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GREEN FUNCTIONS AND MARTIN COMPACTIFICATION FOR KILLED RANDOM WALKS RELATED TO SU(3)

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Abstract
We consider the random walks killed at the boundary of the quarter plane, with homogeneous non-zero jump probabilities to the eight nearest neighbors and drift zero in the interior, and which admit a positive harmonic polynomial of degree three. For these processes, we find the asymptotic of the Green functions along all infinite paths of states, and from this we deduce that the Martin compactification is the one-point compactification.

1 Introduction and main results

First introduced for Brownian motion by R. Martin in 1941, the concept of Martin compactification has then been extended for countable discrete time Markov chains by J. Doob and G. Hunt at the end of the fifties. The purpose of this theory is to describe the asymptotic behavior of the Markov chains and also to characterize all their non-negative superharmonic and harmonic functions, see *e.g.* [6].

For a transient Markov chain with state space E, the *Martin compactification* of E is the smallest compactification $\hat{E}$ of E for which the Martin kernels $z \mapsto k_z^{z_0} = G_z^{z_0}/G_z^{z_1}$ extend continuously – by $G_z^{z_0}$ we mean the *Green functions* of the process, *i.e.* the mean number of visits made by the process at z starting at z_0, and we note z_1 a reference state. $\hat{E} \setminus E$ is usually called the *full Martin boundary*. Clearly, for $\alpha \in \hat{E}$, $z_0 \mapsto k_\alpha^{z_0}$ is superharmonic ; then $\partial_m E = \{\alpha \in \hat{E} \setminus E : z_0 \mapsto k_\alpha^{z_0} \text{ is minimal harmonic}\}$ is called the *minimal Martin boundary* – a harmonic function h is said minimal if $0 \leq \tilde{h} \leq h$ with $\tilde{h}$ harmonic implies $\tilde{h} = ch$ for some constant c. Then, every superharmonic (resp. harmonic) function h can be written as $h(z_0) = \int_{\hat{E}} k_z^{z_0} \mu(\mathrm{d}z)$ (resp. $h(z_0) = \int_{\partial_m E} k_z^{z_0} \mu(\mathrm{d}z)$), where μ is some finite measure, uniquely characterized in the second case above.

The case of the *homogeneous* random walks in $\mathbb{Z}^d$ is now completely understood. First, their minimal Martin boundary is found in [5], thanks to Choquet-Deny theory. Furthermore, in the case of a *non-zero drift*, P. Ney and F. Spitzer find, in their well-known paper [11], the asymptotic of the Green functions, by using exponential changes of measure and the local limit theorem ; this gives consequently the concrete realization of the Martin compactification, in that case the sphere. Additionally, in the case of a *drift zero*, the asymptotic of the Green functions is computed in [14] ; it follows that the Martin compactification consists in the one-point compactification.

Results on Martin boundary for *non-homogeneous* random walks are scarcer and more recent. We concentrate here our analysis on important and recently extensively studied examples that are the random walks in $\mathbb{Z}^d$ killed at the boundary of cones. They are related to many areas of probability, as *e.g.* to non-colliding random walks or quantum processes.

On the one hand, the case of the *non-zero drift* is now rather well studied.
In [3], P. Biane considers quantum random walks on the dual of compact Lie groups and, by restriction, arrives at classical random walks with non-zero drift killed at the boundary of the Weyl chamber of the dual. Solving an equation of Choquet-Deny type, he finds the minimal Martin boundary of these processes.
When the compact Lie group is SU(d) and the associated random walk has non-zero drift, the Martin compactification is obtained in [4], by finding the asymptotic of the Green kernels.
Recently, in [8], I. Ignatiouk-Robert obtains the Martin compactification of the random walks in $\mathbb{Z}_+^d$ with non-zero drift and killed at the boundary. She uses there an original approach based on large deviations theory in order to compute the asymptotic of the Martin kernels. Unfortunately, her methods seem quite difficult to extend up to the case of the drift zero. Also, they do not provide the asymptotic of the Green functions.
This asymptotic in the case of the dimension $d = 2$ is found in [10].

On the other hand, results on Martin boundary for killed random walks with *drift zero* are quite rare. The simplest example of the cartesian product is due to [12]. A more interesting case comes again from quantum processes : in [2], P. Biane shows that the minimal Martin boundary of the random walk with zero drift and killed at the boundary of the Weyl chamber of the dual of SU(d) is reduced to one point.

It is immediate from [1] that this classical random walk in the Weyl chamber of the dual of SU(d) is, for $d = 3$, the random walk spatially homogeneous on the lattice $\{i + j\exp(\imath\pi/3),\ (i, j) \in \mathbb{Z}^2\}$ with jump probabilities as represented on the left of Picture 1.

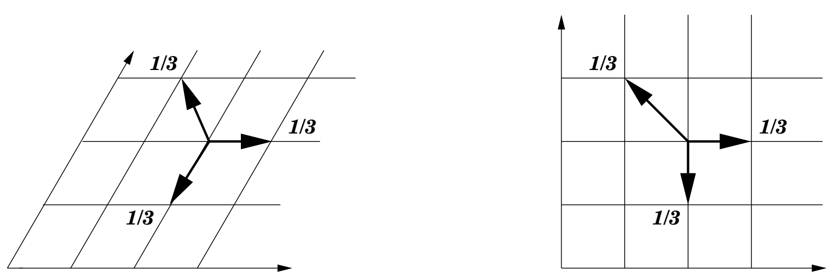

Figure 1: Random walk in the Weyl chamber of the dual of SU(3)

Obviously, a suitable linear transformation maps the lattice $\{i + j\exp(\imath\pi/3) : (i, j) \in \mathbb{Z}^2\}$ into $\mathbb{Z}^2$, see Picture 1 ; in this way, the Weyl chamber $\{i+j\exp(\imath\pi/3) : (i, j) \in \mathbb{Z}_+^2\}$ becomes $\mathbb{Z}_+^2$. For $d = 3$, the killed random walk considered by P. Biane in [1] can therefore be viewed as an element of

$\mathscr{P} = \{$random walks in $\mathbb{Z}_+^2$ with non-zero jump probabilities $(p_{i,j})_{-1\leq i,j\leq 1}$

to the eight nearest neighbors, with drift zero and killed at the boundary}

with jump probabilities as represented on the right of Picture 1 – above, "drift zero" means that $p_{1,1}+p_{1,0}+p_{1,-1}=p_{-1,1}+p_{-1,0}+p_{-1,-1}$ and $p_{1,1}+p_{0,1}+p_{-1,1}=p_{1,-1}+p_{0,-1}+p_{-1,-1}$. In this setting, P. Biane proves, in [2], that $(i_0,j_0)\mapsto i_0j_0(i_0+j_0)$ is the only positive harmonic function for this process.

By the same methods, it can certainly be shown that there is only one positive harmonic function for the "dual" walk, namely for the random walk with jump probabilities $p_{-1,0}=p_{0,1}=p_{1,-1}=1/3$. In particular, if we set $\mathscr{P}^c=\{$random walks of $\mathscr{P}$ such that $p_{0,-1}=p_{-1,1}=p_{1,0}=\mu$, $p_{-1,0}=p_{0,1}=p_{1,-1}=\nu$, $\mu+\nu=1/3\}$ – in other words, $\mathscr{P}^c$ is the set of all cartesian products of the random walk on the dual of SU(3) with its dual, see on the left of Picture 2 below –, it follows from [12] that any process of $\mathscr{P}^c$ has also a minimal Martin boundary reduced to one point.

