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A counterexample to monotonicity of relative mass in random walks*

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Abstract

For a finite undirected graph $G = (V, E)$, let $p_{u,v}(t)$ denote the probability that a continuous-time random walk starting at vertex u is in v at time t. In this note we give an example of a Cayley graph G and two vertices $u, v \in G$ for which the function

$$r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} \qquad t \geq 0$$

is not monotonically non-decreasing. This answers a question asked by Peres in 2013.

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1 Introduction

Let $G = (V, E)$ be a finite undirected regular graph. Let $p_{u,v}(t)$ denote the probability that a continuous-time random walk starting at vertex u is in v at time t. In this note we are interested in the function

$$r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} \qquad t \geq 0 \ .$$

Clearly, in regular connected graphs for any $u \neq v$, we have $r_{u,v}(0) = 0$ and $\lim_{t\to\infty} r_{u,v}(t) = 1$. One might wonder if the function is monotonically non-decreasing. It is not difficult to see that there are regular graphs for which this is *not* the case. In fact, there are regular graphs such that $r_{u,v}(t) > 1$ for some vertices u, v and time t; in particular, $r_{u,v}(t)$ is not monotonically non-decreasing. We give an example of such a graph in Appendix A. We thank Jeff Cheeger [1] for pointing this out to us.

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For vertex-transitive graphs, however, it holds that $r_{u,v}(t) \le 1$ for all vertices u, v and all $t \ge 0$. Indeed, using Cauchy-Schwarz and the reversibility of the walk,

$$\begin{aligned} p_{u,v}(t) &= \sum_{w \in V} p_{u,w}(t/2) p_{w,v}(t/2) \\ &\le \Big(\sum_{w} p_{u,w}(t/2)^2 \Big)^{1/2} \cdot \Big(\sum_{w} p_{w,v}(t/2)^2 \Big)^{1/2} \\ &= p_{u,u}(t)^{1/2} \cdot p_{v,v}(t)^{1/2} = p_{u,u}(t) \ . \end{aligned}$$

This motivates the following question, asked in 2013 by Peres [5]:

> Is the function $r_{u,v}$ monotonically non-decreasing in t for all vertex-transitive graphs and all vertices u, v?

More recently, a special case of that question was asked independently by Price [7]. Namely, Price asked whether for Brownian motion on flat tori (i.e., on $\mathbb{R}^n$ modulo a lattice), it holds that for any point x, the density at x divided by the density at the starting point x_0 is monotonically non-decreasing in time. This would follow from a positive answer to Peres's question through a limit argument. Price gave a positive answer to his question for the case of a cycle ($n = 1$) and recently, a positive answer for arbitrary flat tori was found [8]. This can be seen as further evidence for a positive answer to Peres's question.

In this note we give a negative answer to Peres's question. In fact, we do so through a Cayley graph.

Theorem 1.1. *There exists a Cayley graph $G = (V, E)$ and two vertices $u, v \in V$ such that the function $r_{u,v}$ is not monotonically non-decreasing.*

One remaining open question is whether $r_{u,v}$ is monotonically non-decreasing for *Abelian* Cayley graphs. The positive result of [8] is a special case of that.

1.1 Some basic facts about continuous-time random walks

Given a weighted finite graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}_+$ a continuous-time random walk $X = (X_t)_{t \ge 0}$ on G is defined by its heat kernel H_t, that at time $t > 0$ is equal to

$$H_t = e^{-t \cdot L},$$

where L is the Laplacian matrix of G given by $L_{u,v} = -w(u, v)$ for $u \ne v$, and $L_{u,u} = \sum_v w(u, v)$. As a result, for a random walk X starting at a vertex u the probability that X is in v at time t is equal to $p_{u,v}(t) := H_t(u, v)$. When G is a d-regular unweighted simple graph, we think of the edges as all having weight $1/d$, in which case the Laplacian of G is given by

$$L_{u,v} = \begin{cases} -1/d & \text{if } (u, v) \in E \\ 1 & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

In this note we consider only vertex-transitive graphs, for which the sum $\sum_v w(u, v)$ is the same for all vertices u of the graph. Note that we do not insist that this sum is equal to 1, though this can be achieved by normalizing L, which corresponds to changing the speed of the random walk. For basic facts about continuous-time random walks see, e.g., [4].

