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A note on tamed Euler approximations

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Abstract

Strong convergence results on tamed Euler schemes, which approximate stochastic differential equations with superlinearly growing drift coefficients that are locally one-sided Lipschitz continuous, are presented in this article. The diffusion coefficients are assumed to be locally Lipschitz continuous and have at most linear growth. Furthermore, the classical rate of convergence, i.e. one–half, for such schemes is recovered when the local Lipschitz continuity assumptions are replaced by global and, in addition, it is assumed that the drift coefficients satisfy polynomial Lipschitz continuity.

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1 Introduction

It is a well-known result that a stochastic differential equation (SDE) with a superlinearly growing drift coefficient has a unique solution if, its drift and diffusion coefficients satisfy a suitable monotone growth condition, so-called coercivity, and a linear growth condition respectively. Typically, suitable local Lipschitz continuity conditions are also required of the coefficients. One could refer to Krylov [8] and the references therein for more details. Moreover, the almost sure convergence and convergence in probability of the corresponding (explicit) Euler approximations were proved by Gyöngy [3]. However, Hutzenthaler, Jentzen and Kloeden [6] showed recently that the absolute moments of (the aforementioned) Euler approximations at a finite time could diverge to infinity as implied in Higham, Mao and Stuart [4]. In other words, the essential property of uniform integrability may not hold for such sequences. Thus, one could not obtain results on strong (in an $\mathcal{L}^p$-sense) approximations although such results exist in the cases of almost sure convergence and convergence in probability. One further realises that the introduction of accelerated Monte Carlo schemes provides a strong incentive for the study of strong approximations of SDEs since, results on the latter are required for the efficient implementation of the former. More information on this topic can be found in Gile's seminal paper [1], Giles and Szpruch [2] and the references therein.

Recently Hutzenthaler, Jentzen and Kloeden [7] introduced the notion of tamed Euler schemes in which the drift term is modified so that it is uniformly bounded. With such an approach, they are able to prove that the tamed Euler scheme converges strongly (with rate one-half) to the exact solution of the SDE if the drift coefficient is globally

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one-sided Lipschitz continuous and has a derivative which grows (at most) polynomially. In addition, they assume that the diffusion coefficient of the SDE satisfies a global Lipschitz condition and it grows at most linearly. Furthermore, they offer a detailed review of the use of implicit schemes and compare them with tamed Euler approximations. Their comparison demonstrates that the implementation of implicit schemes requires significantly more computational effort than the tamed version.

One however observes in [7] that for the “taming" of the drift coefficient, the term n^{-1} is used when it is known from classical literature that the standard strong convergence rate is one-half. In other words, one expects that the use of $n^{-1/2}$ should be sufficient in order to control the drift coefficient and achieve strong convergence of the numerical scheme. In this article, a generalisation of the results of Hutzenthaler, Jentzen and Kloeden [7] is presented by using a variant of their tamed Euler method while a simpler proof is provided. It is proved that, even when global Lipschitz continuity conditions are replaced by local conditions, the tamed Euler scheme converges in $\mathcal{L}^p$ to the exact solution of the SDE. Moreover, as a consequence of the aforementioned generalisation, the classical rate of convergence is obtained under the same assumptions as in [7]. In fact, one further observes that the use of $n^{-\alpha}$, where $\alpha \in (0, 1/2]$, is also suitable for proving $\mathcal{L}^p$ convergence of such tamed schemes, e.g. see Theorem 2.2 below. Naturally, this implies that the proposed tamed coefficients/schemes belong to a large class of functions/schemes which satisfy certain properties. For example, (2.4) from below represents a suitable condition on tamed coefficients so as to achieve uniform moment bounds. Similarly, Hutzenthaler and Jentzen [5] offer results on a class of suitably “tamed" numerical schemes by applying space truncation techniques, e.g. corollary 2.19 in [5].

We conclude this section by introducing some basic notation. The norm of a vector $x \in \mathbb{R}^d$ and the Hilbert-Schmidt norm of a matrix $A \in \mathbb{R}^{d\times m}$ are respectively denoted by $|x|$ and $|A|$. The transpose of a matrix $A \in \mathbb{R}^{d\times m}$ is denoted by A^T and the scalar product of two vectors $x, y \in \mathbb{R}^d$ is denoted by xy. The integer part of a real number x is denoted by $[x]$. Moreover, $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of random variables X with a norm $\|X\|_p := \big(\mathbb{E}\big[|X|^p\big]\big)^{1/p} < \infty$ for $p > 0$. Finally, $\mathcal{B}(V)$ denotes the σ-algebra of Borel sets of a topological space V.

