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A generalized Pólya's urn with graph based interactions: convergence at linearity

Jun Chen* Cyrille Lucas†

Abstract

We consider a special case of the generalized Pólya's urn model introduced in [3]. Given a finite connected graph G, place a bin at each vertex. Two bins are called a pair if they share an edge of G. At discrete times, a ball is added to each pair of bins. In a pair of bins, one of the bins gets the ball with probability proportional to its current number of balls. A question of essential interest for the model is to understand the limiting behavior of the proportion of balls in the bins for different graphs G. In this paper, we present two results regarding this question. If G is not balanced-bipartite, we prove that the proportion of balls converges to some deterministic point $v = v(G)$ almost surely. If G is regular bipartite, we prove that the proportion of balls converges to a point in some explicit interval almost surely. The question of convergence remains open in the case when G is non-regular balanced-bipartite.

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1 Introduction and statement of results

As a special case of the generalized Pólya's urn model introduced in [3], the model with linear reinforcement is defined as follows. Let $G = (V, E)$ be a finite connected graph with $V = [m] = \{1, \dots, m\}$ and $|E| = N$, and assume that on each vertex i there is a bin initially with $B_i(0) \geq 1$ balls. Consider the random process of adding N balls to these bins at each step, according to the following law: if the numbers of balls after step $n-1$ are $B_1(n-1), \dots, B_m(n-1)$, step n consists of adding, for each edge $\{i, j\} \in E$, one ball either to i or to j, and the probability that the ball is added to i is

$$\mathbb{P}\left[i \text{ is chosen among } \{i,j\} \text{ at step } n\right] = \frac{B_i(n-1)}{B_i(n-1) + B_j(n-1)} \cdot \tag{1.1}$$

Let $N_0 = \sum_{i=1}^{m} B_i(0)$ denote the initial total number of balls, and let

$$x_i(n) = \frac{B_i(n)}{N_0 + nN}\,, \quad i \in [m], \tag{1.2}$$

be the proportion of balls at vertex i after step n. Let $x(n) = (x_1(n), \dots, x_m(n))$. We are interested in the limiting behavior of $x(n)$ for different graphs G.

*California Institute of Technology, USA. E-mail: chenjun851009@gmail.com
†Weizmann Institute of Science, Israel and Université Paris Diderot, France.
E-mail: lucas@math.univ-paris-diderot.fr

We call G *balanced-bipartite* if there is a bipartition $V = A \cup B$ with $\#A = \#B$. In [3], the authors proved that when G is not balanced-bipartite, the limit of $x(n)$ exists, and it can only take finitely many possible values. Here, we improve this result and prove that almost surely the limit of $x(n)$ is in fact one deterministic point, thus confirming the conjecture in Section 11 of [3].

Theorem 1.1. *Let G be a finite, connected, not balanced-bipartite graph. Then there exists a deterministic point $v = v(G)$ such that $x(n)$ converges to v almost surely.*

In the proof of this theorem, we will give a characterization of $v(G)$, that enables us to explicitly compute its value for some graphs, like regular nonbipartite graphs, star graphs, and other small graphs. Hence, Theorem 1.1 will imply Theorems 1.1(a) and 1.5(a) of [3].

When G is regular bipartite, the authors in [3] proved that the limit set of $x(n)$ is contained in Ω where Ω is the subset of the $(m-1)$-dimensional closed simplex defined as follows: if $V = A \cup B$ is the bipartition of G, then

$$\Omega = \{(x_1, \ldots, x_m) : \exists p, q \geq 0, p + q = 2/m, \text{ s.t. } x_i = p \text{ on } A, x_i = q \text{ on } B\}. \tag{1.3}$$

Nevertheless, the question whether $x(n)$ has a limit was left open (see Problem 11.2 in [3]). The following theorem provides the answer to this question.

Theorem 1.2. *Let G be a finite, regular and bipartite graph. Then $x(n)$ almost surely converges to a point in Ω.*

The question of the distribution of this random limit in Ω is left open.

The main technique used in [3] is the dynamical system approach (see e.g. [1, 2]), by which one can analyze the limiting behavior of $x(n)$ via an approximating ordinary differential equation (ODE). Under some conditions on $x(n)$ and on the ODE, it was shown that the limit set of $x(n)$ is contained in the *equilibria set* of the ODE. Depending on G, the equilibria set can be either finite or infinite. By a probabilistic argument, the authors in [3] also proved that $x(n)$ has probability zero to converge to an unstable equilibrium (see Definition 2.5).

Our results and proofs in this paper are continuation of those in [3]. To prove Theorem 1.1, the main work is to prove the uniqueness of a non-unstable equilibrium for any not balanced-bipartite G. The difficulty is that for a general graph, there is no explicit formula for the equilibria and hence it is impossible to calculate eigenvalues of the jacobian matrix at equilibria. We overcome this difficulty by constructing a Lyapunov function. To prove Theorem 1.2, one main difficulty is that the limit set Ω attracts exponentially in the interior, but not at its two endpoints. Thus one cannot directly apply the theorem proved in [4] for dealing with the case where there is a uniform exponential attractor. Then our strategy to prove the convergence in Theorem 1.2 is to treat the convergence to the two endpoints of Ω and to its interior separately. More precisely, we will prove that the random process (*interpolated process*) has to converge to some point in the interior of Ω if it does not converge to the endpoints of Ω. The proof uses ideas similar to shadowing techniques [4, 5, 9]. One main reason for our technique to work is due to the special structure of Ω, which is a segment of equilibria that loses exponential attraction only at its two endpoints. Naturally, our technique can be applied to a setting where a segment of equilibria attracts exponentially everywhere but not at finitely many points.

