Pure jump increasing processes and the change of variables formula

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Abstract

Given an increasing process $(A_t)_{t \geq 0}$, we characterize the non-decreasing right-continuous functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ that map A to a pure-jump process. As an example of application, we show for instance that functions with bounded variation belong to the domain of the extended generator of any subordinator with no drift and infinite Lévy measure.

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1 Introduction

We make first some simple observations about the composition of non-decreasing functions which partly motivate the present work. Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function, which, as usual, we further assume to be right-continuous with $a(0) = 0$. It is well-known that a can be decomposed canonically into the sum of two non-decreasing functions, $a = a^c + a^d$, where a^c is continuous and a^d purely discontinuous. More precisely, the latter is given by

$$a^d(t) = \sum_{0 < s \leq t} \Delta a(s) \,,$$

where $\Delta a(s) = a(s) - a(s-)$ and the sum above only takes into account the instants s of effective jumps, that is such that $\Delta a(s) > 0$.

We next recall the Stieltjes change of variables formula, a particular case of Itō's formula. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function; then there is the identity

$$f \circ a(t) - f \circ a(0) = \int_0^t f' \circ a(s) \mathrm{d}a^c(s) + \sum_{0 < s \leq t} \Delta(f \circ a)(s) \,. \tag{1.1}$$

Note that when f is non-decreasing, (1.1) specifies the canonical decomposition of the non-decreasing function $f \circ a$.

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We assume henceforth that a is purely discontinuous, i.e. $a^c \equiv 0$; the change of variables formula (1.1) then reduces to

$$f \circ a(t) - f \circ a(0) = \sum_{0 < s \leq t} \Delta(f \circ a)(s), \qquad (1.2)$$

and it is tempting to think that this identity still holds when f has only finite variation. However, this is incorrect in full generality as we shall now see.

We further suppose that a is strictly increasing; in other words, the Stieltjes measure $\mathrm{d}a$ is purely atomic and its support coincides with $\mathbb{R}_+$. The left-inverse a^{-1} of a is given by

$$a^{-1}(x) = \inf\{y > 0 : a(y) > x\}, \qquad x \geq 0;$$

one sometimes calls a^{-1} a Devil's staircase due to the fact that it is a continuous, non-decreasing function that remains constant on the neighborhood of Lebesgue almost every $x > 0$. In particular the Stieltjes measure $\mathrm{d}a^{-1}$ is continuous and singular with respect to Lebesgue measure. Then $a^{-1} \circ a(x) = x$ for all $0 \leq x < a(\infty)$, as a consequence $a^{-1} \circ a$ has no jump at all. Thus the formula (1.2) fails for $f = a^{-1}$, precisely because in this case the composition $f \circ a$ is not purely discontinuous.

Nonetheless, this note is concerned with the validity of the change of variables formula (1.2) in the situation where the deterministic function a above is replaced by an increasing random process $(A_t)_{t \geq 0}$. Our main result specifies when the latter holds true in terms of a notion of left-accessibility of points. This enables us to observe that, for a large class of increasing processes A which includes subordinators with no drift and infinite Lévy measure, the version of (1.2) with A replacing a holds a.s. for any function f with finite variation. As an example of application, we deduce that functions with bounded variation belong to the domain of the extended generator of such subordinators.

2 Increasing processes and left-accessibility of points

Throughout this note, an increasing process $A = (A_t)_{t \geq 0}$ is a random process with values in $\mathbb{R}_+$, such that with probability one, $A_0 = 0$ and the sample path $t \mapsto A_t$ is right-continuous and non-decreasing. For every $x > 0$, we write

$$L(x) = \sup\{t \geq 0 : A_t < x\}$$

for the last passage time of A below the level x. Observe that $L(x)$ coincides with the left-limit at x of the right-continuous process $A^{-1}(y) = \inf\{t \geq 0 : A_t > y\}$. In particular, L is non-decreasing and left-continuous, $A_{L(x)} \geq x$ and $A_t < x$ for all $t < L(x)$.

