_{g_{S^d}}$ for the associated Laplacian. Let $\mathcal{H}_\ell^d$ be the eigenspace spanned by eigenfunctions (1.1) of degree $\ell \geq 0$. Then the dimension of $\mathcal{H}_\ell^d$ is k_ℓ^d given in (1.2). Since $L^2(S^d) = \oplus_{\ell \geq 0} \mathcal{H}_\ell^d$, if we normalize the eigenfunctions so that their L^2-norm is 1, the expansion of $L^2(S^d)$ functions in the orthonormal basis of spherical harmonics provides a natural generalization of Fourier series expansions. Let $\mathcal{H}_{\ell \leq L}^d \triangleq \bigoplus_{\ell=0}^L \mathcal{H}_\ell^d$ be the space of spherical harmonics of degree at most L.

For L large enough, the map (1.5) is an embedding [11] i.e. $i_L^d(S^d) \cong S^d$. Furthermore, it follows from the properties of spectral projection kernels that the $\mathbb{R}^{\pi_L^d}$ norm of $i_L^d(x)$ is identically 1, so that i_L^d is actually a map between spheres; viz.

$$i_L^d: \;\; S^d \to S^{\pi_L^d - 1}. \tag{2.1}$$

In addition, the pull-back of the Euclidean metric satisfies [11]

$$(i_L^d)^*(g_E) \cong c_d L^2 g_{S^d} \tag{2.2}$$

where c_d is a constant depending only on d.

Our interest in this section lies in the critical radius of $i_L(S^d)$ in $\mathbb{R}^{\pi_L^d}$.

Recall that if M is a smooth manifold embedded in an ambient manifold $\widetilde{M}$, then the local critical radius, or reach, at a point $x \in M$ is the furthest distance one can travel, along any geodesic in $\widetilde{M}$ based at x but normal to M in $\widetilde{M}$, without meeting a similar vector originating at another point in M. The (global) critical radius of M is then the infimum of all the local ones. We refer the reader to Section 3 of [6] for additional background and for formal definitions.

We can now state our first result.

Theorem 2.1. *For sufficiently large L, the critical radius of the embedding $i_L(S^d)$ in $\mathbb{R}^{\pi_L^d}$ has a strictly positive, uniform in L, lower bound which depends only on d.*

Let

$$\mathrm{Tube}_{\pi_L^d}(i_L^d(S), \rho) \;\triangleq\; \left\{ x \in \mathbb{R}^{\pi_L^d} : \; \min_{p \in i_L^d(S)} \|x - p\| \;\leq\; \rho \right\}$$

be the tube around $i_L(S^d)$ in $\mathbb{R}^{\pi_L^d}$, where ρ is less than the critical radius of $i_L^d(S)$ in $\mathbb{R}^{\pi_L^d}$. Then, by (2.1), the intersection $\mathrm{Tube}_{\pi_L^d}(i_L^d(S), \rho) \cap S^{\pi_L^d - 1}$ will be a tube of $i_L(S^d)$ in $S^{\pi_L^d - 1}$ without self-intersection. This fact immediately implies

Corollary 2.2. *Theorem 2.1 continues to hold, with a similar lower bound, when $i_L(S^d)$ is considered as an embedding in $S^{\pi_L^d - 1}$.*

2.2 Random spherical harmonics and exceedence probabilities

In this section we turn our attention to the random spherical harmonics under the spherical ensemble, which are defined by (1.7). As opposed to the simpler $\widetilde{\Phi}_L^d$ of (1.4), and as explained in the final paragraph of the Introduction, Φ_L^d has a broader and flatter spectrum than does $\widetilde{\Phi}_L^d$.

If we now let ρ_d denote the uniform lower bound for the critical radius of $i_L^d(S^d)$ in the ambient space $S^{\pi_L^d-1}$ appearing in Corollary 2.2, then this corollary and the same arguments as adopted in Section 6 of [6] prove the following result.