The organization of this paper is as follows. In Section 2, we do some preparation work for the later proofs: we describe the dynamical system approach in our setting and cite the necessary results from [3]. In Sections 3 and 4, we prove Theorem 1.1 and 1.2 respectively. In Section 5, we discuss the model on non-regular balanced-bipartite graphs.

2 Some results from [3]

We will first describe the evolution of the model in a way that highlights the underlying deterministic ODE. Let $\mathcal{F}_n = \sigma(x(i) : 0 \le i \le n)$ be the filtration generated by $x(i)$ up to step n. Then we have the following lemma, which was proved in Sections 2 and 3 of [3].

Lemma 2.1. *The evolution of $\{x(n)\}_{n\ge 0}$ follows a recursive equation of the form*

$$x(n+1) - x(n) = \gamma_n \left[F(x(n)) + u(n)\right], \tag{2.1}$$

where $F : \mathbb{R}^m \to \mathbb{R}^m$ is a deterministic map, $u(n)$ is a random sequence of vectors with zero conditional mean ($\mathbb{E}(u(n)|\mathcal{F}_n) = 0$) and γ_n is a normalizing factor with $\gamma_n = O(1/n)$.

Proof. Recall that $x_i(n)$ is the fraction of the total number of balls contained in the i-th bin at time n:

$$x_i(n) = \frac{B_i(n)}{N_0 + nN}, \quad i \in [m]. \tag{2.2}$$

Let $\delta_{i\leftarrow j}(n+1)$ be the indicator of the event that the new ball added on the edge $\{i, j\}$ at step $n+1$ is added to the i-th bin. By the definition of the process, we have

$$\mathbb{E}\left[\delta_{i\leftarrow j}(n+1)|\mathcal{F}_n\right] = \frac{x_i(n)}{x_i(n) + x_j(n)}\,. \tag{2.3}$$

Now observe that

$$\begin{aligned}
x_i(n+1) - x_i(n) &= \frac{B_i(n) + \sum_{j\sim i}\delta_{i\leftarrow j}(n+1)}{N_0 + (n+1)N} - \frac{B_i(n)}{N_0 + nN} \\
&= \frac{-Nx_i(n) + \sum_{j\sim i}\delta_{i\leftarrow j}(n+1)}{N_0 + (n+1)N} \\
&= \frac{1}{N_0/N + (n+1)}\left(-x_i(n) + \frac{1}{N}\sum_{j\sim i}\frac{x_i(n)}{x_i(n) + x_j(n)}\right) \\
&\quad + \frac{1}{N_0/N + (n+1)}\cdot\frac{1}{N}\sum_{j\sim i}\left(\delta_{i\leftarrow j}(n+1) - \frac{x_i(n)}{x_i(n) + x_j(n)}\right).
\end{aligned}$$

Let

$$\gamma_n = \frac{1}{\frac{N_0}{N} + (n+1)}. \tag{2.4}$$

Thus, defining the sequence of random vectors $u(n) = (u_i(n))_{i\in[m]} \subset \mathbb{R}^m$ by

$$u_i(n) = \frac{1}{N}\sum_{j\sim i}\left(\delta_{i\leftarrow j}(n+1) - \frac{x_i(n)}{x_i(n) + x_j(n)}\right) \tag{2.5}$$

and $F = (F_1, \ldots, F_m)$ by

$$F_i(x_1, \ldots, x_m) = -x_i + \frac{1}{N}\sum_{j\sim i}\frac{x_i}{x_i + x_j}\,, \tag{2.6}$$

our random process takes the form

$$x(n+1) - x(n) = \gamma_n \left[F(x(n)) + u(n)\right], \tag{2.7}$$

where $\mathbb{E}(u(n)|\mathcal{F}_n) = 0$, which concludes the proof of Lemma 2.1. □

Following a limit set theorem (see e.g. [1]), we can analyze the limiting behavior of $x(n)$ by considering its underlying ODE $dv/dt = F(v)$, $v \in \mathbb{R}^m$:

$$\begin{cases} \dfrac{dv_1(t)}{dt} & = & -v_1(t) + \dfrac{1}{N} \displaystyle\sum_{j\sim 1} \dfrac{v_1(t)}{v_1(t) + v_j(t)} \\ & \vdots & \\ \dfrac{dv_m(t)}{dt} & = & -v_m(t) + \dfrac{1}{N} \displaystyle\sum_{j\sim m} \dfrac{v_m(t)}{v_m(t) + v_j(t)}. \end{cases} \tag{2.8}$$

Let us specify the domain of the vector field F. Fix $c < 1/N$, and let Δ be the set of m-tuples $(x_1, \ldots, x_m) \in \mathbb{R}^m$ such that:

(1) $x_i \geq 0$ and $\sum_{i=1}^m x_i = 1$, and

(2) $x_i + x_j \geq c$ for all $\{i, j\} \in E$.