We then say that x is *left-accessible* for a sample path of $A(\omega)$ if

$$A_{L(x)-}(\omega) = x,$$

that is to say that the sample path takes values strictly less than x which can be arbitrarily close to x. We write $\mathcal{L}(A)$ for the random set of left-accessible points for A; note that $\mathcal{L}(A)$ is necessarily closed on the left, i.e. the limit of an increasing sequence of left-accessible points is again left-accessible. We also define

$$\mathbb{L}(A) = \{x > 0 : \mathbb{P}(x \in \mathcal{L}(A)) > 0\}.$$

In words, $\mathbb{L}(A)$ is the set of $x > 0$ which are left-accessible for A with positive probability.

Finally, let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous non-decreasing function. For every Borel set $B \subseteq \mathbb{R}_+$, we write $f(B)$ for the mass assigned to B by the Stieltjes measure $\mathrm{d}f$. In particular $f(y) - f(x) = f((x, y])$ for $0 \leq x \leq y$.

We now claim the main result of this note.

Theorem 2.1. (i) *There is the identity*

$$f(A_t) - f(0) = f(\mathcal{L}(A) \cap (0, A_t]) + \sum_{0 < s \le t} (f(A_s) - f(A_{s-}))$$

for all $t \ge 0$, a.s. As a consequence :
(ii) *If $f(\mathbb{L}(A)) = 0$, then we have*

$$f(A_t) - f(0) = \sum_{0 < s \le t} (f(A_s) - f(A_{s-}))$$

for all $t \ge 0$, a.s.

Proof. (i) We observe that for every $x \notin \mathcal{L}(A)$, i.e. x is not left-accessible for $A(\omega)$, there is the identity

$$\mathbf{1}_{[x,\infty)}(A_t(\omega)) = \sum_{0 < s \le t} \left(\mathbf{1}_{[x,\infty)}(A_s(\omega)) - \mathbf{1}_{[x,\infty)}(A_{s-}(\omega)) \right).$$

Indeed the left-hand side equals 1 if $t \ge L(x)$ and 0 otherwise. Note that since $A(\omega)$ is non-decreasing, the summand in the right-hand side is 0 for all $s > 0$, except for $s = L(x)$ where it equals 1 since x is not left-accessible.

Similarly, for every $x \in \mathcal{L}(A)$ and $t \ge 0$, we have

$$\sum_{0 < s \le t} \left(\mathbf{1}_{[x,\infty)}(A_s(\omega)) - \mathbf{1}_{[x,\infty)}(A_{s-}(\omega)) \right) = 0,$$

because the summand is always zero for $s < L(x)$ and for $s > L(x)$, and also for $s = L(x)$ since x is left-accessible for $A(\omega)$.

We can now reformulate these two observations through the identity

$$\mathbf{1}_{(0, A_t(\omega)]}(x) - \mathbf{1}_{\mathcal{L}(A) \cap (0, A_t(\omega)]}(x) = \sum_{0 < s \le t} \left(\mathbf{1}_{[x,\infty)}(A_s(\omega)) - \mathbf{1}_{[x,\infty)}(A_{s-}(\omega)) \right).$$

We also point out that the series in the right-hand side *de facto* only involves instants $s > 0$ at which $A(\omega)$ is discontinuous, and since the set of discontinuities of $A(\omega)$ is at most countable, the counting measure on that set is sigma-finite.

We now integrate the preceding equality with respect to the Stieltjes measure $\mathrm{d}f(x)$. We get that the left-hand side equals

$$f(A_t(\omega)) - f(0) - f(\mathcal{L}(A) \cap (0, A_t(\omega)]).$$

We apply the Fubini-Tonelli theorem on the right-hand side, which is legitimate by the observation which was made above. We obtain

$$\sum_{0 < s \le t} \int_{(0,\infty)} \left(\mathbf{1}_{[x,\infty)}(A_s(\omega)) - \mathbf{1}_{[x,\infty)}(A_{s-}(\omega)) \right) \mathrm{d}f(x)$$
$$= \sum_{0 < s \le t} (f(A_s(\omega)) - f(A_{s-}(\omega))),$$

which proves our formula.