If G is a weighted Cayley graph with a generating set S and a weight function $w : S \to \mathbb{R}_+$, then a continuous-time random walk $X = (X_t)_{t \ge 0}$ on G is described by mutually independent Poisson processes of rate $w(g)$ for each group generator $g \in S$, where each process indicates the times when X jumps along the corresponding edge.

2 Non-monotonicity of time spent at the origin in the hypercube graph

For an integer $d \geq 1$ denote by Q_d the d-dimensional hypercube graph. The vertices of Q_d are $\{0,1\}^d$ and there is an edge between two vertices u and v if and only if they differ in exactly one coordinate. Let $X = (X_t)_{t>0}$ be a continuous-time random walk on Q_d starting at the origin, denoted by $\mathbf{0} = (0,\dots,0) \in \{0,1\}^d$. Denote by $C_d(t)$ the expected time spent at the origin until time t, conditioned on the event that $X_t = \mathbf{0}$. That is,

$$C_d(t) = \int_0^t \Pr[X_s = \mathbf{0} | X_t = \mathbf{0}] ds.$$

In this section we show that for d sufficiently large $C_d(t)$ is not monotonically non-decreasing.

Lemma 2.1. *Let $d \in \mathbb{N}$ be sufficiently large. Then, there are some $t_1 < t_2$ such that $C_d(t_1) > C_d(t_2)$, and in particular, the function C_d is not monotonically non-decreasing in t.*

Remark 2.2. Numerically, one can see that the function C_d is not monotone for $d \geq 5$. See Figure 1. Since C_d has a closed form expression (as can be seen from the calculations below), one can probably show non-monotonicity directly for C_5 by analyzing the function, though doing so would likely be messy and not too illuminating.

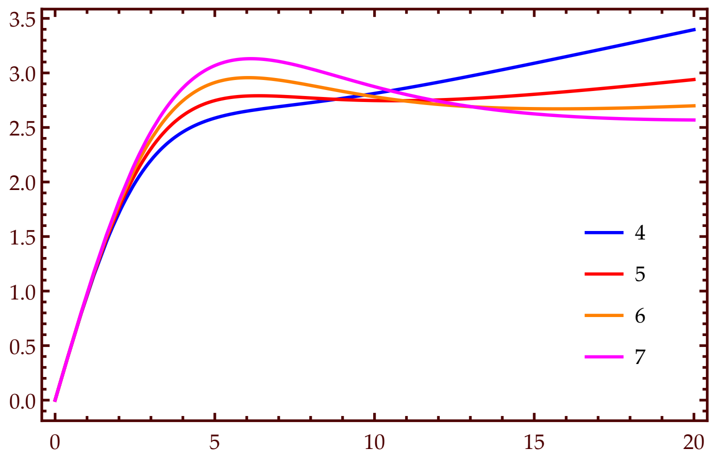

Figure 1: $C_d(t)$ for $d = 4,5,6,7$ (from top right to bottom right).

Before proving Lemma 2.1 we prove the following claim.

Claim 2.3. *Let $d \geq 1$, and let Q_d be the d-dimensional hypercube graph. Let $X = (X_t)_{t>0}$ be a continuous-time random walk on Q_d starting at $\mathbf{0}$. Then,*

$$\Pr[X_t = \mathbf{0}] = \left(\frac{1+e^{-2t/d}}{2}\right)^d.$$

Proof. Since X moves in each coordinate with rate $1/d$, it follows that for each $i \in [d]$ the number of steps in direction i up to time t is distributed like $\mathrm{Pois}(t/d)$. Therefore,

$$\Pr[(X_t)_i = 0] = \Pr[\mathrm{Pois}(t/d) \text{ is even}] = (1+e^{-2t/d})/2,$$

where we used that the probability that $\mathrm{Pois}(\lambda)$ is even is

$$\Pr[\mathrm{Pois}(\lambda) \text{ is even}] = e^{-\lambda} \cdot \sum_{j \text{ even}} \frac{\lambda^j}{j!} = e^{-\lambda} \cdot \frac{1}{2}\Big(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!}\Big) = e^{-\lambda} \cdot \frac{1}{2}\Big(e^{\lambda} + e^{-\lambda}\Big).$$

Since the coordinates of X move independently the result follows. $\square$

We now prove Lemma 2.1.

