Electron. Commun. Probab. **18** (2013), no. 62, 1–13.
DOI: 10.1214/ECP.v18-2717
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Exact simulation of Hawkes process with exponentially decaying intensity

Angelos Dassios* Hongbiao Zhao†

Abstract

We introduce a numerically efficient simulation algorithm for Hawkes process with exponentially decaying intensity, a special case of general Hawkes process that is most widely implemented in practice. This computational method is able to exactly generate the point process and intensity process, by sampling interarrival-times directly via the underlying analytic distribution functions without numerical inverse, and hence avoids simulating intensity paths and introducing discretisation bias. Moreover, it is flexible to generate points with either stationary or non-stationary intensity, starting from any arbitrary time with any arbitrary initial intensity. It is also straightforward to implement, and can easily extend to multi-dimensional versions, for further applications in modelling contagion risk or clustering arrival of events in finance, insurance, economics and many other fields. Simulation algorithms for one dimension and multi-dimension are represented, with numerical examples of univariate and bivariate processes provided as illustrations.

Keywords: Contagion risk; Stochastic intensity model; Self-exciting point process; Hawkes process; Hawkes process with exponentially decaying intensity; Exact simulation; Monte Carlo simulation.
AMS MSC 2010: Primary: 60G55; Secondary: 60H35; 65C05; 60G17.
Submitted to ECP on April 3, 2013, final version accepted on July 10, 2013.

1 Introduction

Throughout the recent subprime mortgage crisis and current European debt crisis, financial deterioration and losses can easily spread through the business network and financial market, and the risk of contagion has created enormous instability and uncertainty substantially threatening the financial systems, both regionally and globally. Financial models capable to capture the contagion or clustering effect are essentially needed. Self-excited point processes, in particular, Hawkes processes early introduced by [16], nowadays become very popular and viable mathematical tools for modelling contagion risk and clustering arrival of events in finance, insurance and economics, see examples of applications in [13], [11], [8], and more recently, [2] and [1]. They also have various applications in many other fields such as seismology and neurophysiology, see [24] and [5].

For statistical analysis or practical implementations, simulation is a crucial issue as the procedure of the parameter estimation or calibration, in particular, for higher

*Department of Statistics, London School of Economics, United Kingdom. E-mail: a.dassios@lse.ac.uk
†Corresponding author, Department of Finance, School of Economics & Wang Yanan Institute for Studies in Economics, Xiamen University, China. E-mail: hongbiao.z@gmail.com

dimensional processes, heavily relies on it. Risk management and asset pricing for complex path-dependent financial derivatives, such as collateralized debt obligations (CDOs), require a high level of computational efficiency of simulation, as the analytic formulas are rather limited and often difficult to obtain. For instance, the model parameters for pricing CDOs can be calibrated to the market data via numerically solving optimisation problem using simulation, see [14].

There are two major simulation approaches in the literature: so-called *intensity-based* and *cluster-based*, since a Hawkes process can either be defined via a conditional stochastic intensity representation (e.g. *Definition 2.1*), or, via a Poisson cluster representation (e.g. *Definition 2.2*). As recently surveyed by [19], the current standard algorithms for simulating Hawkes processes mostly remain the conventional thinning method via the acceptance/rejection procedure early introduced by [18][1] that is generally used for simulating any point process with stochastic intensity. [20] provided algorithms named *perfect simulation*[2] for simulating (marked) Hawkes processes (on $\mathbb{R}$-time) using a cluster-based approach with a branching structure. Their algorithms start from time minus infinity and require certain stationarity condition, also see [21]. [14] introduced an algorithm named *exact simulation*[3] for Hawkes process (on $\mathbb{R}_+$-time) which is capable to sample interarrival-times directly via their underlying analytic distribution functions and hence avoid generating intensity paths and introducing discretisation bias for associated estimators, also see [15]. However, their method requires numerical evaluation of the inverse of these distribution functions via Brent's method which involves intensive computations.

In this paper, we focus on simulation algorithms rather than associated applications. We have tailored a numerically efficient algorithm specifically for a Hawkes process (on $\mathbb{R}_+$-time) with exponentially decaying intensity, a special case of general Hawkes process that is most widely implemented in practice. Especially for this special case, our approach has many advantages, in particular,

1. it is able to exactly generate Hawkes processes by sampling interarrival-times directly via the underlying analytic distribution functions without numerical inverse, as each of these interarrival-times can be simulated exactly by splitting into two independent simple random variables;
2. it is straightforward to implement, and easily to generalise into higher dimensions;
3. it is flexible to simulate Hawkes processes starting from any time, conditional on any arbitrary fixed initial intensities, or, with any arbitrary distributions for the initial intensities;
4. it does not require stationarity condition for intensity dynamics, and hence Hawkes processes with unbounded intensities in a finite time can also be simulated;
5. it also can be easily adjusted to simulate Hawkes processes with stationary intensities.

The paper is organised as follows. Section 2 briefly reviews the Hawkes process with exponentially decaying intensity via definitions of intensity-based and cluster-based representations respectively, and some basic distributional properties are given. Section

[1] Also see Ogata's modified thinning algorithm by [23] and *Algorithm 7.5.IV* in [6].

[2] Here 'perfect' refers to the fact that the simulation on a finite time interval takes the past into account (without edge effects), also see [4].

[3] Here the 'exact' simulation means a method of drawing an unbiased associated estimator throughout the entire simulation process.

3 provides the numerical algorithm of exact simulation for a Hawkes process in one dimension. For verification, numerical examples and comparison with exact theoretical results of means and variances are also given in Section 4. Section 5 represents how straightforward our method can be extended to a multi-dimensional version, and a numerical example of bivariate case is provided as an illustration.

2 Preliminaries

[16] introduced a general type of Hawkes process, where the current intensity of the arrival of points is determined by points in the past with associated weights. In this paper, we focus on a simple and slightly modified version, when the response function is decaying exponentially, and sizes of self-excited jumps in the intensity process are allowed to be either fixed or random. The randomness in jump sizes could provide higher flexibility for measuring the self-exciting effect, and this is a minor generalisation of the Markovian self-exciting process represented by [22].