2 Main Result

Let $(\Omega, \{\mathcal{F}_t\}_{\{t\geq 0\}}, \mathcal{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e. the filtration is increasing, right continuous and complete. Let $\{W(t)\}_{\{t\geq 0\}}$ be an m-dimensional Wiener martingale. Furthermore, it is assumed that $b(t, x)$ and $\sigma(t, x)$ are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions which take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$ respectively. For a fixed $T > 0$, let us consider an SDE given by

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \qquad \forall\, t \in [0, T], \tag{2.1}$$

with initial value $X(0)$ which is an almost surely finite $\mathcal{F}_0$-measurable random variable.

For every $n \geq 1$, and any $t \in [0, T]$, the following tamed Euler scheme is defined

$$dX_n(t) = b_n(t, X_n(\kappa_n(t)))dt + \sigma(t, X_n(\kappa_n(t)))dW(t) \tag{2.2}$$

with the same initial value $X(0)$ as SDE (2.1) and $\kappa_n(t) := [nt]/n$. Moreover, it is assumed that

$$b_n(t, x) := \frac{1}{1 + n^{-\alpha}|b(t, x)|} b(t, x), \tag{2.3}$$

for any $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\alpha \in (0, 1/2]$. One then observes that

$$|b_n(t, x)| \leq \min(n^\alpha, |b(t, x)|). \tag{2.4}$$

Moreover, for every $n \geq 1$, one deduces immediately that $b_n(t,x)$ is a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function which take values in $\mathbb{R}^d$.

Let $\mathbb{L}^p$ denote the set of nonnegative p-th integrable functions on $[0,\, T]$, i.e. to say if $f \in \mathbb{L}^p$ then

$$\int_0^T |f(t)|^p dt < \infty.$$

We make the following assumptions.

A-1. There exists a positive constant K such that,

$$2xb(t,x) \vee |\sigma(t,x)|^2 \leq K(1+|x|^2)$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^d$.

A-2. For every $R > 0$, there exists a positive constant L_R such that, for any $t \in [0,T]$,

$$2(x-y)(b(t,x)-b(t,y)) \vee |\sigma(t,x)-\sigma(t,y)|^2 \leq L_R|x-y|^2$$

for all $|x|, |y| \leq R$.

A-3. For every $R \geq 0$ and $p > 0$, there exists $N_R \in \mathbb{L}^p$, such that

$$\sup_{|x|\leq R} |b(t,x)| \leq N_R(t)$$

for any $t \in [0,T]$.

A-4. For every $p > 0$, $\mathbb{E}[|X(0)|^p] < \infty$.

Remark 2.1. *Note that due* (2.4)*, for each $n \geq 1$, the norm of b_n is a bounded function of t and x and, due to **A-1**, the norm of σ has at most linear growth. This fact guarantees the existence of a unique solution to* (2.2)*. Moreover, it guarantees that for each $n \geq 1$, all moments exist, each of which is bounded above by some value that depends on n, i.e.*

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_n(t)|^p] \leq N \tag{2.5}$$

for any $p > 0$, where $N := N(n,p,T,\mathbb{E}|X(0)|^p)$ is a positive constant.

Theorem 2.2. *Suppose **A-1** – **A-4** hold, then the tamed Euler scheme* (2.2) *converges to the true solution of SDE* (2.1) *in $\mathcal{L}^p$-sense, i.e.*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)-X_n(t)|^p\right] = 0$$

for all $p > 0$.

A-5. There exists positive constants l and L such that, for any $t \in [0,T]$,

$$(x-y)(b(t,x)-b(t,y)) \vee |\sigma(t,x)-\sigma(t,y)|^2 \leq L|x-y|^2$$

and

$$|b(t,x)-b(t,y)| \leq L(1+|x|^l+|y|^l)|x-y|$$

for all $x,\, y \in \mathbb{R}^d$.

Corollary 2.3. *Suppose **A-1** and **A-3**–**A-5** hold, then the tamed Euler scheme* (2.2) *with $\alpha = 1/2$ converges to the true solution of SDE* (2.1) *in $\mathcal{L}^p$-sense with order 1/2, i.e.*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)-X_n(t)|^p\right] \leq Cn^{-p/2}$$

for all $p > 0$, where C is a constant independent of n.