Theorem 2.3. *Let Φ_L^d be the random spherical harmonics under the spherical ensemble as in* (1.7)*. Then there exists constants $\rho_d > 0$ such that, for sufficiently large L, and for all $u > \sqrt{\frac{\pi_L^d}{s_d}}\cos(\rho_d)$,*

$$\begin{aligned}
&\mathbb{P}_{\mu_L^d}\left\{\sup_{S^d}\Phi_L^d(x) > u\right\} \\
&\quad = \frac{1}{s_{\pi_L^d-1}}\sum_{j=0}^{d} f_{\pi_L^d,j}\left(\cos^{-1}\left(u/\sqrt{\frac{\pi_L^d}{s_d}}\right)\right)\left(\frac{L(L+d)}{d+2}\right)^{j/2}\mathcal{L}_j(S^d),
\end{aligned} \tag{2.3}$$

where $s_{\pi_L^d-1}$ is the surface area of the unit sphere $S^{\pi_L^d-1}$, the $f_{\pi_L^d,j}(\rho)$ are explicit functions given in Theorem 10.5.7 of [1] and (6.6) *of [6], and the $\mathcal{L}_j(S^d)$ are the standard j-th Lipschitz-Killing curvatures of the unit sphere S^d, given explicitly, for example, in* (6.10) *of [6].*

Note that although Theorem 2.3 gives an exact result under the spherical ensemble, analogous (but approximate) results can also be formulated under the Gaussian ensemble (i.e. when the a_j^ℓ are all independent, standard, Gaussian random variables). This follows from the rich literature relating mean Euler characteristics of excursion sets and exceedence probabilities for Gaussian processes; e.g. [1, 5, 7, 8, 10]. We will not repeat this here.

3 Proof of Theorem 2.1 for S^2

In this section, we will prove Theorem 2.1 for the 2-sphere.

3.1 Spectral projection kernels

We will drop the index 2 in this section whenever it does not lead to ambiguities. Thus $\mathcal{H}_{\ell\le L}$ is now the space of spherical harmonics of S^2 of degree at most L. Its dimension is $\pi_L = (L+1)^2$ by (1.6). The spectral projection kernel is now given, for $x, y \in S^2$, by the Christoffel-Darboux formula [2],

$$K_L(x,y) \;=\; \sum_{\ell=0}^{L}\sum_{j=-\ell}^{\ell}\phi_j^\ell(x)\phi_j^\ell(y) \;=\; \frac{L+1}{4\pi}P_L^{(1,0)}(\cos\Theta(x,y)),$$

where $\Theta(x,y)$ is the angle between the vectors x and y on S^2, and $P_L^{(1,0)}$ is a Jacobi polynomial. In general, the Jacobi polynomials are defined by

$$P_L^{(\alpha,\beta)}(x) \;=\; \sum_{s=0}^{L}\binom{L+\alpha}{s}\binom{L+\beta}{L-s}\left(\frac{x-1}{2}\right)^{L-s}\left(\frac{x+1}{2}\right)^{s}.$$

We will need the following facts about $P^{(1,0)}$:

$$P_L^{(1,0)}(1) \;=\; L+1, \qquad P_L^{(1,0)\prime}(1) \;=\; \frac{L(L+1)(L+2)}{4}.$$

We will also need the fact that, on the diagonal, the kernel K_L is given by

$$K_L(x,x) \;=\; \frac{L+1}{4\pi}P_L^{(1,0)}(1) \;=\; \frac{(L+1)^2}{4\pi}.$$

Defining the normalized kernel

$$\Pi_L(x,y) \;\triangleq\; \frac{4\pi}{(L+1)^2}K_L(x,y) \;=\; \frac{1}{L+1}P_L^{(1,0)}(\cos\Theta(x,y)),$$

we have that the norm of $i_L(x)$, defined at (1.5), is given by

$$\|i_L(x)\|^2 \;=\; \frac{4\pi}{(L+1)^2}\sum_{j,\ell}|\phi_j^\ell(x)|^2 \;=\; \Pi_L(x,x) \;=\; 1,$$

so that i_L is actually a map $i_L:\ S^2 \to S^{L^2+2L}$. Following the computations in [6], the pull back of the Euclidean metric under this mapping is $g_L \;=\; i_L^*(g_E) \;=\; \frac{L^2+2L}{4}g_{S^2}$.