We equip Δ with the distance d induced by the L^1 norm in $\mathbb{R}^m$. Note that Δ is positively invariant (see Lemma 3.4 in [3] for a detailed proof), and that the restriction of F to Δ is Lipschitz. A point $x \in \Delta$ is called an *equilibrium* if $F(x) = 0$. Let Λ be the equilibria set of F in Δ. The following result gives the relation between the limit set of $x(n)$ and Λ.

Proposition 2.2. *[3, Theorem 3.3] The limit set of $\{x(n)\}_{n\geq 0}$ is a connected subset of Λ almost surely.*

For the sake of completeness, we sketch the proof of this proposition. It requires the construction of a Lyapunov function. Let $U \subset \mathbb{R}^m$ be a closed set and $F : U \to \mathbb{R}^m$ be a continuous vector field with unique integral curves.

Definition 2.3 (Lyapunov function)**.** *A (strict) Lyapunov function for $W \subset U$ is a continuous map $L : U \to \mathbb{R}$ which is (strictly) monotone along any integral curve of F in $U \setminus W$.*

Proof of Proposition 2.2. We refer the reader to Section 3 of [3] for a detailed proof. We will use the limit set theorem stated therein, which requires the following conditions:

(i) for any $T > 0$,

$$\lim_{n\to\infty} \left(\sup_{\{k: 0\leq \tau_k - \tau_n \leq T\}} \left\| \sum_{i=n}^{k-1} \gamma_i u(i) \right\| \right) = 0 \text{ a.s.}$$

where $\tau_n = \sum_{i=0}^{n-1} \gamma_i$, and

(ii) F admits a strict Lyapunov function L for Λ.

We remark that (i) controls the noise perturbation between the random process $x(n)$ and its associated ODE, and (ii) guarantees the convergence of the ODE to its equilibria.

For (i), let $M_n = \sum_{i=0}^n \gamma_i u(i)$. $\{M_n\}_{n\geq 0}$ is a martingale with bounded quadratic variation, hence it converges almost surely to a finite random vector (see e.g. Theorem 5.4.9 of [6]). In particular, it is a Cauchy sequence and so (i) holds almost surely.

For (ii), let $L : \Delta \to \mathbb{R}$ be given by

$$L(v_1, \ldots, v_m) = -\sum_{i=1}^m v_i + \frac{1}{N} \sum_{\{i,j\}\in E} \log(v_i + v_j). \tag{2.9}$$

Thus

$$\frac{dv_i}{dt} = v_i \left(-1 + \frac{1}{N} \sum_{i \sim j} \frac{1}{v_i + v_j} \right) = v_i \frac{\partial L}{\partial v_i}. \tag{2.10}$$

If $v = (v_1(t), \ldots, v_m(t))$, $t \geq 0$, is an integral curve of F, then (2.10) implies

$$\frac{d}{dt}(L \circ v) = \sum_{i=1}^{m} \frac{\partial L}{\partial v_i} \frac{dv_i}{dt} = \sum_{i=1}^{m} v_i \left(\frac{\partial L}{\partial v_i} \right)^2 \geq 0.$$

In particular, the last expression is zero if and only if $v_i \left(\frac{\partial L}{\partial v_i} \right)^2 = 0$ for all $i \in [m]$, which is equivalent to $F(v) = 0$. Hence, L is a strict Lyapunov function for Λ.

The rest of the proof is a straightforward application of the limit set theorem. □

Define a face Δ_S of Δ as its subset such that $v_i = 0$ if and only if $i \notin S \subset [m]$. Let $L|_{\Delta_S}$ be the restriction of L to Δ_S. Since an equilibrium v satisfies $v_i(\partial L / \partial v_i) = 0$ for any $i \in [m]$, we can decompose the equilibria set Λ into the union of the sets of critical points of $L|_{\Delta_S}$ over all faces Δ_S.

When G is not balanced-bipartite, L is strictly concave (see Corollary 1.3 in [3]). So for any face Δ_S, $L|_{\Delta_S}$ is strictly concave, and hence has at most one critical point. Therefore, Λ is finite. Then it immediately follows from Proposition 2.2 that the limit of $x(n)$ exists in this case. We have the corollary below.

Corollary 2.4. *[3, Corollary 1.3] Let G be a finite, connected, not balanced-bipartite graph. Then Λ is finite and $x(n)$ converges to an element of Λ almost surely.*

After proving that the limit set of $x(n)$ is contained in Λ in Proposition 2.2, we want to understand which equilibrium $x(n)$ can actually converge to. First we give the following definition.

Definition 2.5 (Unstable/non-unstable equilibrium)**.** *An equilibrium x is called unstable if at least one of the eigenvalues of $JF(x)$, the jacobian matrix of F at x, has positive real part. Otherwise, we call it non-unstable.*

The following lemma rules out the possibility that $x(n)$ converges to an unstable equilibrium.

Lemma 2.6. *Let G be a finite and connected graph. Let v be an unstable equilibrium. Then*

$$\mathbb{P}\left[\lim_{n \to \infty} x(n) = v \right] = 0. \tag{2.11}$$

The proof of Lemma 2.6 follows from Lemma 5.2 in [3] and the characterization of an unstable equilibrium as shown in the following lemma.