Our aim is to develop algorithms for practical implementation and we assume the Hawkes processes start from time zero. The definition of this univariate Markovian Hawkes process on $\mathbb{R}_+$-time via *conditional stochastic intensity representation* is given in *Definition 2.1*.

Definition 2.1 (Intensity-based)**.** *Hawkes process with exponentially decaying intensity is a point process $N_t \equiv \{T_k\}_{k=1,2,\dots}$ on $\mathbb{R}_+$ with non-negative $\mathcal{F}_t-$stochastic intensity*

$$\lambda_t = a + (\lambda_0 - a)\, e^{-\delta t} + \sum_{0 \le T_k < t} Y_k e^{-\delta(t-T_k)}, \quad t \ge 0,$$

where

- *$\{\mathcal{F}_t\}_{t\ge 0}$ is a history of the process N_t with respect to which $\{\lambda_t\}_{t\ge 0}$ is adapted;*
- *$a \ge 0$ is the constant reversion level;*
- *$\lambda_0 > 0$ is the initial intensity at time $t = 0$;*
- *$\delta > 0$ is the constant rate of exponential decay;*
- *$\{Y_k\}_{k=1,2,\dots}$ are sizes of self-excited jumps, a sequence of i.i.d. positive random variables with distribution function $G(y), y > 0$;*
- *$\{T_k\}_{k=1,2,\dots}$ and $\{Y_k\}_{k=1,2,\dots}$ are assumed to be independent of each other.*

Jump-sizes $\{Y_k\}_{k=1,2,\dots}$ in the intensity measure the levels of compact of contagion and can also be fixed constants; δ captures the persistence of contagion with a common rate of exponentially decaying after each jump. A sample figure of joint process (N_t, λ_t) is represented in *Figure 1*.

On the other hand, this Hawkes point process $\{N_t\}_{t\ge 0}$ can also be alternatively constructed as a marked Poisson cluster process on $\mathbb{R}_+$-time with the clusters following a recursive branching structure, see [17], [6] and [25]. The definition via *marked Poisson cluster representation* is given by *Definition 2.2*, and offers a nice economic interpretation: the immigrants and offsprings could be considered as primary shocks and associated aftershocks respectively. Note that, as this point process is defined on $\mathbb{R}_+$, there are no immigrants before time 0 and hence no edge effects as concerned by [20].

Definition 2.2 (Cluster-based)**.** *Hawkes process with exponentially decaying intensity is a marked Poisson cluster process $C = \{(T_i, Y_i)\}_{i=1,2,\dots}$ with times $T_i \in \mathbb{R}_+$ and marks*

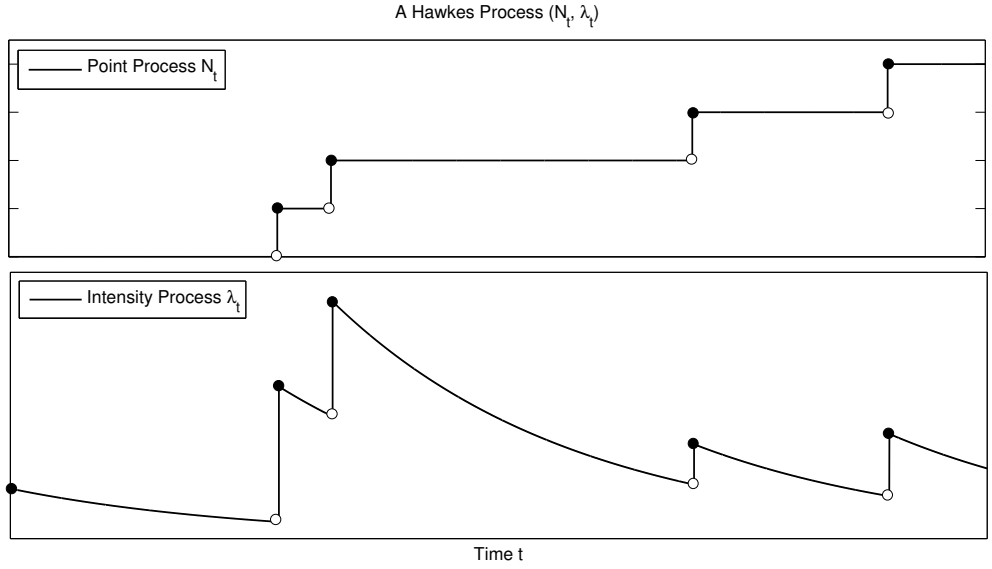

Figure 1: A Hawkes Process with Exponential Decaying Intensity (N_t, λ_t)

Y_i: the number of points in $(0,t]$ is defined by $N_t = N_{C(0,t]}$; the cluster centers of C are the particular points called immigrants, the rest of the points are called offspring, and they have the following structure:

(a) *The immigrants $I = \{T_m\}_{m=1,2,...}$ on $\mathbb{R}_+$ are distributed as an inhomogeneous Poisson process of rate $a + (\lambda_0 - a)\, e^{-\delta t}$, $t \geq 0$.*

(b) *The marks $\{Y_m\}_{m=1,2,...}$ associated to immigrants I are i.i.d. with $Y_m \sim G$, and are independent of the immigrants.*

(c) *Each immigrant T_m generates one cluster C_m, and these clusters are independent.*

(d) *Each cluster C_m is a random set formed by marked points of generations of order $n = 0, 1, ...$ with the following branching structure:*

- *The immigrant and its mark (T_m, Y_m) is said to be of generation 0.*
- *Recursively, given generations $0, 1, ..., n$ in C_m, each $(T_j, Y_j) \in C_m$ of generation n generates a Poisson process of offspring of generation $n+1$ on (T_j, ∞) with intensity $Y_j e^{-\delta(t-T_j)}, t > T_j$, where mark $Y_j \sim G$, independent of generations $0, 1, ..., n$.*

(e) *C consists of the union of all clusters, i.e. $C = \bigcup_{m=1,2,...} C_m$.*

Definition 2.1 and *Definition 2.2* are equivalent. The stationarity condition for this Hawkes process is $\delta > \mu_{1_G}$ where $\mu_{1_G} = \int_0^\infty y \mathrm{d}G(y)$, although this is not required in the simulation algorithm we developed later. We provide the expectation and variance of λ_t and expectation of N_t conditional on λ_0 in Proposition 2.3, which will be used later in Section 4 for numerically validating our algorithm. Fundamental distributional properties including Proposition 2.3, Laplace transform of λ_t, and probability generating function of N_t and the size of clusters can be easily derived by using formulas in [7] for a more generalised point process (so-called *dynamic contagion process*), also see further extensions in [9] and [10].