3.2 Critical radius of $i_L(S^2)$

A useful formula for the critical radius of a smooth manifold embedded in Euclidean space was derived in [10].

Following the calculations in Section 3 of [6], the critical radius of the embedding of $i_L(S^2)$ can be rewritten as

$$r_L \;=\; \inf_{\theta\in[0,\pi]} \frac{1-\frac{1}{L+1}P_L^{(1,0)}(\cos\theta)}{\sqrt{2-\frac{2}{L+1}P_L^{(1,0)}(\cos\theta)-\frac{1}{L+1}\frac{[P_L^{(1,0)\prime}(\cos\theta)\sin\theta]^2}{P_L^{(1,0)\prime}(1)}}}. \tag{3.1}$$

3.3 Proof of Theorem 2.1 for S^2

The proof is then based on classical asymptotic estimates for Jacobi polynomials.

We start with an asymptotic formula of Hilb's type ([9], Theorem 8.21.12):

$$\left(\sin\frac{\theta}{2}\right)P_L^{(1,0)}(\cos\theta) \;=\; \left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} J_1((L+1)\theta) + R_{1,L}(\theta), \tag{3.2}$$

where

$$R_{1,L}(\theta) \;=\; \begin{cases} \theta^3 O(L), & 0 \le \theta \le c/L, \\ \theta^{\frac{1}{2}} O(L^{-\frac{3}{2}}), & c/L \le \theta \le \pi-\epsilon, \end{cases} \tag{3.3}$$

and c, ϵ are uniform constants, independent of L.

We also have the Darboux formula ([9], Theorem 8.21.13):

$$P_L^{(1,0)}(\cos\theta) \;=\; L^{-\frac{1}{2}}k(\theta)\cos\left((L+1)\theta-\frac{3}{4}\pi\right) + R_{2,L}(\theta), \tag{3.4}$$

if $c'/L \le \theta \le \pi - c'/L$, where c' is a uniform constant and

$$k(\theta) \;=\; \frac{1}{\sqrt{\pi}}\left(\sin\frac{\theta}{2}\right)^{-3/2}\left(\cos\frac{\theta}{2}\right)^{-1/2}, \tag{3.5}$$

$$R_{2,L}(\theta) \;=\; L^{-\frac{1}{2}}k(\theta)(L\sin\theta)^{-1}O(1). \tag{3.6}$$

Near the right end point π, the asymptotic behavior of the Jacobi polynomials is given by the Mehler-Heine formula

$$\lim_{L\to\infty} P_L^{(1,0)}\left(\cos\left(\pi-\frac{x}{L}\right)\right) \;=\; J_0(x), \tag{3.7}$$

where the limit is uniform on compact subsets of $\mathbb{R}$ (see equation (3) in [3]) and the J_ν are the Bessel functions of the first kind of order ν.

In order to prove Theorem 2.1, we divide $[0,\pi]$ into the four subintervals

$$[0,c/L],\quad [c/L,L^{-4/5}],\quad [L^{-4/5},\pi-c'/L],\quad [\pi-c'/L,\pi], \tag{3.8}$$

and replace the global infimum in (3.1) as

$$\min\{I_L, II_L, III_L, IV_L\} \triangleq \min\left\{\inf_{[0,c/L]}, \inf_{[c/L,L^{-4/5}]}, \inf_{[L^{-4/5},\pi-c'/L]}, \inf_{[\pi-c'/L,\pi]}\right\}. \tag{3.9}$$

We now treat each of the four infima in (3.9), starting with

$$I_L = \inf_{[0,c/L]} \frac{1-\frac{1}{L+1}P_L^{(1,0)}(\cos\theta)}{\sqrt{2-\frac{2}{L+1}P_L^{(1,0)}(\cos\theta)-\frac{1}{L+1}\frac{[P_L^{(1,0)\prime}(\cos\theta)\sin\theta]^2}{P_L^{(1,0)\prime}(1)}}}.$$