Lemma 2.7. *An equilibrium v is unstable if and only if there exists some coordinate $i \in [m]$ with $v_i = 0$ and $\partial L / \partial v_i > 0$.*

Lemma 2.7 was proved in Section 5 of [3]. For the sake of completeness, we give its proof here.

Proof. We look at the jacobian matrix $JF(v)$:

$$\frac{\partial F_i}{\partial v_j} = \begin{cases} v_i \dfrac{\partial^2 L}{\partial v_i \partial v_j} & \text{if } i \sim j, \\ \dfrac{\partial L}{\partial v_i} + v_i \dfrac{\partial^2 L}{\partial v_i^2} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, assume that $v_i = 0$ iff $1 \le i \le k$ (k can be zero). Thus

$$JF(v) = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{2.12}$$

where A is a $k \times k$ diagonal matrix with $a_{ii} = \partial L / \partial v_i$, $i \in [k]$. The spectrum of $JF(v)$ is the union of the spectra of A and B. With respect to the inner product $(x, y) = \sum_{i=k+1}^m x_i y_i / v_i$, B is self-adjoint and negative semidefinite (by the concavity of L), hence the eigenvalues of B are real and nonpositive. Therefore, $JF(v)$ has at least one real positive eigenvalue if and only if at least one of the a_{ii} is positive. □

Let $w = (w_1, \ldots, w_m)$ be a non-unstable equilibrium. Let $P = \{i \in [m] : w_i > 0\}$ and $Z = \{i \in [m] : w_i = 0\} = [m] \setminus P$ denote the coordinates of w with strictly positive and zero values respectively. Notice that Z can be empty. By the definition of equilibrium and (2.10), if $i \in P$ then $\partial L / \partial v_i |_w = 0$. By Lemma 2.7, if $i \in Z$ then $\partial L / \partial v_i |_w \le 0$. Hence, w is non-unstable if and only if it satisfies

$$\left. \frac{\partial L}{\partial v_i} \right|_w \le 0, \ \forall i \in Z; \ \left. \frac{\partial L}{\partial v_i} \right|_w = 0, \ \forall i \in P. \tag{2.13}$$

It can also be seen from these conditions that only boundary equilibria can be unstable.

3 Not balanced-bipartite graphs: Proof of Theorem 1.1

By Corollary 2.4, if G is not balanced-bipartite, then the limit of $x(n)$ exists almost surely and is contained in Λ, i.e.

$$\sum_{v \in \Lambda} \mathbb{P} \left[\lim_{n \to \infty} x(n) = v \right] = 1. \tag{3.1}$$

By Lemma 2.6, the probability that $x(n)$ converges to an unstable equilibrium of F is zero. Then there exists at least one non-unstable equilibrium. To prove Theorem 1.1, it suffices to prove its uniqueness, which is given by the following lemma.

Lemma 3.1. *Let G be a finite, connected, not balanced-bipartite graph. Then all but exactly one equilibrium are unstable.*

Proof. The proof uses properties of the vector field F only (the random sequence $\{x(n)\}_{n \ge 0}$ plays no role). Let $w = (w_1, \ldots, w_m)$ be a non-unstable equilibrium. We claim that for any $v^0 \in \text{int}(\Delta)$, the orbit $\{v(t)\}_{t \ge 0}$ with $v(0) = v^0$ converges to w. Clearly, this implies the uniqueness of w.

Recall the definition of P in the end of the previous section. We prove the claim by constructing a Lyapunov function $H : \{v \in \Delta : v_i > 0, \ \forall i \in P\} \to \mathbb{R}$,

$$H(v) = \sum_{i \in P} w_i \log v_i. \tag{3.2}$$

It is worth noting that $\operatorname{int}(\Delta) \subset \{v \in \Delta : v_i > 0, \forall i \in P\}$ and that $H(v) \le 0$ in Δ. Set $c^0 = \sum_{i\in P} w_i \log v_i(0)$, and consider $H^{-1}[c^0, 0] = \{v \in \Delta : H(v) \ge c^0\}$. Observe that there exists some small $\widetilde{c} > 0$ such that

$$H^{-1}[c^0, 0] \subset \{u \in \Delta : u_i \ge \widetilde{c}, i \in P\}.$$

Hence if the orbit $v(t)$ is located in the set $H^{-1}[c^0, 0]$, it is legitimate to take the derivative of $H(v)$ along it:

$$\begin{aligned} \frac{dH(v(t))}{dt} &= \frac{d}{dt}\left(\sum_{i\in P} w_i \log v_i\right) \\ &= \sum_{i\in P} w_i \frac{1}{v_i}\left(-v_i + \frac{1}{N}\sum_{j\sim i}\frac{v_i}{v_i+v_j}\right) \\ &= \sum_{i\in P} w_i \left(-1 + \frac{1}{N}\sum_{j\sim i}\frac{1}{v_i+v_j}\right). \end{aligned}$$

Since $w_i = 0$ for $i \in Z$, it follows from above that

$$\begin{aligned} \frac{dH(v(t))}{dt} &= \sum_{i=1}^{m} w_i \left(-1 + \frac{1}{N}\sum_{j\sim i}\frac{1}{v_i+v_j}\right) \\ &= -1 + \frac{1}{N}\sum_{i=1}^{m} w_i \left(\sum_{j\sim i}\frac{1}{v_i+v_j}\right) \\ &= -1 + \frac{1}{N}\sum_{\{i,j\}\in E}\frac{w_i+w_j}{v_i+v_j}. \end{aligned} \tag{3.3}$$

By Lemma 3.2 below, (3.3) is non-negative with equality if and only if $v = w$. This immediately implies that $H^{-1}[c^0, 0]$ is positively invariant, and that $v(t)$ converges to w. This completes the proof of the claim. □

In the proof of Lemma 3.1, we made use of the following lemma.