In this paper, we introduce the set

$$\mathscr{P}_{1,0}=\{\text{random walks of } \mathscr{P} \text{ for which } (i_0,j_0)\mapsto i_0j_0(i_0+j_0) \text{ is harmonic}\}.$$

Note that we have $\mathscr{P}^c\subset\mathscr{P}_{1,0}$, but we will see, in Remark 4, that the inclusion is strict. More generally, we define

$$\mathscr{P}_{\alpha,\beta}=\{\text{random walks of } \mathscr{P} \text{ for which } (i_0,j_0)\mapsto i_0j_0(i_0+\alpha j_0+\beta) \text{ is harmonic}\}. \qquad (1)$$

Since any harmonic function for a killed process takes the value zero on the boundary, $\mathscr{P}_{\alpha,\beta}$ is in fact exactly the set of *all* killed random walks in $\mathbb{Z}_+^2$ to the eight nearest neighbors admitting a harmonic polynomial of degree three.

The description of the set $\mathscr{P}_{\alpha,\beta}$ in terms of the $(p_{i,j})_{i,j}$ is rather cumbersome but not difficult to obtain, it is postponed until Remark 4. Let us just note here that if $\alpha>2$ or $\alpha<1/2$, then for all β, $\mathscr{P}_{\alpha,\beta}=\emptyset$; if $\alpha=1/2$ or $\alpha=2$, then for all $\beta\neq 0$, $\mathscr{P}_{\alpha,\beta}=\emptyset$, and $\mathscr{P}_{\alpha,0}$ is reduced to one walk ; and if $\alpha\in]1/2,2[$ and $|\beta|$ is small enough, then $\mathscr{P}_{\alpha,\beta}$ is a (non-empty) set with two free parameters, properly described in Remark 4. We have represented on the right of Picture 2 an example of a process belonging to $\mathscr{P}_{\alpha,0}$, for any $\alpha\in[1/2,2]$.

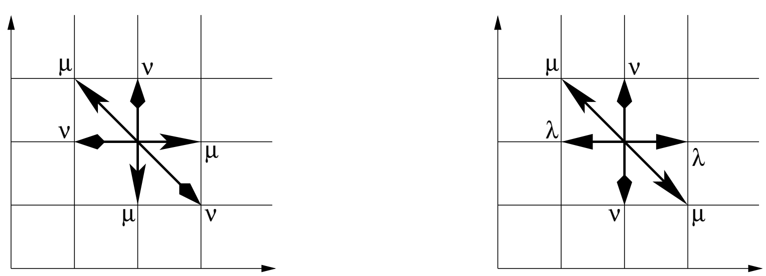

Figure 2: On the left, a generic walk of $\mathscr{P}^c$ $(\mu+\nu=1/3)$; on the right, an example of walk of $\mathscr{P}_{\alpha,0}$ $(\lambda=\alpha(\alpha-1/2)/[2-\alpha+2\alpha^2]$, $\mu=(\alpha/2)/[2-\alpha+2\alpha^2]$ and $\nu=(1-\alpha/2)/[2-\alpha+2\alpha^2])$

Moreover, note that considering in this paper $\mathscr{P}_{\alpha,\beta}$ is all the more natural as the set {random walks of $\mathscr{P}$ for which $(i_0,j_0)\mapsto i_0j_0$ is harmonic} is studied in [13].

Our first result deals with the Green functions – below, (X,Y) denotes the coordinates of the random walk and $\mathbb{E}_{(i_0,j_0)}$ the conditional expectation given $(X(0),Y(0))=(i_0,j_0)$ –

$$G_{i,j}^{i_0,j_0}=\mathbb{E}_{(i_0,j_0)}\left[\sum_{k\geq 0}\mathbf{1}_{\{(X(k),Y(k))=(i,j)\}}\right], \qquad (2)$$

and, more precisely, with their asymptotic along *all* paths of states.

Theorem 1. *Suppose that the process belongs to $\mathscr{P}_{\alpha,\beta}$. Then the Green functions (2) admit the following asymptotic as $i+j\to\infty$ and $j/i\to\tan(\gamma)$, γ lying in $[0,\pi/2]$:*

$$G_{i,j}^{i_0,j_0}\sim C i_0 j_0\left(i_0+\alpha j_0+\beta\right)\frac{ij(i+\alpha j)}{\left[i^2+\alpha ij+\alpha^2 j^2\right]^3}, \tag{3}$$

where $C>0$ depends only on the parameters $(p_{i,j})_{i,j}$ and is made explicit in the proof.

In the particular case of the random walk killed at the boundary of the Weyl chamber of the dual of SU(3), the asymptotic (3) is, for $\gamma\in]0,\pi/2[$, proved in [1]. Theorem 1 actually completes this result for that very particular random walk and, in fact, gives the asymptotic of the Green functions for a much larger class of processes.
In addition, Theorem 1 has the following consequence.

Corollary 2. *The Martin compactification of any process belonging to $\mathscr{P}_{\alpha,\beta}$ is the one-point compactification.*

Furthermore, the asymptotic (3) of the Green functions in the two limit cases $\gamma=0$ and $\gamma=\pi/2$ enables us to obtain the asymptotic of the absorption probabilities

$$\begin{aligned} h_i^{i_0,j_0} &= \mathbb{P}_{(i_0,j_0)}\left[\exists k\geq 1:(X(k),Y(k))=(i,0)\right], \\ \widetilde{h}_j^{i_0,j_0} &= \mathbb{P}_{(i_0,j_0)}\left[\exists k\geq 1:(X(k),Y(k))=(0,j)\right]. \end{aligned} \tag{4}$$

Indeed, the absorption probabilities (4) are related to the Green functions (2) through

$$\begin{aligned} h_i^{i_0,j_0} &= p_{1,-1}G_{i-1,1}^{i_0,j_0}+p_{0,-1}G_{i,1}^{i_0,j_0}+p_{-1,-1}G_{i+1,1}^{i_0,j_0}, \\ \widetilde{h}_j^{i_0,j_0} &= p_{-1,1}G_{1,j-1}^{i_0,j_0}+p_{-1,0}G_{1,j}^{i_0,j_0}+p_{-1,-1}G_{1,j+1}^{i_0,j_0}, \end{aligned}$$

so that, from Theorem 1, we immediately obtain the following result.

Corollary 3. *Suppose that the process belongs to $\mathscr{P}_{\alpha,\beta}$. Then the absorption probabilities (4) admit the following asymptotic as $i\to\infty$:*

$$h_i^{i_0,j_0}\sim C(p_{1,-1}+p_{0,-1}+p_{-1,-1})i_0 j_0\left(i_0+\alpha j_0+\beta\right)\frac{1}{i^4},$$

where $C>0$ is the same constant as is the statement of Theorem 1.
The same asymptotic holds for $\tilde{h}_i^{i_0,j_0}$, after having replaced $(p_{1,-1}+p_{0,-1}+p_{-1,-1})$ above by $(p_{-1,1}+p_{-1,0}+p_{-1,-1})/\alpha^5$.

The asymptotic of the absorption probabilities in the case of a non-zero drift being obtained in [10], Corollary 3 thus gives an example of the behavior of these probabilities in the case of a drift zero.

In order to prove Theorem 1, we are going to develop methods initiated in [7] and based on complex analysis, what will allow us to express *explicitly* the Green functions (2). Indeed, in [7], the authors G. Fayolle, R. Iasnogorodski and V. Malyshev elaborate a profound and ingenious analytic approach for studying the stationary probabilities for random walks to the eight nearest

neighbors in the quarter plane supposed ergodic, *i.e.* such that $p_{1,1}+p_{1,0}+p_{1,-1}<p_{-1,1}+p_{-1,0}+p_{-1,-1}$ and $p_{1,1}+p_{0,1}+p_{-1,1}<p_{1,-1}+p_{0,-1}+p_{-1,-1}$.
We are going to see here that this analytical approach can be extended up to the case of the random walks in the quarter plane with drift zero and killed at the boundary : Section 2 of this paper first broadens the analysis begun in Part 6 of [7] for the drift zero, and then shows how this applies in the case of the random walks of $\mathscr{P}_{\alpha,\beta}$.
It is worth noting that this approach *via* complex analysis is intrinsic to the dimension $d=2$; for this reason, it seems really a difficult task to generalize it in higher dimension.

