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EXPONENTIAL MOMENTS OF FIRST PASSAGE TIMES AND RELATED QUANTITIES FOR RANDOM WALKS

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Abstract
For a zero-delayed random walk on the real line, let $\tau(x)$, $N(x)$ and $\rho(x)$ denote the first passage time into the interval (x,∞), the number of visits to the interval $(-\infty,x]$ and the last exit time from $(-\infty,x]$, respectively. In the present paper, we provide ultimate criteria for the finiteness of exponential moments of these quantities. Moreover, whenever these moments are finite, we derive their asymptotic behaviour, as $x\to\infty$.

1 Introduction and main results

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. real-valued random variables and $X:=X_1$. Further, let $(S_n)_{n\geq 0}$ be the zero-delayed random walk with increments $S_n-S_{n-1}=X_n$, $n\geq 1$. For $x\in\mathbb{R}$, define the *first passage time* into (x,∞)

$$\tau(x) \;:=\; \inf\{n\in\mathbb{N}_0 : S_n>x\},$$

the *number of visits* to the interval $(-\infty,x]$

$$N(x) \;:=\; \#\{n\in\mathbb{N} : S_n\leq x\} \;=\; \sum_{n\geq 1}\mathbb{1}_{\{S_n\leq x\}},$$

and the *last exit time* from $(-\infty,x]$

$$\rho(x) \;:=\; \begin{cases} \sup\{n\in\mathbb{N} : S_n\leq x\}, & \text{if } \inf_{n\geq 1} S_n\leq x,\\ 0, & \text{if } \inf_{n\geq 1} S_n > x.\end{cases}$$

Note that, for $x\geq 0$,

$$\rho(x) \;=\; \sup\{n\in\mathbb{N}_0 : S_n\leq x\}.$$

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For typographical ease, throughout the text we write τ for $\tau(0)$, N for $N(0)$ and ρ for $\rho(0)$.
Our aim is to find criteria for the finiteness of the exponential moments of $\tau(x)$, $N(x)$ and $\rho(x)$, and to determine the asymptotic behaviour of these moments, as $x \to \infty$.
Assuming that $0 < \mathbb{E}X < \infty$, Heyde [11, Theorem 1] proved that

$$\mathbb{E}e^{a\tau(x)} < \infty \text{ for some } a > 0 \qquad \text{iff} \qquad \mathbb{E}e^{bX^-} < \infty \text{ for some } b > 0.$$

See also [3, Theorem 2] and [6, Theorem II] for relevant results.
When $\mathbb{P}\{X \geq 0\} = 1$ and $\mathbb{P}\{X = 0\} < 1$,

$$\tau(x) - 1 \;=\; N(x) \;=\; \rho(x), \quad x \geq 0. \tag{1}$$

Plainly, in this case, criteria for all the three random variables are the same (Proposition 1.1). An intriguing consequence of our results in the case when $\mathbb{P}\{X < 0\} > 0$ and $\mathbb{P}\{X > 0\} > 0$, in which

$$\tau(x) - 1 \;\leq\; N(x) \;\leq\; \rho(x), \quad x \geq 0, \tag{2}$$

is that provided the abscissas of convergence of the moment generating functions of $\tau(x)$, $N(x)$ and $\rho(x)$ are positive there exists a unique value $R > 0$ such that

$$\mathbb{E}e^{a\tau(x)} < \infty,\ \mathbb{E}e^{aN(x)} < \infty \text{ iff } a \leq R, \text{ and } \mathbb{E}e^{a\rho(x)} < \infty \text{ if } a < R$$

whereas $\mathbb{E}e^{R\rho(x)}$ is finite in some cases and infinite in others. Also we prove that whenever the exponential moments are finite they exhibit the following asymptotics:

$$\mathbb{E}e^{a\tau(x)} \;\sim\; C_1 e^{\gamma x},\ \mathbb{E}e^{aN(x)} \;\sim\; C_2 e^{\gamma x},\ \mathbb{E}e^{a\rho(x)} \;\sim\; C_3 e^{\gamma x},\ \ x \to \infty,$$

for an explicitly given $\gamma > 0$ and distinct positive constants C_i, $i = 1, 2, 3$ (when the law of X is lattice with span $\lambda > 0$ the limit is taken over $x \in \lambda\mathbb{N}$). Our results should be compared (or contrasted) to the known facts concerning power moments (see [13, Theorem 2.1 and Section 4.2] and [13, Theorem 2.2], respectively): for $p > 0$

$$\mathbb{E}(\tau(x))^{p+1} < \infty \quad \Leftrightarrow \quad \mathbb{E}(N(x))^p < \infty \quad \Leftrightarrow \quad \mathbb{E}(\rho(x))^p < \infty;$$

$$\mathbb{E}(\tau(x))^p \;\asymp\; \mathbb{E}(N(x))^p \;\asymp\; \mathbb{E}(\rho(x))^p \;\asymp\; \left(\frac{x}{\mathbb{E}\min(X^+, x)}\right)^p,\ \ x \to \infty$$

where $f(x) \asymp g(x)$ means that $0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.
Proposition 1.1 is due to Beljaev and Maksimov [2, Theorem 1]. A shorter proof can be found in [12, Theorem 2.1].

Proposition 1.1. *Assume that $\mathbb{P}\{X \geq 0\} = 1$ and let $\beta := \mathbb{P}\{X = 0\} \in [0, 1)$. Then for $a > 0$ the following conditions are equivalent:*

$$\mathbb{E}e^{a\tau(x)} < \infty \textit{ for some (hence every) } x \geq 0;$$

$$a < -\log\beta$$

where $-\log\beta := \infty$ if $\beta = 0$. The same equivalence holds for $N(x)$ and $\rho(x)$.