To treat this term, we study a rescaling limit via a new parameter x, where $x=L\theta$, so that $x\in[0,c]$. By the Hilb's type asymptotic (3.2) on $0\le\theta\le c/L$,

$$\begin{aligned}\frac{1}{L}P_L^{(1,0)}\left(\cos\frac{x}{L}\right) &= \frac{1}{L}\left(\sin\frac{x}{2L}\right)^{-1}\left(\frac{x/L}{\sin x/L}\right)^{\frac12} J_1\left(x+\frac{x}{L}\right)+O(L^{-2})\\ &= \frac{1}{L}\left(\frac{x}{2L}+O(L^{-3})\right)^{-1}\left(1+O(L^{-2})\right)^{1/2}\\ &\quad\times\left(J_1(x)+O(L^{-1})\right)+O(L^{-2})\\ &= \frac{2}{x}J_1(x)+O(L^{-1}).\end{aligned}$$

Next, for the rescaling of $P_L^{(1,0)\prime}\left(\cos\frac{x}{L}\right)$, note the following two standard facts about Jacobi polynomials and Bessel functions:

$$\begin{aligned}P_L^{(1,0)\prime}(x) &= \frac12(L+2)P_{L-1}^{(2,1)}(x), && \text{([9], (4.5.5))}\\ \left(\frac{2}{x}J_1(x)\right)' &= -\frac{2}{x}J_2(x). && \text{([9], (1.71.5))}\end{aligned}$$

Applying these facts and the Hilb's asymptotic for $P_L^{(2,1)}(\cos\theta)$ given below in (4.3), we have

$$\begin{aligned}&\frac{1}{L^2}P_L^{(1,0)\prime}(\cos\theta)\sin\theta\\ &= \frac{1}{L^2}\left\{\frac12\sin\theta\cdot(L+2)P_{L-1}^{(2,1)}(\cos\theta)\right\}\\ &= \frac{L+2}{L^2}\left(\sin\frac{\theta}{2}\right)^{-1}\left\{\frac{L}{L+1}\left(\frac{\theta}{\sin\theta}\right)^{\frac12}J_2((L+1)\theta)+O(L^{-2})\right\}\\ &= \frac{L+2}{L^2}\left(\sin\frac{x}{2L}\right)^{-1}\left\{\frac{L}{L+1}\left(\frac{x/L}{\sin x/L}\right)^{\frac12}J_2\left(x+\frac{x}{L}\right)+O(L^{-2})\right\}\\ &= \frac{2}{x}J_2(x)+O(L^{-1})\\ &= -\left(\frac{2}{x}J_1(x)\right)'+O(L^{-1}).\end{aligned}$$

We rescale

$$\frac{1}{L+1}\frac{[P_L^{(1,0)\prime}(\cos\theta)\sin\theta]^2}{P_L^{(1,0)\prime}(1)}$$

to obtain

$$\frac{4\left(\left(\frac{2}{x}J_1(x)\right)' L^2+O(L)\right)^2}{L(L+1)^2(L+2)} \;=\; 4\left(\left(\frac{2}{x}J_1(x)\right)'+O(L^{-1})\right)^2.$$

Hence, as $L\to\infty$, I_L is asymptotic to

$$I_\infty \;=\; \inf_{[0,c]}\frac{1-\frac{2}{x}J_1(x)}{\sqrt{2-\frac{4}{x}J_1(x)-4((\frac{2}{x}J_1(x))')^2}}. \tag{3.10}$$

For the second term, II_L, we again use the Hilb's type asymptotic formula (3.2), this time on the interval $[c/L, L^{-4/5}]\subset[c/L,\pi-\epsilon]$. We also rescale to $x=L\theta$, so that $x\in[c,L^{1/5}]$, and will need the following three basic properties of Bessel functions ([9], Pages 15–16), for ν real but $\nu\neq -1,-2,-3,\ldots$.