Let us conclude this introductory part by describing the set $\mathscr{P}_{\alpha,\beta}$ defined in (1) in terms of the jump probabilities $(p_{i,j})_{i,j}$.

Remark 4. *The fact that the two drifts are equal to zero gives two equations and the fact that the sum of the jump probabilities is one yields an other one. Moreover, the harmonicity of $h(i_0,j_0)=i_0j_0(i_0+\alpha j_0+\beta)$, which reads $h(i_0,j_0)=\sum_{i,j}p_{i,j}h(i_0+i,j_0+j)$, leads to ten equations, by identification of the coefficients of the third-degree polynomials above. It turns out that some of these equations are trivial and that some other ones are linearly dependent, we finally obtain six equations linearly independent. We can therefore express all the eight jump probabilities $(p_{i,j})_{i,j}$ in terms of $p_{1,1}$ and $p_{1,0}$ only, and we obtain :*

$* \; p_{-1,0}=-[\alpha(1-2\alpha-\beta)+8p_{1,1}+(4-3\alpha+2\alpha^2+\alpha\beta)p_{1,0}]/[\alpha(1+2\alpha+\beta)]$,
$* \; p_{-1,1}=[\alpha(1-\alpha-\beta)+2(4+3\alpha+2\alpha^2+\alpha\beta)p_{1,1}+2(2+\alpha^2+\alpha\beta)p_{1,0}]/[2\alpha(1+2\alpha+\beta)]$,
$* \; p_{0,1}=-[-(1+\alpha+\beta)+4(2+2\alpha+\beta)p_{1,1}+2(2+\alpha+\beta)p_{1,0}]/[2(1+2\alpha+\beta)]$,
$* \; p_{1,-1}=[\alpha^2+(-1+2\alpha-\beta)p_{1,1}-(1+\beta+2\alpha^2)p_{1,0}]/[1+2\alpha+\beta]$,
$* \; p_{0,-1}=-[(-1-3\alpha-\beta+4\alpha^2)+4(-2+2\alpha-\beta)p_{1,1}+(-4+6\alpha-2\beta-8\alpha^2)p_{1,0}]/[2(1+2\alpha+\beta)]$,
$* \; p_{-1,-1}=[\alpha(1-3\alpha-\beta+2\alpha^2)+2(4-3\alpha+2\alpha^2-\alpha\beta)p_{1,1}+2(2-3\alpha+3\alpha^2-2\alpha^3)p_{1,0}]/[2\alpha(1+2\alpha+\beta)]$.

The properties of $\mathscr{P}_{\alpha,\beta}$ mentioned below (1) are immediately obtained by studying the sign of the jump probabilities above in terms of α, β, $p_{1,1}$ and $p_{1,0}$.

2 Explicit expression of the Green functions

Section 2 aims at obtaining an explicit expression of the Green functions (2) – what we will succeed in doing in Theorem 8 below. This forthcoming expression of the Green functions will be, in turn, the starting point of Section 3, where we will find their asymptotic.
In order to prove Theorem 8, we need to state two results, namely Equation (6) and Proposition 6 : Equation (6) is a functional equation between the generating function of the Green functions (2) and the ones of the absorption probabilities (4), and Proposition 6 establishes some quite important properties of the generating functions of the absorption probabilities.
The proof of Proposition 6 turns out to require considering the Riemann surface defined by $\{(x,y)\in(\mathbb{C}\cup\{\infty\})^2:\sum_{i,j}p_{i,j}x^iy^j=1\}$, for this reason we begin Section 2 by studying – and, in fact, by uniformizing – this surface.
It seems of interest to us to introduce this Riemann surface in whole generality ; this is why, at the beginning of Section 2, we are going to suppose that the process belongs to $\mathscr{P}$ – and not necessarily to $\mathscr{P}_{\alpha,\beta}$.

To begin with, we define the generating functions of the Green functions (2) and of the absorption probabilities (4) by

$$G^{i_0,j_0}(x,y)=\sum_{i,j\geq 1}G^{i_0,j_0}_{i,j}x^{i-1}y^{j-1},\ \ h^{i_0,j_0}(x)=\sum_{i\geq 1}h^{i_0,j_0}_i x^i,\ \ \widetilde{h}^{i_0,j_0}(y)=\sum_{j\geq 1}\widetilde{h}^{i_0,j_0}_j y^j \tag{5}$$

and $h^{i_0,j_0}_{0,0}=\mathbb{P}_{(i_0,j_0)}[\exists k\geq 1:(X(k),Y(k))=(0,0)]$. Of course, G^{i_0,j_0}, h^{i_0,j_0} and $\tilde{h}^{i_0,j_0}$ are holomorphic in their unit disc. With these notations, we can state the following functional equation on $\{(x,y)\in\mathbb{C}^2:|x|<1,|y|<1\}$:

$$Q(x,y)\,G^{i_0,j_0}(x,y)=h^{i_0,j_0}(x)+\widetilde{h}^{i_0,j_0}(y)+h^{i_0,j_0}_{0,0}-x^{i_0}y^{j_0}, \tag{6}$$

where $Q(x,y)=xy\left[\sum_{i,j}p_{i,j}x^i y^j-1\right]$. Equation (6) is obtained exactly as in Subsection 2.1 of [10].

The polynomial $Q(x,y)$ defined above can obviously be written as

$$Q(x,y)=a(x)y^2+b(x)y+c(x)=\widetilde{a}(y)x^2+\widetilde{b}(y)x+\widetilde{c}(y),$$

with

$$a(x)=p_{1,1}x^2+p_{0,1}x+p_{-1,1},\ b(x)=p_{1,0}x^2-x+p_{-1,0},\ c(x)=p_{1,-1}x^2+p_{0,-1}x+p_{-1,-1},$$

$$\widetilde{a}(y)=p_{1,1}y^2+p_{1,0}x+p_{1,-1},\ \widetilde{b}(y)=p_{0,1}y^2-y+p_{0,-1},\ \widetilde{c}(y)=p_{-1,1}y^2+p_{-1,0}y+p_{-1,-1}.$$

Let us also define the polynomials

$$d(x)=b(x)^2-4a(x)c(x),\qquad \widetilde{d}(y)=\widetilde{b}(y)^2-4\widetilde{a}(y)\widetilde{c}(y).$$

It is proved in Part 2.3 of [7] that for any random walk of $\mathscr{P}$, d (resp. $\tilde{d}$) has one simple root in $]-1,1[$, that we call x_1 (resp. y_1), a double root at 1, and a simple root in $\mathbb{R}\cup\{\infty\}\setminus[-1,1]$, that we note x_4 (resp. y_4).

For example, in the case of SU(3), *i.e.* for the random walk with transition probabilities as in Picture 1, we immediately obtain $x_1=0$, $y_1=1/4$, $x_4=4$ and $y_4=\infty$.

From a general point of view, it is shown in Part 2.3 of [7] that x_1 (resp. y_1) is positive, zero or negative depending on whether ${p_{-1,0}}^2-4p_{-1,1}p_{-1,-1}$ (resp. ${p_{0,-1}}^2-4p_{1,-1}p_{-1,-1}$) is positive, zero or negative, and that x_4 (resp. y_4) is positive, infinite or negative depending on whether ${p_{1,0}}^2-4p_{1,1}p_{1,-1}$ (resp. ${p_{0,1}}^2-4p_{1,1}p_{-1,1}$) is positive, zero or negative.

