Electron. Commun. Probab. **18** (2013), no. 10, 1–6.
DOI: 10.1214/ECP.v18-2236
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Explicit formula for the supremum distribution of a spectrally negative stable process

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Abstract

In this article we get simple formulas for $\mathbb{E}\sup_{s\le t} X(s)$ where X is a spectrally positive or negative Lévy process with infinite variation. As a consequence we derive a generalization of the well-known formula for the supremum distribution of Wiener process that is we obtain $\mathbb{P}(\sup_{s\le t} Z_\alpha(s) \ge u) = \alpha\,\mathbb{P}(Z_\alpha(t) \ge u)$ for $u \ge 0$ where Z_α is a spectrally negative α-stable Lévy process with $1 < \alpha \le 2$ which also stems from Kendall's identity for the first crossing time. Our proof uses a formula for the supremum distribution of a spectrally positive Lévy process which follows easily from the elementary Seal's formula.

Keywords: Lévy process; distribution of the supremum of a stochastic process; α-stable Lévy process.
AMS MSC 2010: Primary 60G51, Secondary 60G52, 60G70.
Submitted to ECP on August 12, 2012, final version accepted on January 26, 2013.

1 Introduction

Lévy processes appear in many theoretical and practical fields where they serve as a basic skeleton for a description of certain phenomena. They are applied in physics, economics, finance, insurance, queueing systems and other branches of knowledge. Their features like independence and stationarity of increments or self-similarity in certain cases permit to apply them to model for instance returns of stock prices, claims to insurance companies or an inflow (outflow) to the buffer in queueing (telecommunications) systems. Moreover Lévy processes serve as a starting point for more complicated models e.g. based on stochastic differential equations.

We will investigate real valued Lévy processes. Lévy-Itô representation shows their stochastic construction which is the following (see e.g. Sato [17])

$$X(t) = B(t) + \int_{|x|<1} x\,(N_t(dx) - tQ(dx)) + \int_{|x|\ge 1} x\,N_t(dx) + at\,,$$

where $B(t)$ is Wiener process, N is a point process generated by the jumps of X that is $N = \sum_{\{t:\Delta X(t)\neq 0\}} \delta_{(t,\Delta X(t))}$. N is a random Poisson measure on $[0,\infty) \times \{\mathbb{R}\setminus 0\}$ with the mean $ds \times Q(dx)$, where $Q(dx)$ is the so-called Lévy measure on $\mathbb{R}\setminus 0$ and $a \in \mathbb{R}$.

In this note we consider spectrally one-sided Lévy processes without Wiener component. We find expected value of the supremum on a finite interval for any spectrally

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positive or negative Lévy process. Then as a corollary we derive a generalization of the famous formula

$$\mathbb{P}(\sup_{s\le t} B(s) \ge u) = 2\mathbb{P}(B(t) \ge u)$$

for $u \ge 0$, where B is Wiener process that is we show that

$$\mathbb{P}(\sup_{s\le t} Z_\alpha(s) \ge u) = \alpha\mathbb{P}(Z_\alpha(t) \ge u) \tag{1.1}$$

for $u \ge 0$, where Z_α is an α-stable Lévy process with $1 < \alpha \le 2$, the scale parameter $\sigma > 0$, the skewness parameter $\beta = -1$ and the shift parameter $\mu = 0$ that is Z_α is a Lévy process with the following characteristic function of one dimensional distributions

$$\mathbb{E}\exp(i\theta Z_\alpha(t)) = \exp\left(-\sigma^\alpha t|\theta|^\alpha \left[1 + i\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2}\right]\right), \tag{1.2}$$

where $\theta \in \mathbb{R}$ and sign is the sign function (see e.g. Janicki and Weron [10] or Samorodnitsky and Taqqu [16] for the definition of a stable distribution with four parameters). This Lévy process has no positive jumps which means it is spectrally negative and it has infinite variation. The formula (1.1) also stems from Kendall's identity for the first crossing time see Kendall [11] or e.g. Bertoin [7] or Borovkov and Burq [8] and the references therein. Let us recall that the proofs of Kendall's identity are analytical (using Laplace transforms) or are using limit and combinatorial arguments or factorization identities except the proof of Borovkov and Burq [8] which is straightforward by the change of measure technique. The identity (1.1) may also be verified by inverting time-space Laplace transforms (see Bernyk, Dalang and Peskir [5] for a similar treatment). The above formula for Wiener process follows easily from the reflection principle. Here we give a straightforward proof of the identity (1.1) based on the formula from Michna [12] and [13] which is simply derived from the elementary Seal's formula for a compound Poisson process see Seal [18] (which is going back to Cramér and Prabhu see e.g. Asmussen and Albrecher [3] and Prabhu [15]). Regardless the theoretical importance of the above formula, the supremum distribution is the key value in many practical problems in insurance, finance and queueing systems. The distribution of the supremum of spectrally one-sided Lévy processes has been investigated in many papers see e.g. Albin [1], Avram, Kyprianou and Pistorius [4], Bernyk, Dalang and Peskir [5], Bertoin [6], Michna [12], Pistorius [14] and many others. Explicit formulas for the supremum distribution of stochastic processes on finite intervals are known only in few cases. Most papers are concerned with an asymptotic behavior of the tail distribution of the supremum for stochastic processes see e.g. Albin and Sunden [2] and the references therein. In some articles one can find the distribution of the supremum but in the form of Laplace transforms of the first passage times see Bertoin [6], Avram, Kyprianou and Pistorius [4] and Pistorius [14].