Proof of Lemma 2.1. We show below that for all $d \geq 1$, it holds that

1. $C_d(\sqrt{d}) \geq e^{-1}\sqrt{d}$,
2. $C_d(d) \leq 6$.

This clearly proves the lemma for d sufficiently large.

To prove Item 1, we show that if a walk starting from the origin is at the origin at time $\sqrt{d}$, then with constant probability it stayed at the origin throughout that time interval. Intuitively, this is because the probability of a coordinate flipping twice during that time is of order only $1/d$ and so with constant probability none of the d coordinates flips. In more detail, by Claim 2.3,

$$\Pr[X_{\sqrt{d}} = \mathbf{0}] = \Big(\frac{1+e^{-2/\sqrt{d}}}{2}\Big)^d \leq \Big(1 - \frac{1}{\sqrt{d}} + \frac{1}{d}\Big)^d = \Big(1 - \frac{\sqrt{d}-1}{d}\Big)^d \leq e^{-\sqrt{d}+1} \ ,$$

where we used the inequality $e^{-x} \leq 1 - x + x^2/2$ valid for all $x \geq 0$. On the other hand, by definition of a continuous-time random walk the probability that X stays in $\mathbf{0}$ during the entire time interval $[0, \sqrt{d}]$ is equal to $\Pr[X_{[0,\sqrt{d}]} \equiv \mathbf{0}] = e^{-\sqrt{d}}$. Therefore,

$$\Pr[X_{[0,\sqrt{d}]} \equiv \mathbf{0} | X_{\sqrt{d}} = \mathbf{0}] \geq e^{-1},$$

and hence the expected time spent at the origin conditioned on $X_{\sqrt{d}} = \mathbf{0}$ is as claimed in Item 1.

We next prove Item 2. Intuitively, here there is enough time for coordinates to flip twice, and only a very small part of the time will be spent at the origin. By definition of C_d and Claim 2.3 we have

$$\begin{aligned} C_d(t) &= \int_0^t \frac{\Pr[X_s = \mathbf{0}] \cdot \Pr[X_{t-s} = \mathbf{0}]}{\Pr[X_t = \mathbf{0}]} ds \\ &= \int_0^t (h_d(t,s))^d ds, \end{aligned}$$

where

$$h_d(t,s) = \frac{(1+e^{-2s/d})(1+e^{-2(t-s)/d})}{2(1+e^{-2t/d})} = \frac{1 + e^{-2s/d} + e^{-2(t-s)/d} + e^{-2t/d}}{2(1+e^{-2t/d})}.$$

Since $h_d(t,s)$ is convex as a function of s, for all $0 \leq s \leq t/2$ we have $h_d(t,s) \leq \ell(s)$ where ℓ is the unique linear function satisfying $\ell(0) = h_d(t,0)$ and $\ell(t/2) = h_d(t,t/2)$. Therefore, taking $t = d$, we get

$$h_d(d,s) \leq \ell(s/d) = 1 - \frac{cs}{d} \ ,$$

where $c = (1-e^{-1})^2/(1+e^{-2})$. Noting that $h_d(t,s) = h_d(t,t-s)$, we get

$$C_d(d) = \int_0^d (h_d(d,s))^d ds = 2\int_0^{d/2} (h_d(d,s))^d ds < 2\int_0^{d/2} \big(1 - \frac{cs}{d}\big)^d ds < 2\int_0^{d/2} e^{-cs} ds \leq \frac{2}{c} \ .$$

This completes the proof of Lemma 2.1. $\square$

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first give a proof for a weighted graph, and then remark on how to convert it into an unweighted graph. For $d \in \mathbb{N}$ sufficiently large we define the weighted graph G to be the *lamplighter graph* on Q_d, whose edges corresponding to steps on Q_d are of weight $1/d$, and edges corresponding to toggling a lamp are of weight ε, for some $\varepsilon > 0$ sufficiently small that depends on d and t_1, t_2 from Lemma 2.1.