Let us now have a look to the surface defined by $\{(x,y)\in(\mathbb{C}\cup\{\infty\})^2:Q(x,y)=0\}$, that we note $\mathscr{Q}$ for the sake of briefness. Note first that $Q(x,y)=0$ is equivalent to $[b(x)+2a(x)y]^2=d(x)$ or to $[\tilde{b}(y)+2\tilde{a}(y)x]^2=\tilde{d}(y)$. As a consequence, it follows from the particular form of d or of $\tilde{d}$ (two distinct simple roots different from 1 and one double root at 1) that the surface $\mathscr{Q}$ has genus zero, and is thus homeomorphic to a sphere $\mathbb{C}\cup\{\infty\}$, see *e.g.* Parts 4.17 and 5.12 of [9]. Therefore, this Riemann surface can be rationally uniformized, in the sense that it is possible to find two rational functions $x(z)$ and $y(z)$, such that $\mathscr{Q}=\{(x(z),y(z)):z\in\mathbb{C}\cup\{\infty\}\}$; moreover, a standard uniformization (for an account of the concept of uniformization, see Part 4.9 of [9]) is :

$$x(z)=\frac{(z-z_1)(z-1/z_1)}{(z-z_0)(z-1/z_0)},\qquad y(z)=\frac{(z-Kz_3)(z-K/z_3)}{(z-Kz_2)(z-K/z_2)}, \tag{7}$$

where

$$
\begin{aligned}
z_0 &= \left[2-(x_1+x_4)+2[(1-x_1)(1-x_4)]^{1/2}\right]/\left[x_4-x_1\right],\\
z_1 &= \left[x_1+x_4-x_1x_4+2[x_1x_4(1-x_1)(1-x_4)]^{1/2}\right]/\left[x_4-x_1\right],\\
z_2 &= \left[2-(y_1+y_4)+2[(1-y_1)(1-y_4)]^{1/2}\right]/\left[y_4-y_1\right],\\
z_3 &= \left[y_1+y_4-y_1y_4+2[y_1y_4(1-y_1)(1-y_4)]^{1/2}\right]/\left[y_4-y_1\right],
\end{aligned}
$$

and where K is a complex number of modulus 1. Note that z_0 and z_1 (resp. z_2 and z_3) have a modulus equal to one or are real, according to the signs of x_1 and x_4 (resp. y_1 and y_4).
For example, in the case of SU(3), it follows from a direct calculation that

$$z_0 = \exp(-2\imath\pi/3),\ \ z_1 = 1,\ \ z_2 = \exp(-\imath\pi/3),\ \ z_3 = \exp(\imath\pi/3),\ \ K = \exp(-\imath\pi/3).$$

Above and throughout the paper, we note $\imath$ the usual complex number verifying $\imath^2 = -1$.
In the general case, in order to find K, we need to introduce a group of automorphisms naturally associated with the surface $\mathcal{Q}$. To begin with, let us remark that, with the previous notations, $Q(x,y)=0$ entails $Q(x,[c(x)/a(x)]/y)=0$ and $Q([\tilde{c}(y)/\tilde{a}(y)]/x,y)=0$; it is therefore natural to consider the group generated by the two bilinear transfor-mations $\hat{\xi}(x,y)=(x,[c(x)/a(x)]/y)$ and $\hat{\eta}(x,y)=([\tilde{c}(y)/\tilde{a}(y)]/x,y)$, which is called, in [7], the *group of the random walk*.
These automorphisms $\hat{\xi}$ and $\hat{\eta}$ define two automorphisms ξ and η of the uniformization space $\mathbb{C}\cup\{\infty\}$, characterized by :

$$\xi\circ\xi=1,\ \ x\circ\xi=x,\ \ y\circ\xi=[c(x)/a(x)]/y,\ \ \eta\circ\eta=1,\ \ y\circ\eta=y,\ \ x\circ\eta=[\widetilde{c}(y)/\widetilde{a}(y)]/x. \quad (8)$$

With (7) and (8), we obtain that they are equal to :

$$\xi(z)=1/z, \qquad \eta(z)=K^2/z. \quad (9)$$

In particular, it is immediate that the group $W=\langle\xi,\eta\rangle$ generated by ξ and η is isomorphic to the dihedral group of order $2\inf\{n>0 : K^{2n}=1\}$. For example, in the case of SU(3) for which $K=\exp(-\imath\pi/3)$, W is of order six – this fact is (differently) proved in Part 4.1 of [7].
A crucial fact is that this property is actually verified by *any* random walk of $\mathscr{P}_{\alpha,\beta}$, since we have the following.

Proposition 5. *For any process of $\mathscr{P}_{\alpha,\beta}$, $K=\exp(-\imath\pi/3)$.*

Proof. With (7), we have $y(K)=y_1$; in addition, by (9), $\eta(K)=K$, so that with (8), we obtain $x(K)^2=\tilde{c}(y_1)/\tilde{a}(y_1)$. This implies that $x(K)=-[\tilde{c}(y_1)/\tilde{a}(y_1)]^{1/2}$ – indeed, we easily show that the roots of $Q(x,y_1)$ have to be negative. By using again (7), we get $K+1/K=\left[\tilde{a}(y_1)^{1/2}(z_1+1/z_1)+\tilde{c}(y_1)^{1/2}(z_0+1/z_0)\right]/\left[\tilde{a}(y_1)^{1/2}+\tilde{c}(y_1)^{1/2}\right]$. In particular, $K+1/K$ can be expressed explicitly in terms of the jump probabilities $(p_{i,j})_{i,j}$. By using then Remark 4 and after simplification, we get $K=\exp(-\imath\pi/3)$. □

From now on, we suppose that the process belongs to $\mathscr{P}_{\alpha,\beta}$.

For a better understanding of the surface $\mathcal{Q}$ as well as for a coming use, we are now going to be interested in the transformations through the uniformization (x,y) of some important cycles, namely the branch cuts $[x_1,x_4]$, $[y_1,y_4]$ and the unit circles $\{|x|=1\}$, $\{|y|=1\}$. First, by using (7) and Proposition 5, we immediately obtain :

$$x^{-1}([x_1,x_4])=\mathbb{R}\cup\{\infty\}, \quad y^{-1}([y_1,y_4])=\exp(-\imath\pi/3)\mathbb{R}\cup\{\infty\}. \quad (10)$$

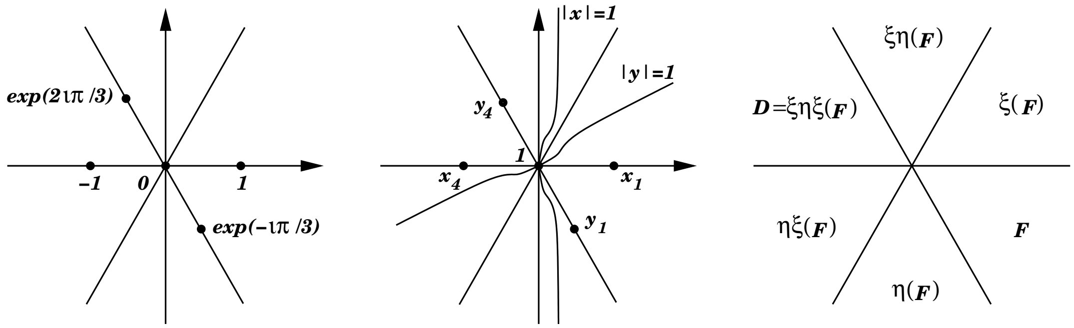

Figure 3: The uniformization space $\mathbb{C} \cup \{\infty\}$, with on the left some important elements of it, in the middle the corresponding elements through the uniformization (x, y), and on the right the images of the cone $F = \{x \exp(\imath\theta) : x \geq 0, -\pi/3 \leq \theta \leq 0\}$ through the six elements of the group $W = \langle \xi, \eta \rangle$