Proposition 2.3. *The expectation and variance of λ_t conditional on λ_0 are given by*

$$\begin{aligned}\mathbb{E}[\lambda_t|\lambda_0] &= \frac{a\delta}{\kappa}+\left(\lambda_0-\frac{a\delta}{\kappa}\right)e^{-\kappa t},\\ \mathrm{Var}[\lambda_t|\lambda_0] &= \frac{\mu_{2_G}}{\kappa}\left[\left(\frac{a\delta}{2\kappa}-\lambda_0\right)e^{-2\kappa t}+\left(\lambda_0-\frac{a\delta}{\kappa}\right)e^{-\kappa t}+\frac{a\delta}{2\kappa}\right],\end{aligned}$$

where $\kappa=\delta-\mu_{1_G}>0$, $\mu_{2_G}=\int_0^\infty y^2\mathrm{d}G(y)$; and the expectation of N_t is given by

$$\mathbb{E}[N_t|\lambda_0]=\frac{a\delta}{\kappa}t+\frac{1}{\kappa}\left(\lambda_0-\frac{a\delta}{\kappa}\right)\left(1-e^{-\kappa t}\right).$$

3 Simulation Algorithm

The algorithm of exact simulation is given by *Algorithm 3.1* for a univariate Hawkes process with exponentially decaying intensity (defined by *Definition 2.1* or *Definition 2.2*) and random sizes of self-excited jumps in the intensity process. This algorithm is very easy to implement, because each of the random interarrival-times of jumps in the Hawkes process can be simulated exactly by decomposing it into two independent and simpler random variables without inverting the underlying cumulative distribution function. In addition, simulating intensity processes does not require stationarity conditions both for the cases of one-dimension in *Algorithm 3.1* and multi-dimension in *Algorithm 5.1*.

Algorithm 3.1 (Univariate)**.** *The simulation algorithm for one sample path of one-dimensional Hawkes process with exponentially decaying intensity $\{(N_t,\lambda_t)\}_{t\geq 0}$ conditional on λ_0 and $N_0=0$, with jump-size distribution $Y\sim G$ and $\bar{K}$ jump-times $\{T_1,T_2,...,T_{\bar{K}}\}$:*

1. *Set the initial conditions $T_0=0$, $\lambda_{T_0^\pm}=\lambda_0>a$, $N_0=0$ and $k\in\{0,1,2,...,\bar{K}-1\}$.*

2. *Simulate the $(k+1)^{\mathrm{th}}$ interarrival-time S_{k+1} by*

$$S_{k+1}=\begin{cases}S_{k+1}^{(1)}\wedge S_{k+1}^{(2)}, & D_{k+1}>0,\\ S_{k+1}^{(2)}, & D_{k+1}<0,\end{cases}$$

where

$$D_{k+1}=1+\frac{\delta\ln U_1}{\lambda_{T_k^+}-a},\quad U_1\sim\mathrm{U}[0,1],$$

and

$$S_{k+1}^{(1)}=-\frac{1}{\delta}\ln D_{k+1},\quad S_{k+1}^{(2)}=-\frac{1}{a}\ln U_2,\quad U_2\sim\mathrm{U}[0,1].$$

3. *Record the $(k+1)^{\mathrm{th}}$ jump-time T_{k+1} in the intensity process λ_t by*

$$T_{k+1}=T_k+S_{k+1}.$$

4. *Record the change at the jump-time T_{k+1} in the intensity process λ_t by*

$$\lambda_{T_{k+1}^+}=\lambda_{T_{k+1}^-}+Y_{k+1},\quad Y_{k+1}\sim G, \tag{3.1}$$

where

$$\lambda_{T_{k+1}^-}=\left(\lambda_{T_k^+}-a\right)e^{-\delta(T_{k+1}-T_k)}+a.$$

5. Record the change at the jump-time T_{k+1} in the point process N_t by

$$N_{T_{k+1}^+} = N_{T_{k+1}^-} + 1. \tag{3.2}$$

Proof. Given the k^{th} jump-time T_k, the point process has the intensity process $\{\lambda_t\}_{T_k \le t < T_k + S_{k+1}}$ following the ODE

$$\frac{\mathrm{d}\lambda_t}{\mathrm{d}t} = -\delta\left(\lambda_t - a\right), \tag{3.3}$$

with the initial condition $\lambda_t\big|_{t=T_k} = \lambda_{T_k}$. Obviously, (3.3) has the unique solution

$$\lambda_t = \left(\lambda_{T_k^+} - a\right) e^{-\delta(t-T_k)} + a, \quad T_k \le t < T_k + S_{k+1},$$

and the cumulative distribution function of the $(k+1)^{\text{th}}$ interarrival-time S_{k+1} is given by

$$\begin{aligned}
F_{S_{k+1}}(s) &= P\left\{S_{k+1} \le s\right\} \\
&= 1 - P\left\{S_{k+1} > s\right\} \\
&= 1 - P\left\{N_{T_k+s} - N_{T_k} = 0\right\} \\
&= 1 - \exp\left(-\int_{T_k}^{T_k+s} \lambda_u \mathrm{d}u\right) \\
&= 1 - \exp\left(-\int_0^s \lambda_{T_k^+ + v} \mathrm{d}v\right) \\
&= 1 - \exp\left(-\left(\lambda_{T_k^+} - a\right)\frac{1-e^{-\delta s}}{\delta} - as\right).
\end{aligned} \tag{3.4}$$