$$\begin{aligned}
J_\nu(x) &= \left(\frac{2}{\pi x}\right)^{\frac12}\cos(x-\frac{\pi\nu}{2}-\frac{\pi}{4})+O(x^{-\frac32}), \quad \text{as } x\to\infty,\\
J_\nu(x) &\sim x^\nu, \quad \text{as } x\to 0^+,\\
J_1'(x) &= \frac{1}{x}J_1(x)-J_2(x).
\end{aligned}$$

Applying these properties, we have, uniformly in $x\in[c,L^{1/5}]$,

$$J_1\left(x+\frac{x}{L}\right) \;=\; J_1(x)+\frac{x}{L}J_1'(x)+\ldots \;=\; J_1(x)+O(L^{-4/5}). \tag{3.11}$$

Consequently,

$$\begin{aligned}
\frac{1}{L}P_L^{(1,0)}\left(\cos\frac{x}{L}\right) &= \frac{1}{L}\left(\sin\frac{x}{2L}\right)^{-1}\left(\frac{x/L}{\sin x/L}\right)^{\frac12}J_1\left(x+\frac{x}{L}\right)+O(L^{-\frac{19}{10}})\\
&= \frac{1}{L}\left(\frac{x}{2L}+O(L^{-12/5})\right)^{-1}\left(1+O(L^{-8/5})\right)^{1/2}\\
&\qquad\times\left(J_1(x)+O(L^{-4/5})\right)+O(L^{-\frac{19}{10}})\\
&= \frac{2}{x}J_1(x)+O(L^{-4/5}).
\end{aligned}$$

Hence, we have the rescaling limit

$$\frac{1}{L+1}P_L^{(1,0)}\left(\cos\frac{x}{L}\right) \;=\; \frac{2}{x}J_1(x)+O(L^{-4/5}), \tag{3.12}$$

for $x\in[c,L^{1/5}]$. Similarly, the rescaling of $\frac{1}{L+1}\frac{[P_L^{(1,0)\prime}(\cos\theta)\sin\theta]^2}{P_L^{(1,0)\prime}(1)}$ will be dominated by the leading term $4((\frac{2}{x}J_1(x))')^2$ for L large enough. Thus II_L will converge, as $L\to\infty$, to the same expression that we had for I_L, viz.

$$II_\infty \;=\; \inf_{[c,\infty]}\frac{1-\frac{2}{x}J_1(x)}{\sqrt{2-\frac{4}{x}J_1(x)-4((\frac{2}{x}J_1(x))')^2}}. \tag{3.13}$$

For III_L, since $\theta\in[L^{-4/5},\pi-c'/L]\subset[c'/L,\pi-c'/L]$, we can apply the Darboux formula (3.4) on $[L^{-4/5},\pi-c'/L]$ and define

$$M_L(\theta) \;=\; \frac{1}{\sqrt{L}}k(\theta)\cos\left((L+1)\theta-\frac{3}{4}\pi\right),$$

giving $P_L^{(1,0)}(\cos\theta) \;=\; M_L(\theta) + R_{2,L}(\theta)$. Since $k'(\frac{2\pi}{3}) = 0$ and $k''(\theta) > 0$ for $\theta \in (0,\pi)$, it is simple to check that

$$\max k(\theta) \;=\; \max\{k(L^{-4/5}),\; k(\pi - c'/L)\} \;=\; O(L^{6/5}),$$

so that

$$\frac{1}{L} M_L(\theta) \;=\; L^{-3/2} \cdot O(L^{6/5}) \;=\; O(L^{-3/10}).$$

Similarly, we have

$$\frac{1}{L} R_{2,L}(\theta) \;=\; L^{-5/2} \cdot O(L^2) \;=\; O(L^{-1/2}),$$

so that

$$\frac{1}{L+1} P_L^{(1,0)}(\cos\theta) \;= O(L^{-3/10}).$$

Note now the fact that, for $c'/L \le \theta \le \pi - c'/L$, it follows from bounds on the derivatives of Jacobi polynomials ([9], P. 236, 8.8.1) that

$$\frac{d}{d\theta} P_L^{(1,0)}(\cos\theta) \;=\; L^{\frac{1}{2}} k(\theta) \left\{ -\sin((L+1)\theta - 3\pi/4) + (L\sin\theta)^{-1} O(1) \right\}.$$