2 Expected value of the supremum

Let X be a spectrally positive Lévy process and Y a spectrally negative Lévy process both with infinite variation (one can regard that $Y = -X$). Let us recall that a spectrally positive Lévy process has no negative jumps and analogically for a spectrally negative Lévy process. Additionally we assume that their Lévy measure Q has a bounded density on every infinite interval cut off from zero and their one-dimensional distributions are absolutely continuous with respect to Lebesgue measure see Michna [13]. We denote $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$.

Proposition 2.1.

$$\begin{aligned}
&\mathbb{E}\sup_{s\le t} X(s) \\
&= \int_0^\infty \mathbb{P}(X(t)>u)\,du + \int_0^t \frac{P(X(t-s)>0)}{s}\,ds \int_{-\infty}^0 \mathbb{P}(X(s)\le u)\,du\,.
\end{aligned}$$

If $\mathbb{E}Y^-(t)<\infty$ *then*

$$\begin{aligned}
&\mathbb{E}\sup_{s\le t} Y(s) \\
&= \int_0^\infty \mathbb{P}(Y(t)>u)\,du + \int_0^t \frac{P(Y(t-s)<0)}{s}\,ds \int_0^\infty \mathbb{P}(Y(s)\ge u)\,du\,,
\end{aligned}$$

Proof. By Michna [12] and [13] we have

$$\mathbb{P}(\sup_{s\le t} X(s)>u) = \mathbb{P}(X(t)>u) + \int_0^t \frac{f(u,s)}{t-s}\,ds \int_{-\infty}^0 \mathbb{P}(X(t-s)\le x)\,dx\,, \tag{2.1}$$

where $f(u,s)$ is a density function of the random variable $X(s)$. Integrating we get

$$\begin{aligned}
&\mathbb{E}\sup_{s\le t} X(s) \\
&= \int_0^\infty \mathbb{P}(X(t)>u)\,du + \int_0^\infty du \int_0^t \frac{f(u,s)}{t-s}\,ds \int_{-\infty}^0 \mathbb{P}(X(t-s)\le x)\,dx \\
&= \int_0^\infty \mathbb{P}(X(t)>u)\,du + \int_0^t \frac{\mathbb{P}(X(s)>0)}{t-s}\,ds \int_{-\infty}^0 \mathbb{P}(X(t-s)\le x)\,dx \\
&= \int_0^\infty \mathbb{P}(X(t)>u)\,du + \int_0^t \frac{\mathbb{P}(X(t-s)>0)}{s}\,ds \int_{-\infty}^0 \mathbb{P}(X(s)\le u)\,du\,,
\end{aligned}$$

where in the last equality we substitute $s'=t-s$.

To prove the second assertion let us notice that for a fixed t and $0\le s\le t$ we have $X(s)\stackrel{d}{=}X(t)-X(t-s)$ in the sense of finite dimensional distributions. Thus

$$\begin{aligned}
\sup_{s\le t} X(s) &\stackrel{d}{=} \sup_{s\le t}(X(t)-X(t-s)) \\
&= X(t)-\inf_{s\le t} X(s)\,,
\end{aligned}$$

where the equality in distribution is in the sense of the one-dimensional distribution. Hence

$$\sup_{s\le t} X(s) \stackrel{d}{=} X(t)+\sup_{s\le t} Y(s)\,,$$

where $Y=-X$. So by the first formula of the proposition we obtain

$$\begin{aligned}
\mathbb{E}\sup_{s\le t} Y(s) &= -\mathbb{E}X(t)+\mathbb{E}\sup_{s\le t}X(s)\\
&= \mathbb{E}Y(t)+\mathbb{E}\sup_{s\le t}X(s)\\
&= \mathbb{E}Y(t)+\int_0^\infty \mathbb{P}(X(t)>u)\,du+\\
&\qquad \int_0^t \frac{P(X(t-s)>0)}{s}\,ds\int_{-\infty}^0 \mathbb{P}(X(s)\le u)\,du\\
&= \mathbb{E}Y(t)+\int_0^\infty \mathbb{P}(Y(t)<-u)\,du+\\
&\qquad \int_0^t \frac{P(Y(t-s)<0)}{s}\,ds\int_{-\infty}^0 \mathbb{P}(Y(s)\ge -u)\,du\\
&= \int_0^\infty \mathbb{P}(Y(t)>u)\,du+\\
&\qquad \int_0^t \frac{P(Y(t-s)<0)}{s}\,ds\int_0^\infty \mathbb{P}(Y(s)\ge u)\,du
\end{aligned}$$

which finishes the proof. □

Remark 2.2. *One can write the first formula of Prop. 2.1 as*

$$\mathbb{E}\sup_{s\le t}X(s)=\mathbb{E}X^+(t)+\int_0^t \frac{P(X(t-s)>0)}{s}\,\mathbb{E}X^-(s)\,ds$$

and the second formula as

$$\mathbb{E}\sup_{s\le t}Y(s)=\mathbb{E}Y^+(t)+\int_0^t \frac{P(Y(t-s)<0)}{s}\,\mathbb{E}Y^+(s)\,ds\,.$$