Lemma 3.2. *Let G be a finite, connected, not balanced-bipartite graph. If w is a non-unstable equilibrium, then*

$$f(v_1, \dots, v_m) = \sum_{\{i,j\}\in E}\frac{w_i+w_j}{v_i+v_j} \ge N, \ \forall (v_1, \dots, v_m) \in \Delta,$$

with equality if and only if $v = w$.

Proof. The proof of Lemma 3.2 follows from the following two claims:

(a) w is a strict local minimum of $f(\cdot)$ in Δ;

(b) $f(\cdot)$ is strictly convex in Δ.

Let's prove (a).

Let $\epsilon = (\epsilon_1, \dots, \epsilon_m)$. Observe that we can write any point in a neighborhood of w as $w^\epsilon = (w_1 + \epsilon_1, \dots, w_m + \epsilon_m)$ with $\sum_{i=1}^m \epsilon_i = 0$. By the following elementary inequality

$$\frac{x}{x+\epsilon} - 1 \ge -\frac{\epsilon}{x}, \ \forall x > 0, \, \epsilon > -x,$$

we have

$$\begin{aligned} f(w^\epsilon) - f(w) &= \sum_{\{i,j\}\in E} \left[\frac{w_i + w_j}{w_i + \epsilon_i + w_j + \epsilon_j} - 1 \right] \\ &\geq -\sum_{\{i,j\}\in E} \frac{\epsilon_i + \epsilon_j}{w_i + w_j} \\ &= -\sum_{i=1}^{m} \epsilon_i \sum_{j\sim i} \frac{1}{w_i + w_j}. \end{aligned} \tag{3.4}$$

Since $w^\epsilon \in \Delta$, we must have $\epsilon_i \geq 0$ for any $i \in Z$. By (2.13),

$$\begin{aligned} \sum_{i=1}^{m} \epsilon_i \sum_{j\sim i} \frac{1}{w_i + w_j} &= \sum_{i\in P} \epsilon_i \sum_{j\sim i} \frac{1}{w_i + w_j} + \sum_{i\in Z} \epsilon_i \sum_{j\sim i} \frac{1}{w_j} \\ &= \sum_{i\in P} \epsilon_i \cdot N + \sum_{i\in Z} \epsilon_i \sum_{j\sim i} \frac{1}{w_j} \\ &\leq \sum_{i\in P} \epsilon_i \cdot N + \sum_{i\in Z} \epsilon_i \cdot N \\ &= N \sum_{i=1}^{m} \epsilon_i = 0. \end{aligned}$$

Then it follows that $f(w^\epsilon) - f(w) \geq 0$.

Notice that (3.4) has equality if and only if

$$\epsilon_i + \epsilon_j = 0, \quad \forall \{i,j\} \in E. \tag{3.5}$$

If G is a non-bipartite graph, it has an odd cycle, then (3.5) implies $\epsilon = 0$. If G is bipartite but not balanced-bipartite, (3.5) together with $\sum_{i=1}^m \epsilon_i = 0$ implies $\epsilon = 0$. In both cases, i.e. if G is not balanced-bipartite, then $f(w^\epsilon) - f(w) > 0$ for all small $\epsilon \neq 0$. This completes the proof of (a).

Now we will prove (b).

For any $u, v \in \Delta$ and $0 < t < 1$, by the convexity of the function $\frac{1}{x}$ $(x > 0)$,

$$\begin{aligned} f(tu + (1-t)v) &= \sum_{\{i,j\}\in E} \left[\frac{w_i + w_j}{(tu_i + (1-t)v_i) + (tu_j + (1-t)v_j)} \right] \\ &\leq \sum_{\{i,j\}\in E} \left[t\frac{w_i + w_j}{u_i + u_j} + (1-t)\frac{w_i + w_j}{v_i + v_j} \right] \\ &\leq tf(u) + (1-t)f(v). \end{aligned} \tag{3.6}$$

Notice that (3.6) has equality if and only if

$$u_i + u_j = v_i + v_j, \quad \forall \{i,j\} \in E. \tag{3.7}$$

Set $g(i) = u_i - v_i$. Then (3.7) implies

$$g(i) + g(j) = (u_i - v_i) + (u_j - v_j) = 0, \quad \forall \{i,j\} \in E. \tag{3.8}$$

By a similar argument as before, if G is not balanced-bipartite then

$$f(tu + (1-t)v) - (tf(u) + (1-t)f(v)) < 0, \ \forall\, u \neq v.$$

This implies that $f(\cdot)$ is strictly convex. We completes the proof of (b). □

By (3.1) and Lemma 3.1, $x(n)$ then converges almost surely to a *unique* non-unstable equilibrium w. Hence, Theorem 1.1 holds with $v(G) = w$, which is characterized by the conditions in (2.13).