By the inverse transformation method, we have

$$S_{k+1} \overset{\mathcal{D}}{=} F_{S_{k+1}}^{-1}(U), \quad U \sim [0,1].$$

However, we can avoid inverting this function $F_{S_{k+1}}(\cdot)$ of (3.4) by decomposing S_{k+1} into two simpler and independent random variables $S_{k+1}^{(1)}$ and $S_{k+1}^{(2)}$ via

$$S_{k+1} \overset{\mathcal{D}}{=} S_{k+1}^{(1)} \wedge S_{k+1}^{(2)},$$

where

$$\begin{aligned}
P\left\{S_{k+1}^{(1)} > s\right\} &= \exp\left(-\left(\lambda_{T_k^+} - a\right)\frac{1-e^{-\delta s}}{\delta}\right), \\
P\left\{S_{k+1}^{(2)} > s\right\} &= e^{-as},
\end{aligned}$$

since

$$\begin{aligned}
P\{S_{k+1} > s\} &= \exp\left(-\left(\lambda_{T_k^+} - a\right)\frac{1-e^{-\delta s}}{\delta}\right) \times e^{-as} \\
&= P\left\{S_{k+1}^{(1)} > s\right\} \times P\left\{S_{k+1}^{(2)} > s\right\} \\
&= P\left\{S_{k+1}^{(1)} \wedge S_{k+1}^{(2)} > s\right\}.
\end{aligned}$$

- For the simulation of $S_{k+1}^{(1)}$, since

$$F_{S_{k+1}^{(1)}}(s) = P\left\{S_{k+1}^{(1)} \le s\right\} = 1 - \exp\left(-\left(\lambda_{T_k^+} - a\right)\frac{1-e^{-\delta s}}{\delta}\right),$$

we set

$$\exp\left(-\left(\lambda_{T_k^+}-a\right)\frac{1-e^{-\delta S_{k+1}^{(1)}}}{\delta}\right)\overset{\mathcal{D}}{=}U_1.$$

We can invert this simple function explicitly by

$$S_{k+1}^{(1)}\overset{\mathcal{D}}{=}-\frac{1}{\delta}\ln\left(1+\frac{\delta\ln U_1}{\lambda_{T_k^+}-a}\right). \tag{3.5}$$

Note that, $S_{k+1}^{(1)}$ is a defective random variable as

$$\lim_{s\to\infty}F_{S_{k+1}^{(1)}}(s)=P\left\{S_{k+1}^{(1)}\leq\infty\right\}=1-\exp\left(-\frac{\lambda_{T_k^+}-a}{\delta}\right)<1,$$

and the condition for simulating a valid $S_{k+1}^{(1)}$ is $D_{k+1}>0$ where

$$D_{k+1}=:1+\frac{\delta\ln U_1}{\lambda_{T_k^+}-a}.$$

- For the simulation of $S_{k+1}^{(2)}$, since $S_{k+1}^{(2)}\sim\mathrm{Exp}(a)$, we have

$$S_{k+1}^{(2)}\overset{\mathcal{D}}{=}-\frac{1}{a}\ln U_2. \tag{3.6}$$

Hence, for the simulation of S_{k+1}, we have

$$S_{k+1}\overset{\mathcal{D}}{=}\begin{cases}S_{k+1}^{(1)}\wedge S_{k+1}^{(2)}, & D_{k+1}>0,\\ S_{k+1}^{(2)}, & D_{k+1}<0,\end{cases}$$

where $S_{k+1}^{(1)}$ and $S_{k+1}^{(2)}$ are given by (3.5) and (3.6), respectively.

Therefore, the $(k+1)^{\text{th}}$ jump-time T_{k+1} in the Hawkes process is given by

$$T_{k+1}=T_k+S_{k+1},$$

and the change in λ_t and N_t at time T_{k+1} then can be easily derived as given by (3.1) and (3.2), respectively. □

Algorithm 3.1 applies to any arbitrary distribution assumption G for self-excited jump-sizes $\{Y_k\}_{k=1,2,\ldots}$, and of course also to the case when jump-sizes are fixed which is more widely used in the existing literature. By slightly adjusting the algorithm, it is straightforward to simulate process starting from any time with any arbitrary fixed value or distribution for λ_0, and this flexibility is useful for practical implementations, such as applications in modelling credit risk and insurance risk, see [7] and [8]. This algorithm can generate non-stationary process, and also is flexible to extend to simulate the process with stationary intensity by additionally assuming that λ_0 follows the stationary distribution[4]. For instance, if jump-sizes follow exponential distribution of parameter β, to simulate a stationary process, we only need set the initial intensity by

$$\lambda_0\sim a+\mathrm{Gamma}\left(\frac{a}{\delta},\frac{\delta\beta-1}{\delta}\right).^{5}$$

[4] This stationary distribution has been derived in *Theorem 3.3.* by [7].
[5] This result is based on *Remark 4.3.* of [7].