Also, on the interval $[L^{-4/5}, \pi - c'/L]$, we have

$$\frac{d}{d\theta} P_L^{(1,0)}(\cos\theta) \;=\; L^{\frac{1}{2}} \cdot O(L^{6/5}) \;=\; O(L^{17/10}).$$

This implies that

$$\frac{1}{L+1} \frac{[P_L^{(1,0)\prime}(\cos\theta)\sin\theta]^2}{P_L^{(1,0)\prime}(1)} \;=\; O(L^{-3/5}).$$

Thus the $L \to \infty$ limit of III_L is

$$III_\infty \;=\; \frac{1}{\sqrt{2}}. \tag{3.14}$$

We treat the final term, IV_L, a little differently, bounding it from below. Firstly, we have

$$\begin{aligned} IV_L = & \inf_{[\pi - c'/L, \pi]} \frac{1 - \frac{1}{L+1} P_L^{(1,0)}(\cos\theta)}{\sqrt{2 - \frac{2}{L+1} P_L^{(1,0)}(\cos\theta) - \frac{1}{L+1} \frac{[P_L^{(1,0)\prime}(\cos\theta)\sin\theta]^2}{P_L^{(1,0)\prime}(1)}}} \\ \ge & \inf_{[\pi - c'/L, \pi]} \sqrt{\frac{1}{2}\left(1 - \frac{1}{L+1} P_L^{(1,0)}(\cos\theta)\right)} \\ = & \inf_{[0,c']} \sqrt{\frac{1}{2}\left(1 - \frac{1}{L+1} P_L^{(1,0)}\left(\cos(\pi - \frac{x}{L})\right)\right)}. \end{aligned}$$

By the Mehler-Heine formula (3.7), we have the uniform estimate

$$\frac{1}{L+1} P_L^{(1,0)}\left(\cos(\pi - \frac{x}{L})\right) \;\to\; 0$$

for $x \in [0, c']$. Hence, as $L \to \infty$, we have

$$IV_\infty \;\ge\; \frac{1}{\sqrt{2}}. \tag{3.15}$$

The limits and lower bound for I_L–IV_L established above show that the critical radius of the $i_L(S^2)$, as $L \to \infty$, have a non-zero lower bound in the ambient space $\mathbb{R}^{(L+1)^2}$. That a similar lower bound holds in the ambient space S^{L^2+2L} is implied by the discussion before Corollary 2.2, and this completes the proof of Theorem 2.1 for S^2. □

Before moving on to the proof for the general case, it is interesting to actually compute the asymptotic lower bound to the critical radius for the two-dimensional case. The necessary information for this is contained in the three functions in Figure 1.

The red curve in Figure 1 is the plot of

$$f(x) \triangleq \frac{1-\frac{2}{x}J_1(x)}{\sqrt{2-\frac{4}{x}J_1(x)-4((\frac{2}{x}J_1(x))')^2}}. \tag{3.16}$$