3 Moment bounds

Lemma 3.1. *Consider the tamed Euler scheme given by equation (2.2). If for some* $p \geq 2$*,*

$$\sup_{n\geq 1}\sup_{0\leq t\leq T}\mathbb{E}[|X_n(t)|^p] < \infty$$

and ***A-1*** *hold, then*

$$\sup_{0\leq t\leq T}\mathbb{E}[|X_n(t) - X_n(\kappa_n(t))|^p] \leq Cn^{-p/2} \tag{3.1}$$

and

$$\sup_{0\leq t\leq T}\mathbb{E}\Big[|X_n(t) - X_n(\kappa_n(t))|^p|b_n(t, X_n(\kappa_n(t)))|^p\Big] \leq C, \tag{3.2}$$

where C *is a positive constant independent of* n*.*

Proof. One immediately writes

$$\mathbb{E}|X_n(t) - X_n(\kappa_n(t))|^p = \mathbb{E}|\int_{\kappa_n(t)}^t b_n(r, X_n(\kappa_n(r)))dr + \int_{\kappa_n(t)}^t \sigma(r, X_n(\kappa_n(r)))dW(r)|^p$$

for every $t \in [0, T]$, and thus, due to Hölder's inequality,

$$\begin{aligned}\mathbb{E}|X_n(t) - X_n(\kappa_n(t))|^p \leq& 2^{p-1}|t-\kappa_n(t)|^{p-1}\mathbb{E}\int_{\kappa_n(t)}^t |b_n(r, X_n(\kappa_n(r)))|^p dr \\ &+ 2^{p-1}\mathbb{E}|\int_{\kappa_n(t)}^t \sigma(r, X_n(\kappa_n(r)))dW(r)|^p. \end{aligned} \tag{3.3}$$

One then observes that,

$$2^{p-1}|t-\kappa_n(t)|^{p-1}\mathbb{E}\int_{\kappa_n(t)}^t |b_n(r, X_n(\kappa_n(r)))|^p dr \leq 2^{p-1}n^{(\alpha-1)p} \tag{3.4}$$

and since **A-1** holds and $\sup_{n\geq 1}\sup_{t\leq T}\mathbb{E}[|X_n(t)|^p] < \infty$, for some $p \geq 2$, then

$$\mathbb{E}\Big[\Big|\int_{\kappa_n(t)}^t \sigma(r, X_n(\kappa_n(r)))dW(r)\Big|^p\Big] \leq CE\Big[\int_{\kappa_n(t)}^t (1 + |X_n(\kappa_n(r))|^2)dr\Big]^{p/2} \leq Cn^{-p/2}, \tag{3.5}$$

where C denotes some positive (general) constant which is independent of n and t. Substituting (3.4) and (3.5) in (3.3) yields (3.1). Furthermore, (3.2) holds trivially, since

$$\mathbb{E}\Big[|X_n(t) - X_n(\kappa_n(t))|^p|b_n(t, X_n(\kappa_n(t)))|^p\Big] \leq \mathbb{E}\Big[|X_n(t) - X_n(\kappa_n(t))|^p\Big]n^{\alpha p} \leq C$$

is true, for any $t \in [0, T]$, due to (3.1). □

Lemma 3.2. *Suppose that* ***A-1*** *and* ***A-4*** *hold, then for some* $C := C(T, K, \mathbb{E}[|X(0)|^2])$*,*

$$\sup_{n\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\Big[|X_n(t)|^2\Big] < C. \tag{3.6}$$

Proof. Let us define

$$I_n(T) := \mathbb{E}\Big[\int_0^T (X_n(s) - X_n(\kappa_n(s))b_n(s, X_n(\kappa_n(s)))ds\Big].$$

Then, one calculates

$$\begin{aligned} I_n(T) &= \mathbb{E}\Big[\int_0^T \Big(\int_{\kappa_n(s)}^s b_n(r, X_n(\kappa_n(r)))dr + \int_{\kappa_n(s)}^s \sigma(r, X_n(\kappa_n(r)))dW(r)\Big) b_n(s, X_n(\kappa_n(s)))ds\Big] \\ &= \sum_{k=0}^{n([T]+1)} \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge T} \mathbb{E}\Big[b_n(s, X_n(k/n))\mathbb{E}\Big(\int_{\frac{k}{n}}^s b_n(r, X_n(k/n))dr + \int_{\frac{k}{n}}^s \sigma(r, X_n(k/n))dW(r)\Big|\mathcal{F}_{\frac{k}{n}}\Big)\Big]ds \\ &= \mathbb{E}\Big[\int_0^T b_n(s, X_n(\kappa_n(s))) \int_{\kappa_n(s)}^s b_n(r, X_n(\kappa_n(r)))drds\Big] \end{aligned}$$