In more detail, the weighted lamplighter graph G is described by placing a lamp at each vertex of Q_d and a lamplighter walking on Q_d. A vertex of G is described by the location $x \in \{0,1\}^d$ of the lamplighter, and a configuration $f : \{0,1\}^d \to \{0,1\}$ indicating which lamps are currently on. In each step the lamplighter either makes a step in the graph Q_d or toggles the state of the lamp in the current vertex. More formally, we have an edge between (x, f) and (y, g) if and only if either

1. $(x, y) \in E_d$ and $f = g$ (this corresponds to a step in Q_d) or
2. $x = y$ and f and g differ on the input x and are equal on all other inputs (this corresponds to toggling a lamp at x).

The edges of the first type are of weight $1/d$, and those of the second type are of weight ε. Thus, in a random walk on G, the steps of the lamplighter are distributed as in a random walk on Q_d, and the number of times the lamps are toggled in a time interval of length T is distributed like $\mathrm{Pois}(\varepsilon T)$ independently of the lamplighter's walk. Lamplighter graphs are well-studied objects (see, e.g., [6, 3, 2]), and are well known to be Cayley graphs.

Let u be the vertex in G corresponding to the lamplighter being at the origin with all lights off. Let v be the vertex in G corresponding to the lamplighter being at the origin with the light at the origin being on, and all other lights off. We show below that $r_{u,v}$ is not monotonically non-decreasing. More specifically, we show that $r_{u,v}(t_1) > r_{u,v}(t_2)$, where $t_1 < t_2$ are from Lemma 2.1.

Let $X = (X_t)_{t \geq 0}$ be a continuous-time random walk on G starting at $X_0 = u$. Denote by Y_t the number of times a toggle occurred during the time interval $[0, t]$. Denote by $Z = (Z_t)_{t \geq 0}$ the trajectory of the lamplighter, i.e., the projection of X to the first coordinate. Note that by definition Z is a continuous-time random walk on Q_d, and that Z is independent of Y_t.

Claim 3.1. *Let $u, v \in V$ be as above. Then, for all $t > 0$ it holds that*

$$0 \leq p_{u,u}(t) - e^{-\varepsilon t} \cdot \Pr[Z_t = \mathbf{0}] \leq \varepsilon^2 t^2 \ , \tag{3.1}$$

and

$$0 \leq p_{u,v}(t) - \varepsilon e^{-\varepsilon t} \cdot C_d(t) \cdot \Pr[Z_t = \mathbf{0}] \leq \varepsilon^2 t^2 \ . \tag{3.2}$$

Using the claim,

$$r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} = \varepsilon \cdot C_d(t) \pm O(\varepsilon^2),$$

where $O(\cdot)$ hides a constant that depends on d and t. In particular, for $t_1 < t_2$ from Lemma 2.1, and $\varepsilon > 0$ sufficiently small we get that $r_{u,v}(t_1) > r_{u,v}(t_2)$, which proves Theorem 1.1.

Intuitively, (3.1) holds because the probability of toggling a lamp twice is very small, and hence $p_{u,u}(t)$ is approximately equal to the probability that no lamp has changed its state multiplied by the probability that a random walk on Q_d will be at the origin at time t. The intuition for (3.2) is that in order to get from u to v, in addition to getting back to the origin, the lamplighter must toggle the switch while being at the origin, and the probability of that is roughly $\varepsilon \cdot C_d(t)$.