4 Regular bipartite graphs: Proof of Theorem 1.2

Let $x(t)$ denote the interpolated process of $x(n)$:

$$x(t) = \sum_{n\geq 0} \left(x(n) + \frac{t-\tau_n}{\gamma_n}(x(n+1)-x(n)) \right) 1_{[\tau_n,\tau_{n+1})}(t),$$

where $\tau_n = \sum_{k=0}^{n-1} \gamma_k$. To prove the convergence of $x(n)$ in Theorem 1.2, it suffices to prove the convergence of $x(t)$.

Let $\Phi = \Phi_t(x)$ be the semiflow induced by (2.6) where $t \geq 0$ is the time parameter and $\Phi_0(x) = x$. Then the following lemma gives a quantitative estimate on how well the interpolated process can be approximated by the semiflow Φ.

Lemma 4.1. *[2, Proposition 8.3] Almost surely,*

$$\sup_{T>0} \limsup_{t\to\infty} \frac{1}{t} \log \left(\sup_{0\leq h\leq T} d(x(t+h), \Phi_h(x(t))) \right) \leq -1/2.$$

The right-hand side of the inequality above depends on the decrease rate of γ_n[1]. In [3], the authors proved that when G is regular bipartite, the distance between $x(n)$ and Ω converges to zero.

Lemma 4.2. *[3, Theorem 1(b)] Let G be a finite, regular, connected and bipartite graph, then $\lim_{n\to\infty} d(x(n),\Omega) = 0$ almost surely.*

When G is $r-$regular and bipartite, one can explicitly calculate $JF(v)$, the jacobian matrix of F at a point $v = (p,\ldots,p,q,\ldots,q) \in \Omega$. Let A and B denote the bipartition of G as before. If we label the vertices of A from 1 to $m/2$, the vertices of B from 1 to $m/2$, and if we let $M = (m_{ij})$ be the $m/2 \times m/2$ adjacency matrix of the edges connecting vertices of A to vertices of B (i.e. $m_{ij} = 1$ when the i-th vertex of A is adjacent to the j-th vertex of B), then $JF(v)$ takes the form

$$JF(v) = -I + \frac{m}{2r} \begin{bmatrix} rqI & -pM \\ & \\ -qM^t & rpI \end{bmatrix}.$$

Let l be the vector in the tangent space of Δ with coordinates

$$l_i \quad = \quad \begin{cases} 1 & \text{if } i \in A, \\ -1 & \text{if } i \in B. \end{cases}$$

Then it is easy to check that $JF(v)\cdot l = 0$ for any $v \in \Omega$. This implies that the jacobian matrix has zero eigenvalue along the direction of Ω. Let $v_{\pm\infty}$ denote the two endpoints of Ω. One can easily see that $JF(v_{\pm\infty})$ has multiple zero eigenvalues. In the interior of Ω, the authors in [3] proved that in any direction transverse to Ω, the eigenvalues have negative real part.

Lemma 4.3. *[3, Lemma 10.1] Let $v \in \mathrm{int}(\Omega)$. Any eigenvalue of $JF(v)$ different from 0 has negative real part, and 0 is a simple eigenvalue of $JF(v)$.*

Lemma 4.3 says that the interior of Ω attracts exponentially along any direction transverse to Ω. This is a strong property about Ω, and it enables us to effectively work with the dynamics of the ODE at the interior of Ω. For a fixed interval $J \subset \Omega$ not containing $v_{\pm\infty}$, and a small neighborhood U of J in Δ, by Lemma 4.3, there is a submanifold $\mathcal{F}_x$ for each $x \in U$ such that:

[1] More specifically, it equals $\frac{1}{2}\limsup_{n\to\infty} \frac{\log\gamma_n}{\tau_n}$. If $\gamma_n = O(1/n)$, then $\frac{1}{2}\limsup_{n\to\infty}\frac{\log\gamma_n}{\tau_n} = -\frac{1}{2}$.

- $\mathcal{F}_x \pitchfork \Omega$ is one point. We denote this point by $\pi(x)$.
- The dynamics of the ODE on $\mathcal{F}_x$ is exponentially contracting to $\pi(x)$. The speed of convergence depends on the non-zero eigenvalues of $JF(\pi(x))$.

This follows from the theory of invariant manifolds for normally hyperbolic sets (see Theorem 4.1 of [7]).

Thus we have a map $\pi : U \to \Omega$. Notice that π is not a projection (it is not even linear), but $\mathcal{F}_x$ depends smoothly on x. Hence if U is small, then π is 2-Lipschitz:

$$d(\pi(x), \pi(y)) \le 2d(x, y), \forall\, x, y \in U. \tag{4.1}$$

Now fix a small parameter $\varepsilon > 0$ and reduce U, if necessary, so that

$$U = \{x \in \Delta : \pi(x) \in J \text{ and } d(x, \pi(x)) < \varepsilon\}. \tag{4.2}$$

Let $c = \max\{\mathrm{Re}(\lambda) : \lambda \neq 0 \text{ is eigenvalue of } JF(x), x \in J\}$. By Lemma 4.3 , $c < 0$. Thus there is $K > 0$ such that

$$d(\Phi_t(x), \pi(x)) \le Ke^{ct} d(x, \pi(x)), \forall\, x \in U, \forall\, t \ge 0. \tag{4.3}$$

Let $x(t)$ be an orbit that does not converge to $v_{\pm\infty}$. By Lemma 4.2, this orbit has an accumulation point in the interior of Ω. Let $J \subset \Omega$ be an interval containing this point but not $v_{\pm\infty}$, and U as in (4.2).