and thus

$$|I_n(T)| \leq \mathbb{E}\Big[\int_0^T |b_n(s, X_n(\kappa_n(s)))| \int_{\kappa_n(s)}^s |b_n(r, X_n(\kappa_n(r)))|drds\Big] \leq Tn^{2\alpha-1} \leq T. \tag{3.7}$$

Furthermore, Itô's formula gives

$$\begin{aligned} |X_n(t)|^2 =& |X(0)|^2 + 2\int_0^t X_n(s)b_n(s, X_n(\kappa_n(s)))ds + \int_0^t |\sigma(s, X_n(\kappa_n(s)))|^2 ds \\ &+ 2\int_0^t X_n(s)\sigma(s, X_n(\kappa_n(s)))dW(s) \\ \leq& |X(0)|^2 + 2\int_0^t X_n(\kappa_n(s))b_n(s, X_n(\kappa_n(s)))ds + \int_0^t |\sigma(s, X_n(\kappa_n(s)))|^2 ds \\ &+ 2\int_0^t (X_n(s) - X_n(\kappa_n(s))b_n(s, X_n(\kappa_n(s)))ds + 2\int_0^t X_n(s)\sigma(s, X_n(\kappa_n(s)))dW(s) \end{aligned} \tag{3.8}$$

and thus, due to **A-1**, (2.5) and (3.7), for any $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}|X_n(t)|^2 \leq& C(1 + \mathbb{E}|X(0)|^2 + \mathbb{E}\int_0^t |X_n(\kappa_n(s))|^2 ds) \\ \leq& C(1 + \mathbb{E}|X(0)|^2 + \int_0^t \sup_{0\leq u\leq s} \mathbb{E}|X_n(u)|^2 ds) \end{aligned}$$

which implies,

$$\sup_{0\leq u\leq t} \mathbb{E}|X_n(u)|^2 \leq C(1 + \mathbb{E}|X(0)|^2 + \int_0^t \sup_{0\leq u\leq s} \mathbb{E}|X_n(u)|^2 ds) < \infty$$

where the positive (general) constant C is independent of n. One then observes that the application of Gronwall's lemma yields

$$\sup_{0\leq u\leq T} \mathbb{E}|X_n(u)|^2 < C$$

where $C := C(T, K, \mathbb{E}[|X(0)|^2])$. □

Lemma 3.3. *Suppose that **A-1** and **A-4** holds, then for some $C := C(p, T, K, \mathbb{E}[|X(0)|^p])$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^p\right] \vee \sup_{n\geq 1} \mathbb{E}\left[\sup_{0\leq t\leq T} |X_n(t)|^p\right] < C \tag{3.9}$$

for every $p > 0$.

Proof. It is well known in the literature that the result

$$\mathbb{E}\left[\sup_{0\le t\le T}|X(t)|^p\right] < C$$

holds for every $p > 0$. One could consult, for example, Krylov (1980) for more details.

In order to prove the second part of (3.9), an inductive argument is used below. First, one chooses $p = 2$ and observes that due to Lemma 3.2 that

$$\sup_{n\ge 1}\sup_{0\le t\le T}\mathbb{E}|X_n(t)|^2 < C$$

holds for some positive constant $C := C(T, K, \mathbb{E}[|X(0)|^2])$ which is independent of n. Thus, (3.2) from Lemma 3.1 holds true for $p = 2$ and one could use (3.8) to obtain the following estimate for $q = 2p$, i.e. $q = 4$,

$$\begin{aligned}\mathbb{E}[\sup_{0\le s\le t}|X_n(s)|^q] \le & C(1+\mathbb{E}[|X(0)|^q] + \int_0^t \mathbb{E}|X_n(\kappa_n(s))|^q ds \\ & + \int_0^t \mathbb{E}[|X_n(s)-X_n(\kappa_n(s))|^{q/2}|b_n(s,X_n(\kappa_n(s)))|^{q/2}]ds \\ & + \mathbb{E}[\sup_{0\le s\le t}|\int_0^s X_n(u)\sigma(u,X_n(\kappa_n(u)))dW(u)|^{q/2}])\end{aligned} \tag{3.10}$$