(ii) Next, we point out that Fubini-Tonelli Theorem yields the identity

$$\mathbb{E}(f(\mathcal{L}(A))) = \mathbb{E}\left(\int_{(0,\infty)} \mathrm{d}f(x) \mathbf{1}_{\mathcal{L}(A)}(x) \right)$$
$$= \int_{(0,\infty)} \mathrm{d}f(x) \mathbb{P}(x \in \mathcal{L}(A)).$$

Thus $f(\mathbb{L}(A)) = 0$ if and only if $f(\mathcal{L}(A)) = 0$ a.s. In particular, if we assume that $f(\mathbb{L}(A)) = 0$ and denote by Λ the event that $f(\mathcal{L}(A)) = 0$, then $\mathbb{P}(\Lambda) = 1$. *A fortiori* $f(\mathcal{L}(A) \cap (0, A_t(\omega)]) = 0$ for all $\omega \in \Lambda$ and $t \geq 0$, and the formula of the statement derives from (i). $\qquad \square$

Remark 2.2. *The argument of the proof also shows that if $f(\mathbb{L}(A)) > 0$, then for $t > 0$ sufficiently large, the strict inequality*

$$ f(A_t) - f(0) > \sum_{0 < s \leq t} (f(A_s) - f(A_{s-})) $$

holds with positive probability.

We stress that for a large class of increasing processes A, the set $\mathbb{L}(A)$ is empty and therefore the change of variables formula holds a.s. for any function with finite variation, that is which can be expressed as the difference of two non-decreasing right-continuous functions. Indeed, observe that if $x \in \mathbb{L}(A)$, then the probability that there exists $s > 0$ with $A_{s-} = x$ must be positive. Thus $\mathbb{L}(A) = \varnothing$ whenever every single point $x > 0$ is *polar* for A, that is if

$$ \mathbb{P}(\exists s > 0 : A_s = x \text{ or } A_{s-} = x) = 0 \,. $$

According to an important result due to Kesten [4] and Bretagnolle [3] (see also Andrew [1] for a more elementary argument), polarity of single points holds for any subordinator with no drift and infinite Lévy measure. Further examples can then be constructed from driftless subordinators, e.g. by strictly increasing mapping, change of time or locally equivalent change of probability measures. For instance, the well-known correspondence due to Lamperti [6] connecting positive self-similar Markov processes and Lévy processes, entails that single points are polar for any strictly increasing self-similar Markov process which is further pure-jump.

3 Some applications

In this section, we present some direct consequences of Theorem 2.1.

First recall the following fact that has been pointed out by Kingman [5] in the setting of subordinators. A non-decreasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$ is purely discontinuous if and only if its closed range,

$$ \mathcal{R}(a) = \{x \geq 0 : x = a(s) \text{ or } x = a(s-) \text{ for some } s \geq 0\} \,, $$

has zero Lebesgue measure, i.e. $|\mathcal{R}(a)| = 0$. Indeed, for every $t \geq 0$, the complementary set $[a(0), a(t)] \backslash \mathcal{R}(a)$ is open and can be expressed as the union of the disjoint open intervals $(a(s-), a(s))$ for $0 < s \leq t$ (of course, such an interval is non-empty if and only if s is a discontinuity point of a, which forms a set that is at most countable). When $|\mathcal{R}(a)| = 0$, this open set has Lebesgue measure $a(t) - a(0)$, which yields the identity

$$ a(t) - a(0) = \sum_{0 < s \leq t} (a(s) - a(s-)) \,. $$

Conversely, when a is purely discontinuous, the identity above holds for all $t \geq 0$ and this entails that $|\mathcal{R}(a)| = 0$.

Combining with Theorem 2.1 and Remark 2.2, we immediately get the following.