As for the cycles $x^{-1}(\{|x| = 1\})$ and $y^{-1}(\{|y| = 1\})$, their explicit expression (calculated starting from (7)) shows that they are real elliptic curves, which are located as in the middle of Picture 3 below.
Note also that with (9) and Proposition 5, we immediately obtain $\xi(\exp(\imath\theta)\mathbb{R}_+) = \exp(-\imath\theta)\mathbb{R}_+$ and $\eta(\exp(\imath\theta)\mathbb{R}_+) = \exp(-\imath(\theta + 2\pi/3))\mathbb{R}_+$. In particular, if we denote by F the set $\{x \exp(\imath\theta) : x \geq 0, -\pi/3 \leq \theta \leq 0\}$, we have – see also on the right of Picture 3 –

$$\bigcup_{w \in W} w(F) = \mathbb{C}. \tag{11}$$

Thanks to the group $W = \langle \xi, \eta \rangle$ and to (11), we are now going to continue the lifted functions $H^{i_0,j_0}(z) = h^{i_0,j_0}(x(z))$ and $\tilde{H}^{i_0,j_0}(z) = \tilde{h}^{i_0,j_0}(y(z))$; this fact will turn out to be of the highest importance in the proof of Theorem 8 – the latter being crucial, since it will be the starting point of the forthcoming Section 3.
Note that in the sequel, we are often going to write $x^{i_0} y^{j_0}(z)$ instead of $x(z)^{i_0} y(z)^{j_0}$.

Proposition 6. *The functions $H^{i_0,j_0}(z) = h^{i_0,j_0}(x(z))$ and $\tilde{H}^{i_0,j_0}(z) = \tilde{h}^{i_0,j_0}(y(z))$ can be meromorphically continued from respectively $\{z \in \mathbb{C} : |x(z)| \leq 1\}$ and $\{z \in \mathbb{C} : |y(z)| \leq 1\}$ up to respectively $\mathbb{C} \setminus \exp(\imath\pi)[0, \infty]$ and $\mathbb{C} \setminus \exp(2\imath\pi/3)[0, \infty]$. These continuations verify*

$$H^{i_0,j_0}(z) = H^{i_0,j_0}(\xi(z)), \qquad \widetilde{H}^{i_0,j_0}(z) = \widetilde{H}^{i_0,j_0}(\eta(z)) \tag{12}$$

for all $z \in \mathbb{C}$, and

$$H^{i_0,j_0}(z) + \widetilde{H}^{i_0,j_0}(z) + h_{0,0}^{i_0,j_0} - x^{i_0} y^{j_0}(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus D \quad \text{(13a)} \\ -\sum_{w \in W} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) & \text{if } z \in D \quad \text{(13b)} \end{cases}$$

where we have set $D = \{x \exp(\imath\theta) : x \geq 0, 2\pi/3 \leq \theta \leq \pi\}$ and $l(w)$ for the length of w, i.e. the smallest r for which we can write $w = w_1 \cdots w_r$, with $w_1, \ldots, w_r$ equal to ξ or η.

Remark 7. *In $\{z \in \mathbb{C} : |x(z)| \leq 1, |y(z)| \leq 1\} \subset \mathbb{C} \setminus D$, (13a) follows directly from (6).*

Proof of Proposition 6. In order to prove Proposition 6, we are going to use strongly the decomposition (11) : precisely, we are going to define H^{i_0,j_0} and $\tilde{H}^{i_0,j_0}$ piecewise, by defining them on each of the six domains $w(F)$ that appear in the decomposition (11), to be equal to some functions $H_w^{i_0,j_0}$ and $\tilde{H}_w^{i_0,j_0}$. It will then be enough to show that the functions H^{i_0,j_0} and $\tilde{H}^{i_0,j_0}$ so defined verify the conclusions of Proposition 6.

• In $F = \{x\exp(\imath\theta) : x \geq 0, -\pi/3 \leq \theta \leq 0\} \subset \{z \in \mathbb{C} : |x(z)| \leq 1, |y(z)| \leq 1\}$, see Picture 1, we are going to use the most natural way to define H^{i_0,j_0} and $\tilde{H}^{i_0,j_0}$, *i.e.* their power series. So we set, for $z \in F$, $H_1^{i_0,j_0}(z) = h^{i_0,j_0}(x(z))$ and $\tilde{H}_1^{i_0,j_0}(z) = \tilde{h}^{i_0,j_0}(y(z))$ – the subscript 1 standing for the identity element of the group $W = \langle \xi, \eta \rangle$.

• Next, we define $H_\xi^{i_0,j_0}$, $\tilde{H}_\xi^{i_0,j_0}$ on $\xi(F)$ and $H_\eta^{i_0,j_0}$, $\tilde{H}_\eta^{i_0,j_0}$ on $\eta(F)$ by

$$\begin{aligned} \forall z \in \xi(F) \quad &: \quad H_\xi^{i_0,j_0}(z) = H_1^{i_0,j_0}(\xi(z)), \quad \widetilde{H}_\xi^{i_0,j_0}(z) = -H_\xi^{i_0,j_0}(z) - h_{0,0}^{i_0,j_0} + x^{i_0} y^{j_0}(z), \\ \forall z \in \eta(F) \quad &: \quad \widetilde{H}_\eta^{i_0,j_0}(z) = \widetilde{H}_1^{i_0,j_0}(\eta(z)), \quad H_\eta^{i_0,j_0}(z) = -\widetilde{H}_\eta^{i_0,j_0}(z) - h_{0,0}^{i_0,j_0} + x^{i_0} y^{j_0}(z). \end{aligned}$$

• Then, we define $H_{\xi\eta}^{i_0,j_0}$, $\tilde{H}_{\xi\eta}^{i_0,j_0}$ on $\xi\eta(F)$ and $H_{\eta\xi}^{i_0,j_0}$, $\tilde{H}_{\eta\xi}^{i_0,j_0}$ on $\eta\xi(F)$ by

$$\begin{aligned} \forall z \in \xi\eta(F) \quad &: \quad H_{\xi\eta}^{i_0,j_0}(z) = H_\eta^{i_0,j_0}(\xi(z)), \quad \widetilde{H}_{\xi\eta}^{i_0,j_0}(z) = -H_{\xi\eta}^{i_0,j_0}(z) - h_{0,0}^{i_0,j_0} + x^{i_0} y^{j_0}(z), \\ \forall z \in \eta\xi(F) \quad &: \quad \widetilde{H}_{\eta\xi}^{i_0,j_0}(z) = \widetilde{H}_\xi^{i_0,j_0}(\eta(z)), \quad H_{\eta\xi}^{i_0,j_0}(z) = -\widetilde{H}_{\eta\xi}^{i_0,j_0}(z) - h_{0,0}^{i_0,j_0} + x^{i_0} y^{j_0}(z). \end{aligned}$$

• At last, we define $H_{\xi\eta\xi}^{i_0,j_0}$ and $\tilde{H}_{\xi\eta\xi}^{i_0,j_0}$ on $\xi\eta\xi(F) = \eta\xi\eta(F)$ by

$$\forall z \in \xi\eta\xi(F) \quad : \quad H_{\xi\eta\xi}^{i_0,j_0}(z) = H_{\eta\xi}^{i_0,j_0}(\xi(z)), \quad \tilde{H}_{\xi\eta\xi}^{i_0,j_0}(z) = \tilde{H}_{\xi\eta}^{i_0,j_0}(\eta(z)).$$