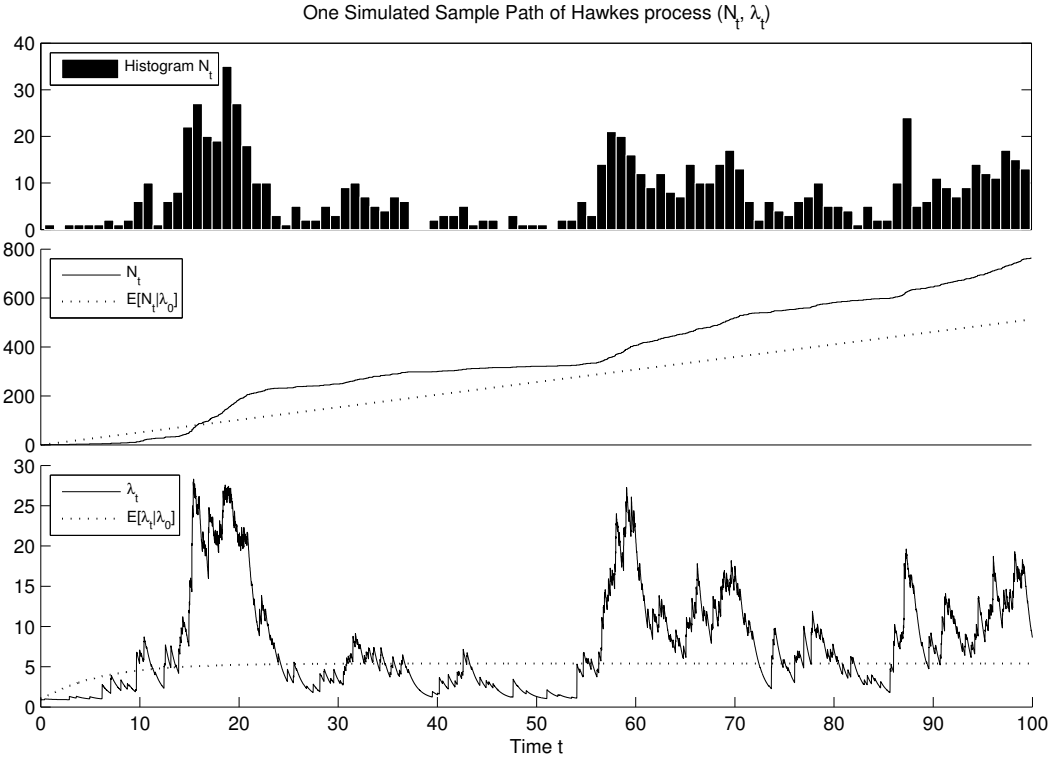

Figure 2: One Simulated Sample Path of Hawkes process (N_t, λ_t)

4 Numerical Examples

For illustration purposes, we assume the sizes of self-excited jumps $\{Y_k\}_{k=1,2,\dots}$ simply follow the exponential distribution, with the density function specified by

$$g(y) = \beta e^{-\beta y}, \quad y, \beta > 0,$$

and then we have the first and second moments $\mu_{1_G} = 1/\beta$ and $\mu_{2_G} = 2/\beta^2$. By setting parameters $(a, \delta; \beta; \lambda_0) = (0.9; 1.0; 1.2; 0.9)$ and using the simulation algorithm in *Algorithm 3.1*, we provide some numerical examples as below.

One Simulated Sample Path of Hawkes process

For instance, one simulated sample path of Hawkes process (N_t, λ_t) with time $T = 100$ is provided in *Figure 2*. We can observe the clustering arrival of points of the Hawkes process by plotting the histogram of N_t, and this contagion effect cannot be captured in classical Poisson models. This would be useful, for instance, for modelling the contagion risk of clustering defaults during economic crisis. For comparison, the theoretical expectations $\mathbb{E}[\lambda_t|\lambda_0]$ and $\mathbb{E}[N_t|\lambda_0]$ (given by *Proposition 2.3*) are also represented.

Comparison between Theoretical Formulas and Simulation Results

To numerically verify our new algorithm, we calculate theoretical formulas of $\mathbb{E}[\lambda_T|\lambda_0]$, $\mathrm{Var}[N_T|\lambda_0]$ and $\mathbb{E}[N_T|\lambda_0]$ explicitly given by *Proposition 2.3*, and compare them with the simulated counterparts, i.e. the mean, variance of simulated λ_T and the mean of

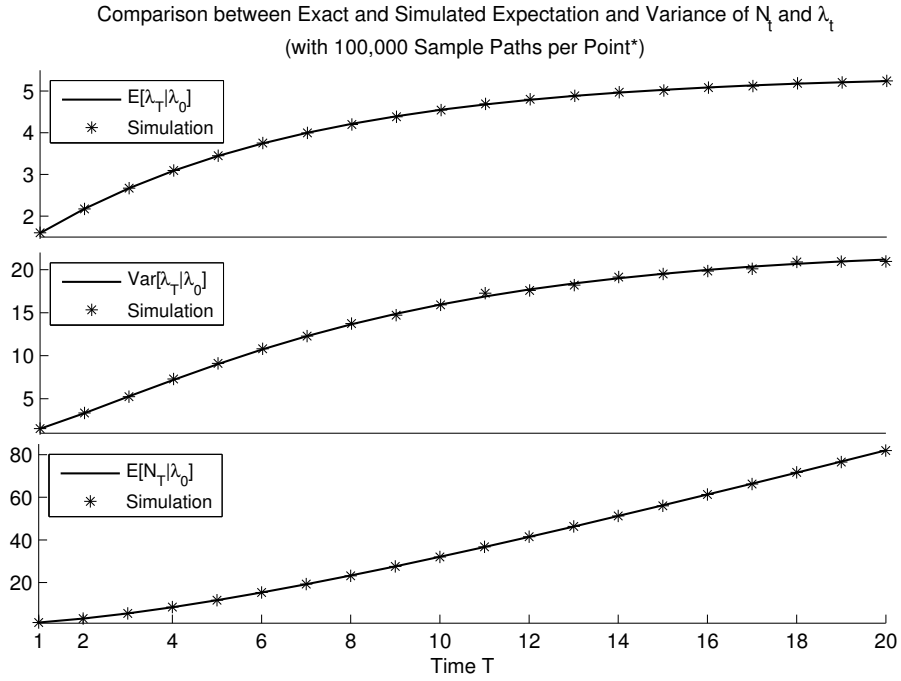

Figure 3: Comparison between Exact and Simulated Expectation and Variance of N_t and λ_t

simulated N_T, respectively, and the results are very close. The comparison figures and error analysis (in error percentage with respect to the exact theoretical result) are represented in *Figure 3* and *Table 1*. Every point (marked by a star $*$) in *Figure 3* is calculated based on 100,000 simulated sample paths of (N_t, λ_t).