Corollary 3.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous non-decreasing function.*
(i) If $f(\mathbb{L}(A)) = 0$, then $|\mathcal{R}(f(A))| = 0$ a.s.
(ii) Conversely, if $f(\mathbb{L}(A)) > 0$ and we assume further that f is continuous, then

$$ \mathbb{P}(|\mathcal{R}(f(A))| > 0) > 0. $$

Remark 3.2. *The continuity of f is required in the second part of the statement to ensure that the only jump times of the process $f(A)$ are those of A.*

We next turn our attention to stochastic calculus. Consider some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ which fulfills the usual conditions. Let $(X_t)_{t \geq 0}$ be a càdlàg semi-martingale and $(A_t)_{t \geq 0}$ a càdlàg increasing (adapted) process which is purely discontinuous. Consider further a function $h : \mathbb{R}_+ \to \mathbb{R}$ of class $\mathcal{C}^1$. Then it is well-known that $h(A)$ is a process with finite variation, and the following integration by parts formula holds:

$$
\begin{aligned}
&X_t h(A_t) - X_0 h(A_0) \\
&= \int_0^t h(A_{s-}) \mathrm{d}X_s + \sum_{0 < s \leq t} X_{s-} \left(h(A_s) - h(A_{s-}) \right) + \sum_{0 < s \leq t} \Delta X_s \left(h(A_s) - h(A_{s-}) \right) \\
&= \int_0^t h(A_{s-}) \mathrm{d}X_s + \sum_{0 < s \leq t} X_s \left(h(A_s) - h(A_{s-}) \right).
\end{aligned}
\tag{3.1}
$$

We now consider a càdlàg function $k : \mathbb{R}_+ \to \mathbb{R}$ with finite variation, that is that k can be expressed as the difference of two right-continuous non-decreasing functions. For every Borel set $B \subset \mathbb{R}_+$, we write $|k|(B)$ for the mass given to B by the total-variation measure $|\mathrm{d}k|$. If we assume that $|k|(\mathbb{L}(A)) = 0$, then it follows from Theorem 2.1 that $k(A)$ is a purely discontinuous process with finite variation, and the classical integration by parts formula yields

$$
X_t k(A_t) - X_0 k(0) = \int_0^t k(A_{s-}) \mathrm{d}X_s + \sum_{0 < s \leq t} X_s \left(k(A_s) - k(A_{s-}) \right).
$$

In particular, when $\mathbb{L}(A) = \varnothing$, and *a fortiori* when single points are polar for A, the integration by parts formula (3.1) holds when one merely assumes that the function h has finite variation.

We now conclude this note with an application to the infinitesimal generator of a subordinator. We consider a subordinator $(S_t)_{t \geq 0}$ with no drift and infinite Lévy measure. That is, S is a random process with values in $\mathbb{R}_+$, with independent and stationary increments, and we assume that its sample path is right-continuous, strictly increasing, and has no continuous component. According to the Lévy-Itō decomposition, the Stieltjes measure $\mathrm{d}S$ is expressed in the form

$$
\mathrm{d}S = \sum_{t \geq 0} \Delta S_t \delta_t
$$

where $\Delta S_t = S_t - S_{t-}$ denotes the possible jump at time t. More precisely, the jump process ΔS is a Poisson point process whose intensity is known as the Lévy measure of S. We refer to [2] for background.

Recall that S is a Feller process on $\mathbb{R}_+$; we write $G : \mathcal{D} \to \mathcal{C}_0$ for its infinitesimal generator, where $\mathcal{D}$ is its domain and $\mathcal{C}_0$ the space of continuous functions on $\mathbb{R}_+$ with limit 0 at $+\infty$. It is well-known that if $\mathcal{C}_0^1$ denotes the subspace of functions $g \in \mathcal{C}_0$ which are continuously differentiable with $g' \in \mathcal{C}_0$, then $\mathcal{C}_0^1 \subset \mathcal{D}$ and there is the identity

$$
Gg(x) = \int_{(0,\infty)} \left(g(x+y) - g(x) \right) \Pi(\mathrm{d}y), \qquad x \geq 0,
$$