Therefore we have, for each of the six domains $w(F)$ of the decomposition (11), defined two functions $H_w^{i_0,j_0}$ and $\tilde{H}_w^{i_0,j_0}$. Then, as said at the beginning of the proof, we set $H^{i_0,j_0}(z) = H_w^{i_0,j_0}(z)$ and $\tilde{H}^{i_0,j_0}(z) = \tilde{H}_w^{i_0,j_0}(z)$ for all $z \in w(F)$ and all $w \in W$.
With this construction, (12) and (13a) are immediately obtained. To prove (13b), we can use the fact that it is possible to express *all* the functions $H_w^{i_0,j_0}$, $\tilde{H}_w^{i_0,j_0}$ in terms *only* of $H_1^{i_0,j_0}, \tilde{H}_1^{i_0,j_0}, h_{0,0}^{i_0,j_0}, x^{i_0} y^{j_0}$: we give *e.g.* the expressions of $H_{\xi\eta\xi}^{i_0,j_0}$ and $\tilde{H}_{\xi\eta\xi}^{i_0,j_0}$ on $\xi\eta\xi(F)$:

$$\begin{aligned} H_{\xi\eta\xi}^{i_0,j_0}(z) &= H_1^{i_0,j_0}(\xi\eta\xi(z)) - x^{i_0} y^{i_0}(\eta\xi(z)) + x^{i_0} y^{i_0}(\xi(z)), \\ \widetilde{H}_{\xi\eta\xi}^{i_0,j_0}(z) &= \widetilde{H}_1^{i_0,j_0}(\xi\eta\xi(z)) - x^{i_0} y^{i_0}(\xi\eta(z)) + x^{i_0} y^{i_0}(\eta(z)). \end{aligned}$$

We therefore obtain (13b), since with (13a) we get $H_1^{i_0,j_0}(\xi\eta\xi(z)) + \tilde{H}_1^{i_0,j_0}(\xi\eta\xi(z)) + h_{0,0}^{i_0,j_0} = x^{i_0} y^{j_0}(\xi\eta\xi(z))$ for $z \in \xi\eta\xi(F)$, and since $W = \{1, \xi, \eta, \eta\xi, \xi\eta, \xi\eta\xi\}$. □

Theorem 8. *For any $i, j, i_0, j_0 > 0$,*

$$G_{i,j}^{i_0,j_0} = \frac{-[z_0 - 1/z_0]/\Omega_x}{2\pi\imath[{p_{1,0}}^2 - 4p_{1,1}p_{1,-1}]^{1/2}} \int_{\exp(\imath\theta)[0,\infty]} \left[\frac{1}{z} \sum_{w \in W} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) \right] \frac{\mathrm{d}z}{x(z)^i \, y(z)^j}, \quad (14)$$

where $\theta \in [2\pi/3, \pi]$ and where we have set $\Omega_x = z_0 + 1/z_0 - [z_1 + 1/z_1] = 4(x_4 - 1)(x_1 - 1)/(x_4 - x_1) < 0$.

Proof. We have already noticed that the generating function of the Green functions G^{i_0,j_0}, defined in (5), is holomorphic in $\{(x,y)\in\mathbb{C}^2 : |x|<1, |y|<1\}$. As a consequence and by using (6), the Cauchy formulas allow us to write its coefficients $G_{i,j}^{i_0,j_0}$ as the following double integrals :

$$G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi\imath]^2}\iint_{\{|x|=|y|=1\}} \frac{h^{i_0,j_0}(x)+\widetilde{h}^{i_0,j_0}(y)+h_{0,0}^{i_0,j_0}-x^{i_0}y^{j_0}}{Q(x,y)\,x^i y^j}\mathrm{d}x\mathrm{d}y.$$

Then, by using the uniformization (7), the location of the cycles $\{|x|=1\}$ and $\{|y|=1\}$, see Picture 3, the residue theorem at infinity and the Cauchy theorem, we obtain that :

$$G_{i,j}^{i_0,j_0} = \frac{1}{2\pi\imath}\int_{\exp(\imath\theta)[0,\infty]} \frac{H^{i_0,j_0}(z)+\widetilde{H}^{i_0,j_0}(z)+h_{0,0}^{i_0,j_0}-x^{i_0}y^{j_0}(z)}{\left[\partial_y Q(x(z),y(z))\right]x(z)^i y(z)^j}x'(z)\,\mathrm{d}z, \tag{15}$$

θ being any angle in $[2\pi/3,\pi]$ – $[2\pi/3,\pi]$ because on the one hand, it is not possible to take $\theta>\pi$, since $\exp(\imath\pi)[0,\infty]$ is a singular curve for H^{i_0,j_0}, and on the other hand, it is not allowed to have $\theta<2\pi/3$, since $\exp(2\imath\pi/3)[0,\infty]$ is a singular curve for $\tilde{H}^{i_0,j_0}$, see Proposition 6.
Then, from (7) and from the fact that $\partial_y Q(x(z),y(z)) = d(x(z))^{1/2}$, we remark that we have

$$x'(z)/\partial_y Q(x(z),y(z)) = \frac{[z_0-1/z_0]/(z_0+1/z_0-[z_1+1/z_1])}{[{p_{1,0}}^2-4p_{1,1}p_{1,-1}]^{1/2}z}.$$

In this way and by using (13b) in (15), we get (14). □

3 Proof of Theorem 1 : asymptotic of the Green functions

Beginning of the proof of Theorem 1. For any $\theta\in[2\pi/3,\pi]$, the function $x(z)^i y(z)^j$ is, on $\exp(\imath\theta)[0,\infty]$, larger than 1 in modulus, see Picture 3. Moreover, it goes to 1 when (and only when) z goes to 0 or to ∞. This is why it seems natural to decompose the contour $\exp(\imath\theta)[0,\infty]$ into a part near 0, an other near ∞ and the remaining part, and to think that the parts near 0 and ∞ will lead to the asymptotic of $G_{i,j}^{i_0,j_0}$, and that the remaining part will lead to a negligible contribution. In this way appears the question of finding the best possible contour in order to achieve this idea ; in other words, it is a matter of finding the value of θ for which the calculation of the asymptotic of (14) on $\exp(\imath\theta)[0,\infty]$ will be the easiest, among all the possibilities $\theta\in[2\pi/3,\pi]$.

For this, we are going to consider with details the function $x(z)^i y(z)^j$, or, equivalently, the function $\chi_{j/i}(z)=\ln(x(z))+(j/i)\ln(y(z))$. Incidentally, this is why, from now on, we suppose that $j/i\in[0,M]$, for some $M<\infty$. Indeed, the function $\chi_{j/i}$ is manifestly not appropriate to the values j/i going to ∞, for such j/i, we will consider later the function $(i/j)\chi_{j/i}(z)=(i/j)\ln(x(z))+\ln(y(z))$. Nevertheless, M can be so large as wished, and, in what follows, we assume that some $M>0$ is fixed.
Now we set $\chi_{j/i}(z)=\sum_{p\geq 0}\nu_p(j/i)z^p$, this function is *a priori* well defined for z in a neighborhood of 0. Moreover, with (7), we obtain that $\nu_0(j/i)=0$ and that for all $p\geq 1$,

$$p\nu_p(j/i) = \left(z_0^p+1/z_0^p-[z_1^p+1/z_1^p]\right)+(j/i)\left(z_2^p+1/z_2^p-[z_3^p+1/z_3^p]\right)/K^p. \tag{16}$$

Likewise, we easily prove, by using (7), that for z near ∞, $\chi_{j/i}(z)=\sum_{p\geq 0}\overline{\nu_p}(j/i)z^{-p}$.