Proof of Claim 3.1. For $p_{u,u}$ we have

$$p_{u,u} = \Pr[X_t = u \wedge Y_t = 0] + \Pr[X_t = u \wedge Y_t \geq 2] \ .$$

Since Y_t is distributed like $\mathrm{Pois}(\varepsilon t)$, the second term satisfies

$$0 \leq \Pr[X_t = u \wedge Y_t \geq 2] \leq \Pr[Y_t \geq 2] \leq \varepsilon^2 t^2 \ ,$$

and for the first term, by independence between Y_t and Z_t we have

$$\Pr[X_t = u \wedge Y_t = 0] = \Pr[Z_t = \mathbf{0} \wedge Y_t = 0] = e^{-\varepsilon t} \cdot \Pr[Z_t = \mathbf{0}] \ ,$$

proving (3.1).

For $p_{u,v}$ we similarly have

$$p_{u,v}(t) = \Pr[X_t = v \wedge Y_t = 1] + \Pr[X_t = v \wedge Y_t \geq 2].$$

As above, the second term is at most $\varepsilon^2 t^2$. For the first term, let E_t be the event that $Y_t = 1$, and the unique lamp that is on at time t is the lamp at the origin. Denote by T_0 the time spent by Z at the origin in the time interval $[0, t]$. Then, conditioning on Z, the event E_t holds if and only if a unique switch happened during T_0 time, and zero switches in the remaining time. Therefore, by independence of a Poisson process in disjoint intervals

$$\Pr[E_t | Z] = \Pr[\mathrm{Pois}(\varepsilon T_0) = 1 | Z] \cdot \Pr[\mathrm{Pois}(\varepsilon(t - T_0)) = 0 | Z] = \varepsilon T_0 \cdot e^{-\varepsilon T_0} \cdot e^{-\varepsilon(t-T_0)} = \varepsilon e^{-\varepsilon t} \cdot T_0.$$

This implies that

$$\Pr[X_t = v \wedge Y_t = 1] = \Pr[E_t | Z_t = \mathbf{0}] \cdot \Pr[Z_t = \mathbf{0}] = \varepsilon e^{-\varepsilon t} \cdot \mathbb{E}[T_0 | Z_t = \mathbf{0}] \cdot \Pr[Z_t = \mathbf{0}].$$

Therefore, since $C_d(t) = \mathbb{E}[T_0 | Z_t = \mathbf{0}]$ we get (3.2), and the claim follows. □

Converting G into an unweighted graph Below we show how to convert a weighted Cayley graph G into an unweighted one, while preserving the property in Theorem 1.1. Let (G, S_G) be a weighted Cayley graph with the generating set $S_G = \{g_1, \dots, g_k\}$, and suppose that all the weights $w(g_1), \dots, w(g_k)$ are integers. For $N \in \mathbb{N}$ sufficiently large define the graph H by replacing each vertex $v \in G$ with a "cloud" of N vertices $\{(v, i) : i \in \mathbb{Z}_N\}$, adding edges between every pair of vertices in the cloud (i.e., replacing each vertex of G with an N-clique), and replacing each edge (u, ug) in G of weight $w(g)$ with $w(g)$ perfect matchings $\{(u, i), (ug, i + j) : i \in \mathbb{Z}_N\}_{j=1}^{w(g)}$. Formally, the graph H is a Cayley graph, whose vertices are $G \times \mathbb{Z}_N = \{(v, i) : v \in G, i \in \mathbb{Z}_N\}$, and the set of generators S_H given by

$$S_H = \{(0, i) : i \in \mathbb{Z}_N \setminus \{0\}\} \bigcup \cup_{g \in S_G} \{(g, j) : j \in \{1, \dots, w(g)\}\}.$$

Note that the projection of a continuous-time random walk on H to the first coordinate is a random walk on G, slowed down by $\deg(H)$. (The walk on the weighted graph G makes $\sum_g w(g)$ steps per unit time on average, whereas by our convention, the walk on the *unweighted* graph H makes one step per unit time on average, a $\sum_g w(g)/\deg(H)$ proportion of which is between the clouds.) Moreover, assuming N is sufficiently large, after constant time the two coordinates become close to independent with the second coordinate being uniform. Therefore, if u, v are vertices in G, and $x = (u, 0), y = (v, 0)$ are the corresponding vertices in H, then for any time $t > 0$ and $t' = \deg(H) \cdot t$ it holds that $p_{x,y}(t') = \frac{1}{N}(p_{u,v}(t) \pm o_N(1))$ and hence $r_{x,y}(t') = r_{u,v}(t') \pm o_N(1)$.