where Π denotes the Lévy measure of S. Recall also that for every $g \in \mathcal{D}$ and $x \geq 0$, the process

$$
g(x + S_t) - \int_0^t Gg(x + S_s) \mathrm{d}s
$$

is a martingale, and conversely, this martingale property together with the assumption that both g and Gg are in $\mathcal{C}_0$, characterize the infinitesimal generator. Further, a measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ belongs to the domain $\mathcal{D}_e$ of the *extended* generator if there exists a measurable function $b : \mathbb{R}_+ \to \mathbb{R}$ such that for every $x \geq 0$ and $t \geq 0$

$$\int_0^t |b(x + S_s)| \mathrm{d}s < \infty \qquad \text{a.s.}$$

and

$$h(x + S_t) - \int_0^t b(x + S_s) \mathrm{d}s$$

is a martingale. By slightly abusing notation (because b is not unique), we then write $G_e h = b$.

Corollary 3.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a non-decreasing right-continuous function, which we further assume to be bounded. Then $f \in \mathcal{D}_e$ and*

$$G_e f(x) = \int_{(0,\infty)} (f(x + y) - f(x)) \, \Pi(\mathrm{d}y), \qquad x \geq 0,$$

where Π denotes the Lévy measure of S.

Note that $G_e f(x)$ is well-defined in $[0, \infty]$ (and thus possibly infinite) since f is non-decreasing.

Proof. Recall that single points are polar for S according to the result of Kesten [4], and thus Theorem 2.1(i) applies to any non-decreasing function f. Replacing the function f there by the function $f(x + \cdot)$, we get

$$
\begin{aligned}
f(x + S_t) - f(x) &= \sum_{0 < s \leq t} (f(x + S_s) - f(x + S_{s-})) \\
&= \sum_{0 < s \leq t} (f(x + S_{s-} + \Delta S_s) - f(x + S_{s-})) \,.
\end{aligned}
$$

Recall that the jump process ΔS of S is a Poisson point process with intensity Π. It follows that the predictable compensator of the increasing process $f(x + S_t) - f(x)$ is

$$\int_0^t \mathrm{d}s \int_{(0,\infty)} \Pi(\mathrm{d}y) \, (f(x + S_s + y) - f(x + S_s)) = \int_0^t G_e f(x + S_s) \mathrm{d}s.$$

In other words,

$$f(x + S_t) - f(x) - \int_0^t G_e f(x + S_s) \mathrm{d}s$$

is a martingale. $\qquad\square$

Of course, by linearity, we deduce that functions $k : \mathbb{R}_+ \to \mathbb{R}$ with bounded variation, i.e. which can be expressed as the difference of two bounded non-decreasing functions, are also in the domain of the extended generator, and then

$$G_e k(x) = \int_{(0,\infty)} (k(x + y) - k(x)) \, \Pi(\mathrm{d}y),$$

where we agree for instance that $G_e k(x) = 0$ when the integral above is not well-defined.

References

[1] Andrew, P. A proof from 'first principles' of Kesten's result for the probabilities with which a subordinator hits points. *Electron. Commun. Probab.* no. **6** (2006), 58-63. MR-2219346

[2] Bertoin, J. Subordinators: examples and applications. In: *Lectures on probability theory and statistics (Saint-Flour, 1997)*, pp. 1-91, Lecture Notes in Math., 1717, Springer, Berlin, 1999. MR-1746300

[3] Bretagnolle, J. Résultats de Kesten sur les processus à accroissements indépendants. In: *Séminaire de Probabilités V*, pp. 21-36. Lecture Notes in Math., Vol. 191, Springer, Berlin, 1971. MR-0368175

[4] Kesten, H. Hitting probabilities of single points for processes with stationary independent increments. *Memoirs Amer. Math. Soc.* **93** (1969). MR-0272059

[5] Kingman, J. F. C. *Regenerative phenomena*. Wiley, London, 1972. MR-0350861

[6] Lamperti, J. (1972). Semi-stable Markov processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **22**, 205–225. MR-0307358