and the application of the Burkholder-Davis-Gundy (BDG) inequality yields

$$\begin{aligned}\mathbb{E}[\sup_{0\le s\le t}|X_n(s)|^q] \le & C\{1+\mathbb{E}[|X(0)|^q] + \int_0^t \mathbb{E}[\sup_{0\le u\le s}|X_n(u)|^q]ds \\ & + \mathbb{E}[(\int_0^t |X_n(s)|^2|\sigma(s,X_n(\kappa_n(s)))|^2 ds)^{q/4}]\},\end{aligned}$$

where C denotes again a general constant which is independent of n. Thus, the application of Young's inequality yields

$$\begin{aligned}\mathbb{E}[\sup_{0\le s\le t}|X_n(s)|^q] \le & C\{1+\mathbb{E}[|X(0)|^q] + \int_0^t \mathbb{E}[\sup_{0\le u\le s}|X_n(u)|^q]ds + \frac{1}{2C}\mathbb{E}[\sup_{0\le s\le t}|X_n(s)|^q] \\ & + \frac{C}{2}\mathbb{E}[(\int_0^t |\sigma(s,X_n(\kappa_n(s)))|^2 ds)^{q/2}]\}\end{aligned}$$

which, due to **A-1** and Hölder's inequality implies that

$$\mathbb{E}[\sup_{0\le u\le t}|X_n(u)|^q] \le C(1+\mathbb{E}[|X(0)|^q] + \int_0^t \mathbb{E}[\sup_{0\le u\le s}|X_n(u)|^q]ds) < \infty$$

and thus the application of Gronwall's lemma yields that

$$\mathbb{E}[\sup_{0\le t\le T}|X_n(t)|^q] < C \tag{3.11}$$

holds for some positive constant $C := C(q, T, K, \mathbb{E}|X(0)|^q)$ which is independent of n. Thus, (3.2) from Lemma 3.1 holds true for $p = 4$ and one could use (3.10) to obtain the estimate (3.11) for $q = 2p$, i.e. $q = 8$. Repeating the same procedure (by induction) one obtains the desired result (3.9). □

4 Proof of Main Result

For every $R > 0$ and $n \geq 1$, let us consider the stopping times

$$\tau_R := \inf\{t \geq 0 : |X(t)| \geq R\},\ \rho_{nR} := \inf\{t \geq 0 : |X_n(t)| \geq R\} \text{ and } \quad \nu_{nR} := \tau_R \wedge \rho_{nR}. \tag{4.1}$$

Lemma 4.1. *Suppose that* ***A-3*** *holds, then for any $R > 0$ and $p > 0$*

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T |b(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - b_n(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds\Big] = 0. \tag{4.2}$$

Proof. One immediately observes that for $p \geq 2$

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T |b(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - b_n(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds\Big] \\
\leq& n^{-\alpha p}\mathbb{E}\Big[\int_0^T \frac{|b(s \wedge \nu_{nR}, X_n(\kappa_n(s) \wedge \nu_{nR}))|^{2p}}{(1 + n^{-\alpha}|b(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|)^p} ds\Big] < \infty
\end{aligned} \tag{4.3}$$

due to **A-3**. Thus the application of the dominated convergence theorem yields the desired result. □

Proof of theorem 2.2. Let $p \geq 2$ and consider

$$\chi_n(s) := X(s \wedge \nu_{nR}) - X_n(s \wedge \nu_{nR}).$$

One observes immediately that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t) - X_n(t)|^p\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |X(t) - X_n(t)|^p\, \mathbf{1}_{\{\tau_R \leq T \text{ or } \rho_{nR}\leq T\}}\right] + \mathbb{E}\left[\sup_{0\leq t\leq T} |\chi_n(s)|^p\right]. \tag{4.4}$$

Then, by the application of Young's inequality for $q > p$ and $\eta > 0$ one obtains