For the graph G given in the proof of Theorem 1.1 above, we may assume that $1/\varepsilon$ is an integer, and so, by multiplying all weights by d/ε we get a Cayley graph with integer weights. Hence, by applying the foregoing transformation we get a simple unweighted Cayley graph H for which $r_{u,v}$ is not monotonically non-decreasing for some $u, v \in H$.

A Appendix: a counterexample in a regular non-transitive graph

Below we give a simple example of a regular non-transitive graph such that $r_{u,v}(t) > 1$ for some vertices u, v and some time t; in particular, $r_{u,v}(t)$ is not monotonically non-decreasing, since $r_{u,v}(t) \to 1$ as $t \to \infty$. We thank Jeff Cheeger [1] for pointing this out to us.

Proposition A.1. *Let L be the Laplacian of a regular graph on vertex set V. Denote its eigenvalues by $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{|V|}$ and by $f_i \in \mathbb{R}^V$ the corresponding normalized eigenvectors. Suppose that $0 < \lambda_2 < \lambda_3$, and that f_2 is such that $f_2(v) > f_2(u) > 0$ for some vertices u, v. Then, there is some $t > 0$ such that $r_{u,v}(t) > 1$.*

Proof. Let $\pi_u \in \mathbb{R}^V$ be the vector with $\pi_u(u) = 1$ and $\pi_u(u') = 0$ for all $u' \neq u$. Writing $\pi_u = \sum \alpha_i f_i$ for $\alpha_i = \langle \pi_u, f_i \rangle = f_i(u)$, for all $w \in V$ we have

$$e^{-tL}\pi_u(w) = \sum_{i=1}^{|V|} e^{-t\lambda_i}\alpha_i \cdot f_i(w) = c + e^{-\lambda_2 t} f_2(u) f_2(w) + O(e^{-\lambda_3 t}),$$

where $O()$ hides some constants that may depend on the graph, but not on t, and $c = \alpha_1 \cdot f_1(w)$ is independent of w since f_1 is a constant function. Using the facts that $f_2(v) > f_2(u) > 0$ and $\lambda_3 > \lambda_2$, it follows that for sufficiently large t,

$$r_{u,v}(t) = \frac{e^{-tL}\pi_u(v)}{e^{-tL}\pi_u(u)} > 1 \ ,$$

as desired. □

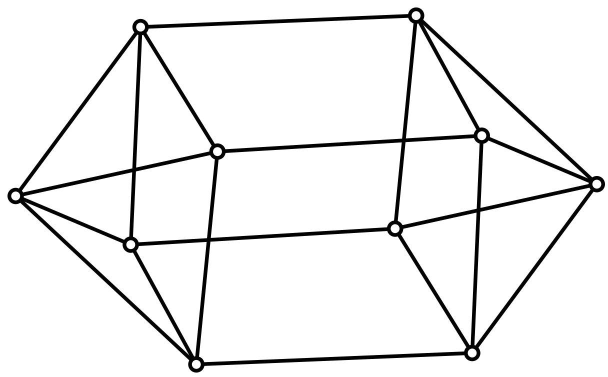

Figure 2: A cube with two square pyramids attached.

Graphs satisfying the constraints in Proposition A.1 are in abundance. As a concrete example, consider the 4-regular graph on 10 vertices shown in Figure 2. Using Mathematica, we see that the second eigenvalue of the Laplacian of this graph is $\lambda_2 = \frac{1}{8}(7-\sqrt{17}) \approx 0.36$, and it is a simple eigenvalue. The corresponding (non-normalized) eigenvector with vertices ordered from left to right is $(c, 1, 1, 1, 1, -1, -1, -1, -1, -c)$, where $c = 3 - \frac{1}{2}(7 - \sqrt{17}) \approx 1.56$. In particular, Proposition A.1 is applicable to this graph.

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