Remark 2.3. *The formulas of Prop. 2.1 are valid for Wiener process as well because the formula (2.1) is true for Wiener process see Michna [12].*

3 The supremum distribution of a spectrally negative stable Lévy process

Now let us consider a spectrally negative α-stable Lévy process Z_α with $1<\alpha\le 2$ (see e.g. Janicki and Weron [10] or Samorodnitsky and Taqqu [16]). A simple proof of a generalization of the famous formula for the supremum distribution of Wiener process we get by Prop. 2.1 which in fact follows from the formula for the supremum distribution of a spectrally positive Lévy process (see Michna [13]).

Theorem 3.1. *Let $u\ge 0$ and Z_α be the spectrally negative α-stable Lévy process given by (1.2). Then*

$$\mathbb{P}(\sup_{s\le t} Z_\alpha(s)\ge u)=\alpha\mathbb{P}(Z_\alpha(t)\ge u)\,.$$

Proof. Let us note that $\mathbb{P}(Z_\alpha(s)>0)=1/\alpha$ and $Z_\alpha(as)\stackrel{d}{=}a^{1/\alpha}Z_\alpha(s)$ for $a>0$ in the sense of finite dimensional distributions (the self-similarity property) see e.g. Samorodnitsky and Taqqu [16]. Since $\mathbb{E}|Z_\alpha(t)| < \infty$ for $1 < \alpha \le 2$ thus by Prop. 2.1 we have

$$\begin{aligned}
&\mathbb{E}\sup_{s\le t} Z_\alpha(s) \\
&= \int_0^\infty \mathbb{P}(Z_\alpha(t) > u)\,du + \int_0^t \frac{P(Z_\alpha(t-s) < 0)}{s}\,ds \int_0^\infty \mathbb{P}(Z_\alpha(s) \ge u)\,du \\
&= \int_0^\infty \mathbb{P}(Z_\alpha(t) > u)\,du + \frac{\alpha-1}{\alpha}\int_0^t \frac{ds}{s}\int_0^\infty \mathbb{P}(Z_\alpha(t) \ge ut^{1/\alpha}/s^{1/\alpha})\,du \\
&= \int_0^\infty \mathbb{P}(Z_\alpha(t) > u)\,du + \frac{\alpha-1}{\alpha}\int_0^t \frac{s^{1/\alpha-1}}{t^{1/\alpha}}\,ds \int_0^\infty \mathbb{P}(Z_\alpha(t) \ge u)\,du \\
&= \int_0^\infty \mathbb{P}(Z_\alpha(t) > u)\,du + (\alpha-1)\int_0^\infty \mathbb{P}(Z_\alpha(t) \ge u)\,du \\
&= \alpha \int_0^\infty \mathbb{P}(Z_\alpha(t) \ge u)\,du\,,
\end{aligned}$$

where in the second equality we use the self-similarity property and in the third equality we substitute $u' = ut^{1/\alpha}/s^{1/\alpha}$. By eq. 1.2 of Albin [1] or Lemma 3 of Furrer, Michna and Weron [9] the following upper bound we can state

$$\mathbb{P}(\sup_{s\le t} Z_\alpha(s) \ge u) \le \alpha\mathbb{P}(Z_\alpha(t) \ge u)\,. \tag{3.1}$$

Since the last calculations give

$$\int_0^\infty \mathbb{P}(\sup_{s\le t} Z_\alpha(s) \ge u)\,du = \int_0^\infty \alpha\mathbb{P}(Z_\alpha(t) \ge u)\,du$$

thus by eq. (3.1) and the continuity and monotonicity with respect to u we obtain the assertion of the theorem. □

Remark 3.2. *The asymptotic behavior of* $\mathbb{P}(Z_\alpha(t) \ge u)$ *for* $u \to \infty$ *can be found in Samorodnitsky and Taqqu [16], eq. 1.2.11 which is not regularly varying but Weibullian.*

Remark 3.3. *In Albin [1] the exact asymptotic for* $\mathbb{P}(\sup_{s\le t} Z_\alpha(s) \ge u)$ *as* $u \to \infty$ *has been derived in the form* $C_\alpha\mathbb{P}(Z_\alpha(t) \ge u)$ *where it was shown that* $C_\alpha > 1$ *for* $1 < \alpha < 2$*. Thus we get that* $C_\alpha = \alpha$ *if* $1 < \alpha \le 2$*.*

Remark 3.4. *The supremum distribution of a spectrally positive Lévy process is qualitatively different than in the spectrally negative case because in the first case the supremum is attained by a jump see e.g. Bernyk, Dalang and Peskir [5].*

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