5 Multi-dimensional Hawkes Process

By slightly modifying *Algorithm 3.1*, our method is straightforward to extend to multi-dimensional cases, for instance, $\bar{D}-$dimendional point process $\left\{N_t^{[\jmath]}\right\}_{\jmath=1,2,...,\bar{D}}$ where $N_t^{[\jmath]} \equiv \left\{T_k^{[\jmath]}\right\}_{k=1,2,...}$ with the underlying intensity process

$$\lambda_t^{[\jmath]} = a^{[\jmath]} + \left(\lambda_0^{[\jmath]} - a^{[\jmath]}\right) e^{-\delta^{[\jmath]} t} + \sum_{\ell=1}^{\bar{D}} \sum_{0 \le T_k^{[\ell]} < t} Y_k^{[\jmath,\ell]} e^{-\delta^{[\jmath]}\left(t - T_k^{[\ell]}\right)}, \quad \jmath \in \{1, 2, ..., \bar{D}\},$$

where $\left\{Y_k^{[\jmath,\ell]}\right\}_{\jmath=\ell}$ are the sizes of *self-excited jumps* and $\left\{Y_k^{[\jmath,\ell]}\right\}_{\jmath\neq\ell}$ are sizes of *cross-excited jumps*, and they are measurements of the impacts of *self-contagion* and *cross-contagion* respectively. Note that, upon the arrival of a jump in point process $N_t^{[\ell]}$, each marginal intensity process $\left\{\lambda_t^{[\jmath]}\right\}_{\jmath=1,2,...,\bar{D}}$ experiences a simultaneous jump of positive random size, and these jump-sizes could be either independent or dependent. The simulation algorithm is provided by *Algorithm 5.1*.

Algorithm 5.1 (Multivariate)**.** *The simulation algorithm for one sample path of a $\bar{D}-$dimensional Hawkes process with exponentially decaying intensity $\left\{\left(N_t^{[\jmath]}, \lambda_t^{[\jmath]}\right)\right\}_{t\ge 0}$ for $\jmath \in \{1, 2, ..., \bar{D}\}$*

Table 1: Comparison between Theoretical Formulas and Simulation Results

Time T	$\mathbb{E}[\lambda_T\|\lambda_0]$	Simulation	Error%	$\mathrm{Var}[N_T\|\lambda_0]$	Simulation	Error%	$\mathbb{E}[N_T\|\lambda_0]$	Simulation	Error%
1	1.5908	1.5988	0.50%	1.5049	1.5281	1.54%	1.2550	1.2679	1.03%
2	2.1756	2.1718	-0.18%	3.3313	3.3485	0.51%	3.1463	3.1484	0.07%
3	2.6706	2.6691	-0.06%	5.2733	5.2403	-0.63%	5.5763	5.5789	0.05%
4	3.0896	3.0936	0.13%	7.2008	7.2906	1.25%	8.4623	8.4913	0.34%
5	3.4443	3.4494	0.15%	9.0357	9.1034	0.75%	11.7342	11.7482	0.12%
6	3.7445	3.7521	0.20%	10.7346	10.8018	0.63%	15.3327	15.3604	0.18%
7	3.9987	3.9975	-0.03%	12.2770	12.2694	-0.06%	19.2079	19.2744	0.35%
8	4.2138	4.1992	-0.35%	13.6574	13.7337	0.56%	23.3171	23.2009	-0.50%
9	4.3959	4.3819	-0.32%	14.8794	14.7119	-1.13%	27.6245	27.5913	-0.12%
10	4.5501	4.5424	-0.17%	15.9523	15.9158	-0.23%	32.0996	31.9812	-0.37%
11	4.6805	4.6932	0.27%	16.8879	17.2706	2.27%	36.7168	36.7424	0.07%
12	4.7910	4.8033	0.26%	17.6997	17.5879	-0.63%	41.4541	41.5309	0.19%
13	4.8845	4.8829	-0.03%	18.4009	18.2099	-1.04%	46.2931	46.3742	0.18%
14	4.9636	4.9716	0.16%	19.0046	19.1359	0.69%	51.2182	51.2477	0.06%
15	5.0306	5.0106	-0.40%	19.5229	19.4930	-0.15%	56.2163	55.9555	-0.46%
16	5.0873	5.0841	-0.06%	19.9668	19.8292	-0.69%	61.2761	61.2907	0.02%
17	5.1353	5.1213	-0.27%	20.3463	20.0887	-1.27%	66.3880	66.2481	-0.21%
18	5.1760	5.1840	0.15%	20.6702	20.8857	1.04%	71.5443	71.6802	0.19%
19	5.2104	5.2071	-0.06%	20.9462	20.9341	-0.06%	76.7379	76.4919	-0.32%
20	5.2395	5.2414	0.04%	21.1813	20.9516	-1.08%	81.9632	81.8746	-0.11%

conditional on $\lambda_0^{[\jmath]}$ *and* $N_0^{[\jmath]} = 0$*, with* $\bar{K}$ *joint jump-times* $\{T_1, T_2, ..., T_{\bar{K}}\}$ *in intensity processes:*

1. *Set the initial conditions* $T_0 = 0$*,* $\lambda^{[\jmath]}_{T_0^{\pm}} = \lambda_0^{[\jmath]} > a^{[\jmath]}$*,* $N_0^{[\jmath]} = 0$*,* $\jmath \in \{1, 2, ..., \bar{D}\}$ *and* $k \in \{0, 1, 2, ..., \bar{K} - 1\}$*.*

2. *Simulate the* $(k+1)^{\text{th}}$ *interarrival-time* W_{k+1} *by*

$$W_{k+1} = \min\left\{S_{k+1}^{[1]}, S_{k+1}^{[2]}, ..., S_{k+1}^{[\bar{D}]}\right\},$$

   where

$$W_{k+1} = S_{k+1}^{[\ell]},$$

   and each $S_{k+1}^{[\jmath]}$ *can be simulated in the same way as* S_{k+1} *as given by Step 2 of Algorithm 3.1.*