The black one is the plot of

$$g(x) \triangleq \frac{1-J_0(x)}{\sqrt{2-2J_0(x)-2[J_0(x)']^2}}, \tag{3.17}$$

and the blue one is the plot of

$$h(x) \triangleq \frac{1+J_0(x)}{\sqrt{2+2J_0(x)-2[J_0(x)']^2}}. \tag{3.18}$$

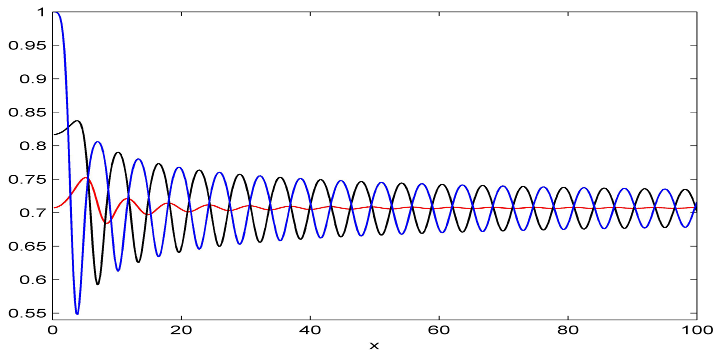

Figure 1: Graphs of the functions (3.16)–(3.18). See text for details.

However, from the proof of Theorem 2.1 above we know that the critical radius of the embeddings of $i_L(S^2)$ in $\mathbb{R}^{(L+1)^2}$ is asymptotic to

$$\min\left\{\inf_{x\in[0,\infty)} f(x),\ \frac{1}{\sqrt{2}}\right\}.$$

On the other hand, we know from (4.9), (4.10) in [6] that, for ℓ odd where we have $\widetilde{i}_\ell(S^2) \cong S^2$ (1.3), the limit is

$$\min\left\{\inf_{x\in[0,\infty)} g(x),\ \inf_{x\in[0,\infty)} h(x),\ \frac{1}{\sqrt{2}}\right\};$$

for ℓ even where the map is an immersion $\widetilde{i}_\ell(S^2) \cong \mathbb{R}P^2$, the limit is

$$\min\left\{\inf_{x\in[0,\infty)} g(x),\ \frac{1}{\sqrt{2}}\right\}.$$

It thus immediately follows from the Figure 1 that the asymptotic lower bounds for the critical radii of the embeddings of S^2 by i_L we established here are larger than those for the embeddings $\widetilde{i}_\ell$ treated in [6]. This is consistent with the expectations discussed in the Introduction.

4 Proof of Theorem 2.1 for the general case

The arguments for S^2 can be generalized to higher dimensions in a straightforward fashion, which we now describe.

The kernel $K_L^d(x,y)$ of $\mathcal{H}^d_{\ell\le L}$ can now be expressed as [3]

$$K_L^d(x,y) \;=\; \sum_{\ell=0}^{L}\sum_{j=1}^{k_\ell^d}\phi_j^\ell(x)\phi_j^\ell(y) \;=\; \frac{\pi_L^d/s_d}{\binom{L+d/2}{L}}P_L^{(1+\lambda,\lambda)}(\cos\Theta(x,y)),$$

for $x,y\in S^d$, where $\lambda=\frac{d-2}{2}$. We first note that [2],

$$P_L^{(1+\lambda,\lambda)}(1) \;=\; \binom{L+d/2}{L} \;=\; \frac{\Gamma(L+d/2+1)}{\Gamma(L+1)\Gamma(d/2+1)};$$

and

$$P_L^{(1+\lambda,\lambda)\prime}(1) \;=\; \frac{L+d}{2}\binom{L+d/2}{L-1}.$$

We define the normalized spectral projection kernel as

$$\Pi_L^d(x,y) \;=\; \frac{s_d}{\pi_L^d}K_L^d(x,y) \;=\; \binom{L+d/2}{L}^{-1}P_L^{(1+\lambda,\lambda)}(\cos\Theta(x,y)).$$

Thus the norm of the map (1.5) is $\|i_L^d(x)\|^2=\Pi_L^d(x,x)=1$, i.e.,

$$i_L^d:\;\; S^d\to S^{\pi_L^d-1}. \tag{4.1}$$

Following the arguments of [6], the pull-back of the Euclidean metric is

$$(i_L^d)^*(g_E) \;=\; \binom{L+d/2}{L}^{-1}P_L^{(1+\lambda,\lambda)\prime}(1)g_{S^d} \;=\; \frac{L(L+d)}{d+2}g_{S^d}. \tag{4.2}$$