Consider now the steepest descent path associated with $\chi_{j/i}$, that is the function $z_{j/i}(t)$ defined by $\chi_{j/i}(z_{j/i}(t)) = t$. By inverting the latter equality, we immediately obtain that the half-line $(1/\nu_1(j/i))[0,\infty]$ is tangent at 0 and at ∞ to the steepest descent path.
Now we set, for the sake of briefness, $\rho_{j/i} = 1/\nu_1(j/i)$. With this notation, let us now answer the question asked above, that dealt with finding the value of θ for which the asymptotic of $G_{i,j}^{i_0,j_0}$ will be the most easily calculated : we are going to choose $\theta = \arg(\rho_{j/i}) \in [2\pi/3,\pi]$, and the decomposition of the contour $\exp(\imath\theta)[0,\infty]$ will be

$$\exp(\imath\theta)[0,\infty] = \left(\rho_{j/i}/|\rho_{j/i}|\right)\left([0,\epsilon]\cup]\epsilon,1/\epsilon[\cup[1/\epsilon,\infty]\right).$$

By using this decomposition in (14), we consider now that the Green functions are the sum of three terms, and we are going to study successively the contribution of each of these terms.
But first of all, we simplify the expression of $\rho_{j/i}$. Setting $\Omega_y = z_2+1/z_2-[z_3+1/z_3] = 4(y_4-1)(y_1-1)/(y_4-y_1)$ and using (16), we immediately obtain that $\nu_1(j/i) = \Omega_x + (j/i)\Omega_y/K$. But it turns out that for all the walks of $\mathscr{P}_{\alpha,\beta}$, we have $\Omega_y = \alpha\Omega_x$ – this follows from a direct calculation starting from the explicit expression of the branch points x_1,x_4,y_1,y_4 in terms of the jump probabilities $(p_{i,j})_{i,j}$ and by using Remark 4. Therefore we have :

$$\rho_{j/i} = \frac{1}{\nu_1\left(j/i\right)} = \frac{1}{\Omega_x}\frac{1}{1+\left(j/i\right)\alpha\exp\left(\imath\pi/3\right)}. \tag{17}$$

Contribution of the neighborhood of 0. In order to evaluate the asymptotic of the integral (14) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$, we are going to use the expansion of the function $(1/z)\sum_{w\in W}(-1)^{l(w)}x^{i_0}y^{j_0}(w(z))$ at 0 – expansion that we will obtain in Equation (21) below. This is why we begin by studying the asymptotic of the following integral, for any non-negative integer k :

$$\int_{(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]} \frac{z^k}{x\left(z\right)^i y\left(z\right)^j}\mathrm{d}z. \tag{18}$$

By using the equality $1/[x(z)^i y(z)^j] = \exp(-i\chi_{j/i}(z))$ as well as the expansion (16) of $\chi_{j/i}$ at 0 and then making the change of variable $z = \rho_{j/i}t$, we obtain that (18) is equal to

$$\rho_{j/i}^{k+1}\int_0^{\epsilon/|\rho_{j/i}|} t^k\exp\left(-it\right)\exp\Big(-i\nu_2\left(j/i\right)\left(\rho_{j/i}t\right)^2\Big)\exp\Big(-i\sum_{p\geq 3}\nu_p\left(j/i\right)\left(\rho_{j/i}t\right)^p\Big)\mathrm{d}t. \tag{19}$$

Now we set $m = \max\{|z_0|,1/|z_0|,|z_1|,1/|z_1|,|z_2|,1/|z_2|,|z_3|,1/|z_3|\}$. Then, with (16), we get $|\nu_p(j/i)| \leq 4m^p(1+M)$. Thus, for all $t\in[0,\epsilon/|\rho_{j/i}|]$, $|-i\sum_{p\geq 3}\nu_p(j/i)(\rho_{j/i}t)^p| \leq 4(1+M)i(m\epsilon)^3/(1-m\epsilon)$. This is why $\exp(-i\sum_{p\geq 3}\nu_p(j/i)(\rho_{j/i}t)^p) = 1+O(i\epsilon^3)$, the O being independent of $j/i\in[0,M]$ and of $t\in[0,\epsilon/|\rho_{j/i}|]$. The integral (19) can thus be calculated as

$$(\rho_{j/i}/i)^{k+1}\left[1+O(i\epsilon^3)\right]\int_0^{i\epsilon/|\rho_{j/i}|} t^k\exp\left(-t\right)\left[1-\nu_2\left(j/i\right)\rho_{j/i}^2t^2/i+O(t^4/i^2)\right]\mathrm{d}t.$$

Let us now choose ϵ such that $i\epsilon/|\rho_{j/i}|\to\infty$ and $O(i\epsilon^3) = o(1/i)$, *e.g.* $\epsilon = 1/i^{3/4}$. For this choice of ϵ, we obtain that the integral (18) is equal to

$$(\rho_{j/i}/i)^{k+1}\left[1+o(1/i)\right]\left[k!-\nu_2\left(j/i\right)\rho_{j/i}^2(k+2)!/i+O(1/i^2)\right], \tag{20}$$

where the o and O above are independent of $j/i \in [0, M]$.

We are now ready to find the asymptotic of the integral (14) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$. To begin with, we have the following expansion in the neighborhood of 0 (directly obtained from (7), (9) and Remark 4) :

$$\sum_{w\in W} (-1)^{l(w)} x^{i_0} y^{j_0} (w(z)) = (\imath 3^{3/2}/2)\alpha\Omega_x^3 i_0 j_0 \left(i_0+\alpha j_0+\beta\right) z^3 + O\left(z^6\right). \tag{21}$$

Equation (21) implies then that the integral (14) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$ equals

$$\frac{-[z_0-1/z_0]/\Omega_x}{2\pi\imath[p_{1,0}{}^2-4p_{1,1}p_{1,-1}]^{1/2}}\int_{(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]}\frac{(\imath 3^{3/2}/2)\alpha\Omega_x^3 i_0 j_0\left(i_0+\alpha j_0+\beta\right)z^2+O(z^5)}{x(z)^i\, y(z)^j}\mathrm{d}z.$$

So, with (18) and (20) applied for $k=2$ and $k=5$, we obtain that the integral (14) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$ is equal to

$$\frac{-[z_0-1/z_0]3^{3/2}\alpha\Omega_x^2}{4\pi[p_{1,0}{}^2-4p_{1,1}p_{1,-1}]^{1/2}} i_0 j_0\left(i_0+\alpha j_0+\beta\right)(\rho_{j/i}/i)^3\left[2-24\nu_2\left(j/i\right)\rho_{j/i}^2/i+o(1/i)\right]. \tag{22}$$

Contribution of the neighborhood of ∞. The part of the contour close to ∞, namely $(\rho_{j/i}/|\rho_{j/i}|)[1/\epsilon,\infty]$, is related to the part $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$ by the transformation $z\mapsto 1/\overline{z}$. Moreover, it is immediate from (7) that for $f=x$, $f=y$, or $f=\sum_{w\in W}(-1)^{l(w)}x^{i_0}y^{j_0}(w)$, $f(1/\overline{z})=\overline{f(z)}$. Therefore, the change of variable $z\mapsto 1/\overline{z}$ immediately entails that the contribution of the integral (14) near ∞ is the complex conjugate of its contribution near 0.