Lemma 4.4. *Let $x(t) \in U$. If t, T are large enough, then*

(i) $d(\pi(x(t + T)), \pi(x(t))) < 2e^{-\frac{t}{4}}$.

(ii) $x(t + T) \in U$.

Proof. To simplify the notation, denote $x(t)$ by x and $x(t + T)$ by $x(T)$.

Let's prove (i). Since $\pi(\Phi_T(x)) = \pi(x)$, and π is 2-Lipschitz,

$$d(\pi(x(T)), \pi(x)) = d(\pi(x(T)), \pi(\Phi_T(x))) \le 2d(x(T), \Phi_T(x)).$$

By Lemma 4.1, $d(x(T), \Phi_T(x)) \le e^{-\frac{t}{4}}$ for large t, therefore $d(\pi(x(T)), \pi(x)) \le 2e^{-\frac{t}{4}}$ for large t. This proves (i). Note that (i) implies that $\pi(x(T)) \in J$ for large t.

For (ii), we just need to estimate $d(x(T), \pi(x(T)))$. By the triangular inequality, (4.1) and (4.3), we have:

$$\begin{aligned} d(x(T), \pi(x(T))) &\le d(x(T), \Phi_T(x)) + d(\Phi_T(x), \pi(\Phi_T(x))) + \\ &\quad\; d(\pi(\Phi_T(x)), \pi(x(T))) \\ &\le 3d(x(T), \Phi_T(x)) + d(\Phi_T(x), \pi(x)) \\ &\le 3e^{-\frac{t}{4}} + Ke^{cT} d(x, \pi(x)) \\ &\le 3e^{-\frac{t}{4}} + Ke^{cT}\varepsilon \\ &< \varepsilon \end{aligned} \tag{4.4}$$

whenever $3e^{-\frac{t}{4}} < \frac{\varepsilon}{2}$ and $Ke^{cT} < \frac{1}{2}$. □

Note that Lemma 4.4(ii) allows us to iteratively apply Lemma 4.4 to the points $x_k := x(t + kT), k \in \mathbb{N}$. Hence $d(\pi(x_{k+1}), \pi(x_k)) < 2e^{-\frac{t+kT}{4}}$ for all $k \ge 0$. Because $\sum_k e^{-\frac{t+kT}{4}} < \infty$, it follows that $\pi(x_k)$ converges. In the above iterative argument, we implicitly used the fact that $\sum_k e^{-\frac{t+kT}{4}}$ can be made arbitrarily small if t and T are large enough.

This fact guarantees that the total drift of $\pi(x_k)$ from $\pi(x)$ is arbitrarily small so that $\pi(x_k) \in J$ for all $k \geq 0$, and thus the iterative argument works.

Also note that (4.4) holds for all $k \geq 0$:

$$d(x_k, \pi(x_k)) \leq 3e^{-\frac{t+(k-1)T}{4}} + Ke^{cT} d(x_{k-1}, \pi(x_{k-1})). \tag{4.5}$$

Let $\lambda = Ke^{cT}$. Iterating (4.5), we get

$$\begin{aligned} d(x_k, \pi(x_k)) &\leq 3e^{-\frac{t}{4}} \left(e^{-\frac{(k-1)T}{4}} + \lambda e^{-\frac{(k-2)T}{4}} + \cdots + \lambda^{k-1} \right) + \lambda^k d(x, \pi(x)) \\ &\leq 3e^{-\frac{t}{4}} k \left(\max\left\{ e^{-\frac{T}{4}}, \lambda \right\} \right)^{k-1} + \lambda^k d(x, \pi(x)). \end{aligned}$$

When T is large, $\max\left\{ e^{-\frac{T}{4}}, \lambda \right\} < 1$, hence $d(x_k, \pi(x_k)) \to 0$ as $k \to \infty$.

Let $x_0 \in J$ be the limit of $\pi(x_k)$. By the triangular inequality

$$d(x_k, x_0) \leq d(x_k, \pi(x_k)) + d(\pi(x_k), x_0).$$

When k tends to infinity, we have just proved that both $d(\pi(x_k), x_0)$ and $d(x_k, \pi(x_k))$ go to zero, thus $d(x_k, x_0)$ goes to zero. This proves that $\lim_{k\to\infty} x_k$ exists, with $\lim_{k\to\infty} x_k = x_0 \in J$.

For any $s \in [t + kT, t + (k+1)T)$, by the triangular inequality and Lemma 4.1

$$\begin{aligned} d(x(s), x_0) &= d(x(s), \Phi_{s-(t+kT)}(x_0)) \\ &\leq d(x(s), \Phi_{s-(t+kT)}(x_k)) + d(\Phi_{s-(t+kT)}(x_k), \Phi_{s-(t+kT)}(x_0)) \\ &\leq e^{-\frac{t+kT}{4}} + c(T) d(x_k, x_0), \end{aligned}$$

where $c(T) > 0$ is the supremum of the Lipschitz constants of $\Phi_\delta, \delta \in [0, T]$. Therefore, $\lim_{t\to\infty} x(t) = x_0$. This completes the proof of Theorem 1.2.