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t) - X_n(t)|^p\, \mathbf{1}_{\{\tau_R \leq T \text{ or } \rho_{nR}\leq T\}}\right] \leq& \frac{\eta p}{q}\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t) - X_n(t)|^q\right] \\
&+ \frac{q-p}{q\eta^{p/(q-p)}}\mathbb{P}(\tau_R \leq T \text{ or } \rho_{nR} \leq T) \\
\leq& \frac{\eta p}{q} 2^q C + \frac{q-p}{q\eta^{p/(q-p)}}\left\{\mathbb{E}\left[\frac{|X(\tau_R)|^p}{R^p}\right] + \mathbb{E}\left[\frac{|X_n(\rho_{nR})|^p}{R^p}\right]\right\} \\
\leq& \frac{\eta p}{q} 2^q C + \frac{q-p}{q\eta^{p/(q-p)} R^p} 2C.
\end{aligned} \tag{4.5}$$

Furthermore, one defines

$$\beta_n(s) := \Big(b(s, X(s)) - b_n(s, X_n(\kappa_n(s)))\Big)\, \mathbf{1}_{[s\leq\nu_{nR}]}$$

and

$$\alpha_n(s) := \Big(\sigma(s, X(s)) - \sigma(s, X_n(\kappa_n(s)))\Big)\, \mathbf{1}_{[s\leq\nu_{nR}]}$$

to obtain

$$|\chi_n(t)|^2 = \int_0^t \Big[2\chi_n(s)\beta_n(s) + |\alpha_n(s)|^2\Big] ds + 2\int_0^t \chi_n(s)\alpha_n(s) dW(s) \tag{4.6}$$

with

$$\chi_n(s)\beta_n(s) = \Big\{(X(s) - X_n(\kappa_n(s))(b(s,X(s)) - b(s,X_n(\kappa_n(s))))$$
$$+ (X(s) - X_n(\kappa_n(s)))(b(s,X_n(\kappa_n(s))) - b_n(s,X_n(\kappa_n(s))))$$
$$+ (X_n(\kappa_n(s)) - X_n(s))(b(s,X(s)) - b_n(s,X_n(\kappa_n(s))))\Big\}\mathbf{1}_{[s\le\nu_{nR}]} \qquad (4.7)$$

which implies, due to **A-2** and **A-3**,

$$\chi_n(s)\beta_n(s) \le J_n(s) := \Big\{(2L_R+1)|\chi_n(s)|^2 + (2L_R+1)|X_n(s) - X_n(\kappa_n(s))|^2$$
$$+ |b(s,X_n(\kappa_n(s))) - b_n(s,X_n(\kappa_n(s)))|^2$$
$$+ 2N_R(t)|X_n(s) - X_n(\kappa_n(s))|\Big\}\mathbf{1}_{[s\le\nu_{nR}]}, \qquad (4.8)$$

whereas

$$|\alpha_n(s)|^p \le 2^{p-1}\Big\{L_R^{p/2}|\chi_n(s)|^p + L_R^{p/2}|X_n(s) - X_n(\kappa_n(s))|^p\Big\}\mathbf{1}_{[s\le\nu_{nR}]}. \qquad (4.9)$$

Furthermore, in view of the above estimate (4.9), one observes that the application of Young's inequality yields

$$|\chi_n(s)\alpha_n(s)|^{p/2} \le 2^{p/2}\Big\{(2L_R^{p/4}+1)|\chi_n(s)|^p + L_R^{p/4}|X_n(s) - X_n(\kappa_n(s))|^p\Big\}\mathbf{1}_{[s\le\nu_{nR}]}. \quad (4.10)$$

Thus, from (4.6) one obtains by applying Hölder and BDG inequalities that

$$\mathbb{E}[\sup_{0\le u\le t}|\chi_n(u)|^p] \le C\mathbb{E}\int_0^t \Big[|J_n(s)|^{p/2} + |\alpha_n(s)|^p + |\chi_n(s)\alpha_n(s)|^{p/2}\Big]ds,$$

where $C := C(p,T)$ is a positive constant, which in view of (4.7), (4.9) and (4.10) yields

$$\mathbb{E}[\sup_{0\le u\le t}|\chi_n(u)|^p] \le C_R\int_0^t \Big[\mathbb{E}|\chi_n(s)|^p + \mathbb{E}|X_n(s\wedge\nu_{nR}) - X_n(\kappa_n(s)\wedge\nu_{nR})|^p\Big]ds,$$
$$+ C\Big[\mathbb{E}\int_0^{t\wedge\nu_{nR}} |b(s,X_n(\kappa_n(s))) - b_n(s,X_n(\kappa_n(s)))|^p ds$$
$$+ \int_0^t N_R^{p/2}(s)\mathbb{E}|X_n(s\wedge\nu_{nR}) - X_n(\kappa_n(s)\wedge\nu_{nR})|^{p/2}ds\Big], \qquad (4.11)$$