Contribution of the intermediate part. We first recall from Proposition 6 that D denotes $\{x\exp(\imath\theta): x\geq 0, 2\pi/3\leq\theta\leq\pi\}$, and we define $A_\epsilon=\{z\in\mathbb{C}:\epsilon\leq|z|\leq 1/\epsilon\}$. Clearly (see Picture 3) there exist $\eta_{x,\epsilon}>0$ and $\eta_{y,\epsilon}>0$ such that for all $z\in D\cap A_\epsilon$, $|x(z)|\geq 1+\eta_{x,\epsilon}$ and $|y(z)|\geq 1+\eta_{y,\epsilon}$. In fact, since $x'(0)=\Omega_x\neq 0$ and $y'(0)=\Omega_y/K\neq 0$, it is possible to take $\eta_{x,\epsilon}\geq\eta\epsilon$ and $\eta_{y,\epsilon}\geq\eta\epsilon$, for some $\eta>0$ independent of ϵ small enough.
Let us now consider the function

$$s(z)=\frac{1}{x^{i_0}y^{j_0}(z)}\left[\sum_{w\in W}(-1)^{l(w)}x^{i_0}y^{j_0}(w(z))\right],$$

and let us show that $\sup_{z\in D}|s(z)|$ is finite. For this, it is sufficient to prove that s has no pole in the closed domain $D\cup\{\infty\}$.
By (7), the only zeros of the denominator of s are at $z_1, 1/z_1, Kz_3, K/z_3$ which, as we easily check, belong to $-(D\cup\overline{D})$. Also, by (7) and (9), the only poles of the numerator of s are at $K^{2k}z_0, K^{2k}/z_0, K^{2k+1}z_2, K^{2k+1}/z_2$, for $k\in\{0,1,2\}$. Next, we verify that both z_0 and Kz_2 belong to D, so that among the twelve previous points, in fact only z_0 and Kz_2 are in D. But in the definition of s, we took care of dividing by $x^{i_0}y^{j_0}$, so that s is in fact holomorphic near these two points. Moreover, s is clearly holomorphic at ∞. Finally, we have proved that the meromorphic function s has no pole in the closed domain $D\cup\{\infty\}$, hence s is bounded in $D\cup\{\infty\}$, in other words $\sup_{z\in D}|s(z)|$ is finite.

In particular, the modulus of the contribution of the integral (14) on the intermediate part $(\rho_{j/i}/|\rho_{j/i}|)]\epsilon, 1/\epsilon[\subset D\cap A_\epsilon$ can be bounded from above by

$$\frac{|z_0-1/z_0|/|\Omega_x|}{2\pi|{p_{1,0}}^2-4p_{1,1}p_{1,-1}|^{1/2}}\frac{1}{\epsilon^2}\frac{\sup_{z\in D}|s(z)|}{(1+\eta\epsilon)^{i-i_0}(1+\eta\epsilon)^{j-j_0}}. \tag{23}$$

Note that the presence of the term $1/\epsilon^2$ in (23) is due to the following : one $1/\epsilon$ appears as an upper bound of the length of the contour, the other $1/\epsilon$ comes from an upper bound of the modulus of the term $1/z$ present in the integrand of (14).
Then, as before, we take $\epsilon = 1/i^{3/4}$, and we use the following straightforward upper bound, valid for i large enough : $1/(1+\eta/i^{3/4})^i \leq \exp(-(\eta/2)i^{1/4})$. We finally obtain that for i large enough, (23) is equal to $O(i^{3/2}\exp(-(\eta/2)i^{1/4}))$. We are going to see soon that this contribution is negligible w.r.t. the sum of the contributions of the integral (14) in the neighborhoods of 0 and ∞.

Conclusion. We have shown that the contribution of the integral (14) in the neighborhood of 0 is given by (22), that the contribution of (14) in the neighborhood of ∞ is equal to the complex conjugate of (22), and that the contribution of the remaining part equals $O(i^{3/2}\exp(-(\eta/2)i^{1/4}))$. Moreover, starting from (17), we immediately get that $(\rho_{j/i}/i)^3 - (\overline{\rho_{j/i}}/i)^3 = -\imath 3^{3/2}\alpha ij(i+\alpha j)/[\Omega_x(i^2+\alpha ij+\alpha^2 j^2)]^3$. In this way, we obtain

$$G_{i,j}^{i_0,j_0} = \frac{-[z_0-1/z_0]3^{3/2}\alpha\Omega_x^2}{4\pi({p_{1,0}}^2-4p_{1,1}p_{1,-1})^{1/2}} i_0 j_0(i_0+\alpha j_0+\beta)\times \tag{24}$$
$$\times\left[\frac{-2\imath 3^{3/2}\alpha ij(i+\alpha j)}{[\Omega_x(i^2+\alpha ij+\alpha^2 j^2)]^3} - 24\frac{\nu_2(j/i){\rho_{j/i}}^5-\overline{\nu_2}(j/i)\overline{\rho_{j/i}}^5}{i^4}+o(1/i^4)\right].$$

If $\gamma\in]0,\pi/2[$ and $j/i\to\tan(\gamma)$, then $ij(i+\alpha j)/(i^2+\alpha ij+\alpha^2j^2)^3\sim C_{\gamma,\alpha}/i^3$ with $C_{\gamma,\alpha}>0$: Theorem 1 for $\gamma\in]0,\pi/2[$ is thus an immediate consequence of (24). In that case, there was in fact no need to make an expansion with two terms in (22) and (24) above, one single term would have been accurate enough.
If $j/i\to\tan(0)=0$, then $ij(i+\alpha j)/(i^2+\alpha ij+\alpha^2j^2)^3\sim(j/i)/i^3$. By using the explicit expressions of $\nu_2(j/i)$ and $\rho_{j/i}$, see respectively (16) and (17), we easily obtain that $\nu_2(j/i){\rho_{j/i}}^5-\overline{\nu_2}(j/i)\overline{\rho_{j/i}}^5 = O(j/i)$. This implies that the sum of the two last terms in the brackets of (24) equals $O((j/i)/i^4)+o(1/i^4)$, which is obviously negligible w.r.t. $(j/i)/i^3$. Theorem 1 is therefore also proved in the case $\gamma=0$.
In order to prove Theorem 1 in the case $\gamma=\pi/2$, we would consider $(i/j)\kappa_{j/i}$ rather than $\kappa_{j/i}$, and we would use then exactly the same analysis, we omit the details.

Finally, in order to prove that the constant C in the statement of Theorem 1 is positive, it is clearly enough to show that $\imath[z_0-1/z_0]/[({p_{1,0}}^2-4p_{1,1}p_{1,-1})^{1/2}\Omega_x]$ is positive.
For this, note first that from its definition, it is immediate that $\Omega_x<0$. Moreover, it follows from the beginning of Section 2 that if $x_4>0$, then $\imath[z_0-1/z_0]<0$ and $({p_{1,0}}^2-4p_{1,1}p_{1,-1})^{1/2}>0$; if $x_4<0$, then $[z_0-1/z_0]>0$ and $\imath/({p_{1,0}}^2-4p_{1,1}p_{1,-1})^{1/2}<0$; and if $x_4=\infty$, by taking the limit in anyone of the two previous cases, we obtain that $\imath[z_0-1/z_0]/[({p_{1,0}}^2-4p_{1,1}p_{1,-1})^{1/2}]<0$.

A few words about the analytical approach used here. The two key steps in the proof of Theorem 1 are *first* the explicit expression for the Green functions (15), and *then* the expansion (13b) of $H^{i_0,j_0}+\tilde{H}^{i_0,j_0}+h_{0,0}^{i_0,j_0}-x^{i_0}y^{j_0}$ at 0, which is the numerator of the integrand in (15).

It is worth noting that for any walk of $\mathscr{P} \supset \mathscr{P}_{\alpha,\beta}$, it is still possible to obtain (15) – without additional technical details, besides. On the other hand, obtaining explicitly the expansion at 0 of $H^{i_0,j_0}+\tilde{H}^{i_0,j_0}+h_{0,0}^{i_0,j_0}-x^{i_0}y^{j_0}$ in the general setting seems us quite difficult – all the more so as this expansion has to comprise several terms, since *a priori* it could happen that several terms lead to non-negligible contributions in the asymptotic of the Green functions.
It is more imaginable (though technically difficult) to obtain this expansion for the walks for which an equality like (13b) holds ; unfortunately, having such an equality is far for being systematic, even for the processes associated with a finite group W : for example, the random walk with jump probabilities $p_{1,1}=p_{0,-1}=p_{-1,0}=1/3$ has manifestly a group W of order six, but does not verify an identity like (13b).

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