5 Non-regular balanced-bipartite graphs

We now discuss non-regular balanced-bipartite graphs. It is the only family of graphs that we do not have precise information on the convergence of $x(n)$.

Lemma 5.1. *Let G be a non-regular balanced-bipartite graph. Then $\Lambda \cap \mathrm{int}(\Delta)$ is either empty or an interval.*

Proof. For a non-regular balanced-bipartite graph, the corresponding ODE can *a priori* have either no or at least one interior equilibrium in Δ. Now suppose that the ODE has an interior equilibrium v. Let $V = A \cup B$ be the bipartition of G. Then for any η with $-\min_{i\in A} v_i < \eta < \min_{i\in B} v_i$, the points $u^\eta = (u_1^\eta, \dots, u_m^\eta)$ defined by

$$u_i^\eta = \begin{cases} v_i + \eta & \text{if } i \in A, \\ v_i - \eta & \text{if } i \in B, \end{cases} \tag{5.1}$$

form an interval of interior equilibria.

Furthermore, if h is another interior equilibrium, we will prove that h is contained in this interval. Recall that

$$L(v) = L(v_1, \dots, v_m) = -\sum_{i=1}^m v_i + \frac{1}{N} \sum_{\{i,j\}\in E} \log(v_i + v_j).$$

By (2.10), h and v are critical points of L. Since L is concave, h and v are global maxima of L in Δ and $L(h) = L(v)$. Then for any $0 < c < 1$, the following holds:

$$L(ch + (1-c)v) = cL(h) + (1-c)L(v). \tag{5.2}$$

Since the $\log$ function is strictly concave, (5.2) yields $h_i+h_j = v_i+v_j$ for every $\{i,j\} \in E$, i.e.

$$h_i - v_i = -(h_j - v_j)\,, \ \ \forall\,\{i,j\} \in E. \tag{5.3}$$

Hence, there exists $\eta \in (-\min_{i\in A} v_i, \min_{i\in B} v_i)$, such that

$$h_i \;\; = \;\; \begin{cases} v_i + \eta & \text{if } i \in A, \\ v_i - \eta & \text{if } i \in B, \end{cases}$$

which completes the proof. □

Observe that the proof of Lemma 5.1 works for any balanced-bipartite graph. Thus, we have proved that for a balanced-bipartite graph, the corresponding F either does not have an interior equilibrium, or has an interval of interior equilibria.

Corollary 5.2. *Let G be a non-regular balanced-bipartite graph. Assume that F does not have an interior equilibrium, then Λ is finite, and $x(n)$ converges to an element of Λ almost surely.*

Proof. By Proposition 2.2, we just need to prove that Λ is finite. Since Λ is the union of the sets of critical points of $L|_{\Delta_S}$ over all faces Δ_S and the total number of faces is finite, it suffices to prove that for each face Δ_S with $S \neq [m]$, $L|_{\Delta_S}$ is strictly concave. Fix a face Δ_S with $S \neq [m]$. Let $u, v \in \Delta_S$ and $c \in (0,1)$. If $L(cu+(1-c)v) = cL(u)+(1-c)L(v)$, then $u_i + u_j = v_i + v_j$ for every $\{i,j\} \in E$, i.e.

$$u_i - v_i = (-1)(u_j - v_j)\,, \ \ \forall\,\{i,j\} \in E. \tag{5.4}$$

Since $S \neq [m]$ and $u, v \in \Delta_S$, there exists some $i \notin S$ such that $u_i - v_i = 0$. Because G is connected, (5.4) implies that $u_i - v_i = 0$ for all $i \in [m]$, i.e. $u = v$. This proves that $L|_{\Delta_S}$ is strictly concave. We complete the proof of the corollary. □

Notice that the proof of Corollary 5.2 is general and only uses the assumption that F does not have an interior equilibrium. We just proved that for any finite connected graph G, $L|_{\Delta_S}$ with $S \neq [m]$ is strictly concave, and hence the corresponding F has at most finitely many boundary equilibria.

For a non-regular balanced-bipartite G, if F does not have an interior equilibrium, by Corollary 5.2, we conjecture that there is a unique non-unstable equilibrium of F such that $x(n)$ almost surely converges to it. If F has an interval of interior equilibria, unlike the case of regular bipartite graphs, we are not able to prove a result similar to Lemma 4.3, and hence not able to prove the convergence of $x(n)$. But we also conjecture that this convergence holds. Combining the results we already proved, we make the conjecture below.

Conjecture 5.3. *Let G be a finite and connected graph. Then there exists either a point $v(G)$ such that $x(n)$ almost surely converges to $v(G)$ or an interval $\Omega(G)$ such that $x(n)$ almost surely converges to a point in $\Omega(G)$.*

Remark 5.4. *We just learned that the convergence of $x(n)$ for non-regular balanced-bipartite graphs was proved in [8], and hence Conjecture 5.3 was confirmed.*

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