Following the computations in [6], the critical radius of the embedding is

$$\inf_{\theta\in[0,\pi]}\frac{1-\binom{L+d/2}{L}^{-1}P_L^{(1+\lambda,\lambda)}(\cos\theta)}{\sqrt{2-2\binom{L+d/2}{L}^{-1}P_L^{(1+\lambda,\lambda)}(\cos\theta)-\binom{L+d/2}{L}^{-1}\frac{[P_L^{(1+\lambda,\lambda)\prime}(\cos\theta)\sin\theta]^2}{P_L^{(1+\lambda,\lambda)\prime}(1)}}}.$$

We still have the following classical asymptotic estimate about the Jacobi polynomials. Firstly, we have the asymptotic formula of Hilb's type below ([9], Theorem 8.21.12):

$$\begin{aligned}&\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta}P_L^{(\alpha,\beta)}(\cos\theta)\\ &=\; N^{-\alpha}\frac{\Gamma(L+\alpha+1)}{L!}\left(\frac{\theta}{\sin\theta}\right)^{\frac12}J_\alpha(N\theta)+R_{1,L}(\theta),\end{aligned} \tag{4.3}$$

where $N=L+(\alpha+\beta+1)/2$,

$$R_{1,L}(\theta) \;=\; \begin{cases}\theta^{\alpha+2}O(L^{\alpha}), & 0\le\theta\le c/L,\\ \theta^{\frac12}O(L^{-\frac32}), & c/L\le\theta\le\pi-\epsilon,\end{cases} \tag{4.4}$$

and c, ϵ are uniform constants, independent of L. In our case $\alpha=1+\lambda$ and $\beta=\lambda$. On the subinterval $c'/L\le\theta\le\pi-c'/L$, another asymptotic estimate is given by the Darboux formula ([9], Theorem 8.21.13)

$$P_L^{(1+\lambda,\lambda)}(\cos\theta) \;=\; L^{-\frac12}k(\theta)\cos((L+\lambda+1)\theta+\gamma)+R_{2,L}(\theta), \tag{4.5}$$

where c is a large enough, but uniform, constant, $\gamma = -(\lambda + 3/2)\pi/2$ and

$$\begin{aligned} k(\theta) &= \frac{1}{\sqrt{\pi}} \left(\sin\frac{\theta}{2}\right)^{-\lambda-3/2} \left(\cos\frac{\theta}{2}\right)^{-\lambda-1/2}, \\ R_{2,L}(\theta) &= L^{-\frac{1}{2}} k(\theta)(L\sin\theta)^{-1} O(1). \end{aligned}$$

Near the end points, the asymptotic behavior of the Jacobi polynomials is given by the Mehler-Heine formulas ([9], p. 192)

$$\begin{aligned} \lim_{L\to\infty} L^{-1-\lambda} P_L^{(1+\lambda,\lambda)}\left(\cos\frac{x}{L}\right) &= \left(\frac{x}{2}\right)^{-1-\lambda} J_{1+\lambda}(x), \\ \lim_{L\to\infty} L^{-\lambda} P_L^{(1+\lambda,\lambda)}\left(\cos\left(\pi - \frac{x}{L}\right)\right) &= \left(\frac{x}{2}\right)^{-\lambda} J_{\lambda}(x), \end{aligned}$$

where the limits are uniform on compact subsets of $\mathbb{R}$ and the J_ν are Bessel functions of the first kind with order ν.

Again, following the arguments in Section 3.3, one can prove the following lower bound for the critical radius, as $L \to \infty$:

$$\inf_{[0,\infty]} \frac{1 - \left(\frac{x}{2}\right)^{-d/2} J_{d/2}(x)}{\sqrt{2 - \left(\frac{x}{2}\right)^{-d/2} J_{d/2}(x) - 2\Gamma(\frac{d}{2}+2)\left(\left(\left(\frac{x}{2}\right)^{-d/2} J_{d/2}(x)\right)'\right)^2}}.$$

This completes the proof of Theorem 2.1 for the general case. □

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