3. *Record the* $(k+1)^{\text{th}}$ *jump-time* T_{k+1} *in the intensity process* $\lambda_t^{[\jmath]}$ *by*

$$T_{k+1} = T_k + W_{k+1}.$$

4. *Record the change at the jump-time* T_{k+1} *in the point process* $N_t^{[\jmath]}$ *by*

$$N^{[\jmath]}_{T^+_{k+1}} = \begin{cases} N^{[\jmath]}_{T^-_{k+1}} + 1, & \jmath = \ell, \\ N^{[\jmath]}_{T^-_{k+1}}, & \jmath \neq \ell, \end{cases} \qquad \jmath \in \{1, 2, ..., \bar{D}\}.$$

5. *Record the change at the jump-time* T_{k+1} *in the intensity process* $\lambda_t^{[\jmath]}$ *by*

$$\lambda^{[\jmath]}_{T^+_{k+1}} = \lambda^{[\jmath]}_{T^-_{k+1}} + Y^{[\jmath,\ell]}_{k+1}, \quad \jmath \in \{1, 2, ..., \bar{D}\},$$

   where

$$\lambda^{[\jmath]}_{T^-_{k+1}} = \left(\lambda^{[\jmath]}_{T^+_k} - a^{[\jmath]}\right) e^{-\delta^{[\jmath]}(T_{k+1} - T_k)} + a^{[\jmath]}.$$

Numerical Example: A Bivariate Hawkes Process

Bivariate point processes are widely used in practice, for instance, for modelling the arrivals of (well-defined) events such as transactions, quote updates or price changes observable in the market, see [12] and [3]. The additional internal and bilateral contagion risk could be further captured by using a bivariate Hawkes process $\left\{\left(N_t^{[1]}, N_t^{[2]}, \lambda_t^{[1]}, \lambda_t^{[2]}\right)\right\}_{t\geq 0}$, for instance, with the joint intensity processes specified by

$$\begin{aligned}
\lambda_t^{[1]} &= a^{[1]} + \left(\lambda_0^{[1]} - a^{[1]}\right) e^{-\delta^{[1]}t} + \sum_{0\leq T_k^{[1]}<t} Y_k^{[1,1]} e^{-\delta^{[1]}\left(t-T_k^{[1]}\right)} + \sum_{0\leq T_k^{[2]}<t} Y_k^{[1,2]} e^{-\delta^{[1]}\left(t-T_k^{[2]}\right)}, \\
\lambda_t^{[2]} &= a^{[2]} + \left(\lambda_0^{[2]} - a^{[2]}\right) e^{-\delta^{[2]}t} + \sum_{0\leq T_k^{[1]}<t} Y_k^{[2,1]} e^{-\delta^{[2]}\left(t-T_k^{[1]}\right)} + \sum_{0\leq T_k^{[2]}<t} Y_k^{[2,2]} e^{-\delta^{[2]}\left(t-T_k^{[2]}\right)}.
\end{aligned}$$

Again, we assume the sizes of jumps in intensity processes follow independent exponential distributions, i.e.

$$Y_k^{[\jmath,\ell]} \sim \mathrm{Exp}(\beta_{\jmath,\ell}), \quad \jmath, \ell \in \{1, 2\},$$

and set parameters by

$$\lambda_0^{[\cdot]} = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}, \quad a^{[\cdot]} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \quad \delta^{[\cdot]} = \begin{bmatrix} 0.8 \\ 1.0 \end{bmatrix}, \quad \beta^{[\cdot,\cdot]} = \begin{bmatrix} 1.5 & 4.0 \\ 8.0 & 2.0 \end{bmatrix}.$$

Here, $\beta^{[1,1]}$, $\beta^{[2,2]}$ provide measurement for self-contagion effect, and $\beta^{[1,2]}$, $\beta^{[2,1]}$ provide measurement for cross-contagion effect. A simulated sample path with $T = 100$ is represented in *Figure 4* by using *Algorithm 5.1*.

References

[1] Yacine Aït-Sahalia, Julio Cacho-Diaz, and Roger J A Laeven, *Modeling financial contagion using mutually exciting jump processes*, Review of Financial Studies (2013), (to appear).

[2] Yacine Aït-Sahalia and Thomas Hurd, *Portfolio choice in markets with contagion*, Working paper. Bendheim Center for Finance at Princeton University, 2012.

[3] Clive G Bowsher, *Modelling security market events in continuous time: Intensity based, multivariate point process models*, Journal of Econometrics **141** (2007), no. 2, 876–912. MR-2413490

[4] Anders Brix and Wilfrid S Kendall, *Simulation of cluster point processes without edge effects*, Advances in Applied Probability **34** (2002), no. 2, 267–280. MR-1909914

[5] E S Chornoboy, L P Schramm, and A F Karr, *Maximum likelihood identification of neural point process systems*, Biological Cybernetics **59** (1988), no. 4-5, 265–275. MR-0961117

[6] D J Daley and D Vere-Jones, *An introduction to the theory of point processes: Volume i: Elementary theory and methods*, Springer, New York, 2002. MR-1950431

[7] Angelos Dassios and Hongbiao Zhao, *A dynamic contagion process*, Advances in Applied Probability **43** (2011), no. 3, 814–846. MR-2858222

[8] ———, *Ruin by dynamic contagion claims*, Insurance: Mathematics and Economics **51** (2012), no. 1, 93–106. MR-2928746

[9] ———, *A dynamic contagion process with diffusion*, Working paper. London School of Economics, 2013.

[10] ———, *A Markov chain model for contagion*, Working paper. London School of Economics, 2013.

[11] Paul Embrechts, Thomas Liniger, and Lu Lin, *Multivariate Hawkes processes: an application to financial data*, Journal of Applied Probability **48A** (2011), 367–378.