where $C_R := C_R(p,T,L_R)$ and (the redefined) $C := C(p,T)$ are positive consants. The application of Grownwall inequality results in

$$\lim_{n\to\infty}\mathbb{E}[\sup_{0\le t\le T}|\chi_n(t)|^p] = 0$$

for every $R > 0$ due to Lemmas 3.1 and 4.1. Finally, given an $\epsilon > 0$, one can choose η small enough so

$$\frac{\eta p}{q}2^q C < \frac{\epsilon}{3},$$

R large enough so

$$\frac{q-p}{q\eta^{p/(q-p)}R^p}2C < \frac{\epsilon}{3}$$

and n large enough so

$$\mathbb{E}[\sup_{0\le t\le T}|\chi_n(t)|^p] < \frac{\epsilon}{3}$$

to obtain due to (4.4) and (4.5) that

$$\mathbb{E}\left[\sup_{0\le t\le T}|X(t) - X_n(t)|^p\right] < \epsilon$$

and thus prove the desired result. □

5 Rate of Convergence

First one observes that if **A-4** and **A-5** hold, then

$$|b(t,x)| \leq |b(t,x) - b(t,0)| + |b(t,0)| \leq L(1+|x|^l)|x| + N_0(t) \leq N(t)(1+|x|^{l+1}) \tag{5.1}$$

for any $t \in [0, T]$ and $x \in \mathbb{R}^d$, where $N(t) \in \mathbb{L}^p$ for any $p > 0$.

Proof of Corollary 2.3. First one rewrites (4.7) in the following way

$$\begin{aligned}\chi_n(s)\beta_n(s) = \Big\{ & (X(s) - X_n(s))(b(s, X(s)) - b(s, X_n(s))) \\ & + (X(s) - X_n(s))(b(s, X_n(s)) - b(s, X_n(\kappa_n(s)))) \\ & + (X(s) - X_n(s))(b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s))))\Big\} \mathbf{1}_{[s \leq \nu_{nR}]}\end{aligned} \tag{5.2}$$

and adjusts accordingly, due to **A-5**, the upper bound J_n from (4.8)

$$\begin{aligned}\chi_n(s)\beta_n(s) \leq J_n(s) := \Big\{ & (L+1)|\chi_n(s)|^2 + L^2(1+|X_n(s)|^l + |X_n(\kappa_n(s))|^l)^2 |X_n(s) - X_n(\kappa_n(s))|^2 \\ & + |b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s)))|^2 \Big\} \mathbf{1}_{[s \leq \nu_{nR}]}\end{aligned} \tag{5.3}$$

and, thus, the last term of (4.11) is replaced by

$$\mathcal{E}(t) := \mathbb{E}\Big[\int_0^{t\wedge \nu_{nR}} C(1+|X_n(s)|^{lp} + |X_n(\kappa_n(s))|^{lp})|X_n(s) - X_n(\kappa_n(s))|^p\Big] ds$$

which is estimated from above by

$$\mathcal{E}(t) \leq C \int_0^t \Big(\sqrt{\mathbb{E}|X_n(s\wedge \nu_{nR}) - X_n(\kappa_n(s)\wedge \nu_{nR})|^{2p}}\Big) ds$$

due to Hölder's inequality and (3.9). Note that the general constant C is independent of t and n. In view of Lemma 3.1, one deduces that

$$\sup_{0\leq t\leq T} \mathcal{E}(t) \leq C n^{-p/2}. \tag{5.4}$$

Furthermore, (4.3), (3.9) and (5.1) imply that

$$\mathbb{E}[\int_0^T |b(s\wedge \nu_{nR}, X_n(\kappa_n(s\wedge \nu_{nR}))) - b_n(s\wedge \nu_{nR}, X_n(\kappa_n(s\wedge \nu_{nR})))|^p] \leq C n^{-p/2} \tag{5.5}$$

which along with (3.1), (5.4) and (5.5) result in

$$\mathbb{E}[\sup_{0\leq t\leq T} |\chi_n(t)|^p] \leq C n^{-p/2} \tag{5.6}$$

due to (4.11). Finally, one chooses $\eta = n^{-\frac{p}{2}}$, $R = n^{\frac{q}{2(q-p)}}$, $q > p \geq 2$, to obtain the desired result due to (4.4), (4.5) and (5.6).

□

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