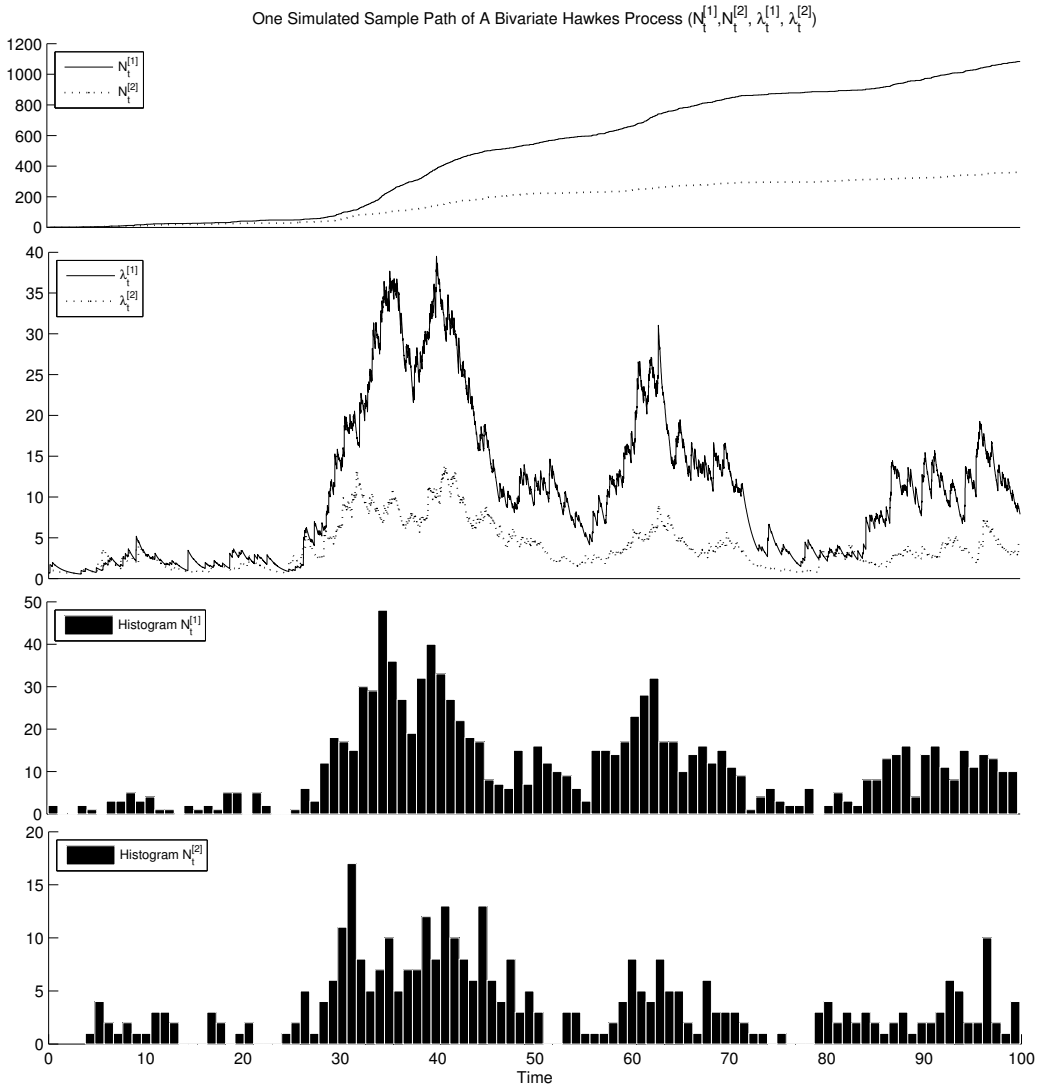

Figure 4: One Simulated Sample Path of Bivariate Hawkes process $\left\{\left(N_t^{[1]}, N_t^{[2]}, \lambda_t^{[1]}, \lambda_t^{[2]}\right)\right\}_{0\leq t\leq 100}$

[12] Robert F Engle and Asger Lunde, *Trades and quotes: a bivariate point process*, Journal of Financial Econometrics **1** (2003), no. 2, 159–188.

[13] Eymen Errais, Kay Giesecke, and Lisa R Goldberg, *Affine point processes and portfolio credit risk*, SIAM Journal on Financial Mathematics **1** (2010), no. 1, 642–665. MR-2719785

[14] Kay Giesecke and Baeho Kim, *Estimating tranche spreads by loss process simulation*, Proceedings of the 2007 Winter Simulation Conference, IEEE Press, 2007, pp. 967–975.

[15] Kay Giesecke, Baeho Kim, and Shilin Zhu, *Monte Carlo algorithms for default timing problems*, Management Science **57** (2011), no. 12, 2115–2129.

[16] Alan G Hawkes, *Spectra of some self-exciting and mutually exciting point processes*, Biometrika **58** (1971), no. 1, 83–90. MR-0278410

[17] Alan G Hawkes and David Oakes, *A cluster process representation of a self-exciting process*, Journal of Applied Probability **11** (1974), 493–503. MR-0378093

[18] Peter A Lewis and Gerald S Shedler, *Simulation of nonhomogeneous Poisson processes by thinning*, Naval Research Logistics Quarterly **26** (1979), no. 3, 403–413. MR-0546120

[19] Thomas Josef Liniger, *Multivariate Hawkes processes*, Ph.D. thesis, Eidgenössische Technische Hochschule (ETH), 2009.

[20] Jesper Møller and Jakob G Rasmussen, *Perfect simulation of Hawkes processes*, Advances in Applied Probability **37** (2005), 629–646. MR-2156552

[21] _____, *Approximate simulation of Hawkes processes*, Methodology and Computing in Applied Probability **8** (2006), no. 1, 53–64. MR-2253076

[22] David Oakes, *The Markovian self-exciting process*, Journal of Applied Probability **12** (1975), 69–77. MR-0362522

[23] Yosihiko Ogata, *On Lewis' simulation method for point processes*, IEEE Transactions on Information Theory **27** (1981), no. 1, 23–31.

[24] _____, *Statistical models for earthquake occurrences and residual analysis for point processes*, Journal of the American Statistical Association **83** (1988), no. 401, 9–27.

[25] Jakob Gulddahl Rasmussen, *Bayesian inference for Hawkes processes*, Methodology and Computing in Applied Probability (2011), 1–20.

Acknowledgments. The authors would like thank anonymous referees for pointing out useful references, and for various helpful suggestions. The corresponding author Hongbiao Zhao also would like thank Baeho Kim (Korea University Business School) for a brief discussion via email, and Xiaoxia Ye (Stockholm University School of Business) for valuable comments.