Our first theorem provides sharp criteria for the finiteness of exponential moments of $\tau(x)$ and $N(x)$ in the case when $\mathbb{P}\{X<0\}>0$. Before we present it, we introduce some notation. Let

$$\varphi:[0,\infty)\to(0,\infty],\quad \varphi(t):=\mathbb{E}e^{-tX} \tag{3}$$

be the Laplace transform of X and

$$R:=-\log\inf_{t\geq 0}\varphi(t). \tag{4}$$

Theorem 1.2. *Let $a>0$ and $\mathbb{P}\{X<0\}>0$. Then the following conditions are equivalent:*

$$\sum_{n\geq 1}\frac{e^{an}}{n}\mathbb{P}\{S_n\leq x\} \ < \ \infty \text{ for some (hence every) } x\geq 0; \tag{5}$$

$$\mathbb{E}e^{a\tau(x)}<\infty \text{ for some (hence every) } x\geq 0; \tag{6}$$

$$\mathbb{E}e^{aN(x)}<\infty \text{ for some (hence every) } x\geq 0; \tag{7}$$

$$a\leq R. \tag{8}$$

Our next theorem provides the corresponding result for the last exit time $\rho(x)$.

Theorem 1.3. *Let $a>0$ and $\mathbb{P}\{X<0\}>0$. Then the following conditions are equivalent:*

$$\sum_{n\geq 0}e^{an}\mathbb{P}\{S_n\leq x\}<\infty \text{ for some (hence every) } x\geq 0; \tag{9}$$

$$\mathbb{E}e^{a\rho(x)}<\infty \text{ for some (hence every) } x\geq 0; \tag{10}$$

$$a<R \quad \text{or} \quad a=R \text{ and } \varphi'(\gamma_0)<0 \tag{11}$$

where γ_0 is the unique positive number such that $\varphi(\gamma_0)=e^{-R}$.

It is worth pointing out that Theorem 1.3 and Proposition 1.1 could be merged into one result. Indeed, if one sets $\gamma_0:=\infty$ and $\varphi'(\infty):=\lim_{t\to\infty}\varphi'(t)(=0)$ in the case that $\mathbb{P}\{X<0\}=0$, then (11) includes the criterion given in Proposition 1.1 for the finiteness of $\mathbb{E}e^{a\tau(x)}$ which, in this case, is equivalent to the finiteness of $\mathbb{E}e^{a\rho(x)}$ due to Eq. (1).
Now we turn our attention to the asymptotic behaviour of $\mathbb{E}e^{a\tau(x)}$, $\mathbb{E}e^{aN(x)}$ and $\mathbb{E}e^{a\rho(x)}$ and start by quoting a known result which, given in other terms, can be found in [12, Theorem 2.2]. In view of equality (1) we only state it for $\mathbb{E}e^{a\tau(x)}$. The phrase 'The law of X is λ-lattice' used in Proposition 1.4 and Theorem 1.5 is a shorthand for 'The law of X is lattice with span $\lambda>0$'.

Proposition 1.4. *Let $a>0$, $\mathbb{P}\{X\geq 0\}=1$ and $\mathbb{P}\{X=0\}<1$. Assume that $\mathbb{E}e^{a\tau(x)}<\infty$ for some (hence every) $x\geq 0$. Then, as $x\to\infty$,*

$$\mathbb{E}e^{a\tau(x)} \ \sim \ e^{\gamma x} \ \times \begin{cases} \frac{1-e^{-a}}{\gamma\mathbb{E}Xe^{-\gamma X}}, & \text{if the law of } X \text{ is non-lattice,}\\ \frac{\lambda(1-e^{-a})}{(1-e^{-\lambda\gamma})\mathbb{E}Xe^{-\gamma X}}, & \text{if the law of } X \text{ is } \lambda\text{-lattice}\end{cases}$$

where γ is the unique positive number such that $\varphi(\gamma)=\mathbb{E}e^{-\gamma X}=e^{-a}$, and in the λ-lattice case the limit is taken over $x\in\lambda\mathbb{N}$.

When $0<a\leq R$ and $\mathbb{P}\{X<0\}>0$, there exists a minimal $\gamma>0$ such that $\varphi(\gamma)=e^{-a}$. This γ can be used to define a new probability measure $\mathbb{P}_\gamma$ by

$$\mathbb{E}_\gamma h(S_0,\ldots,S_n) \ = \ e^{an}\mathbb{E}e^{-\gamma S_n}h(S_0,\ldots,S_n),\quad n\in\mathbb{N}, \tag{12}$$

for each nonnegative Borel function h on $\mathbb{R}^{n+1}$, where $\mathbb{E}_\gamma$ denotes expectation with respect to $\mathbb{P}_\gamma$. Since $\mathbb{E}_\gamma X = \mathbb{E}_\gamma S_1 = -e^a \varphi'(\gamma)$ (where φ' denotes the left derivative of φ) and since φ is decreasing and convex on $[0,\gamma]$, there are only two possibilities:

$$\text{Either} \quad \mathbb{E}_\gamma X \in (0,\infty) \quad \text{or} \quad \mathbb{E}_\gamma X = 0. \tag{13}$$

When $a < R$, then the first alternative in (13) prevails. When $a = R$, then typically $\varphi'(\gamma) = 0$ since γ is then the unique minimizer of φ on $[0,\infty)$. In particular, $\mathbb{E}_\gamma X = 0$. But even if $a = R$ it can occur that $\mathbb{E}_\gamma X > 0$ or, equivalently, $\varphi'(\gamma) < 0$. Of course, then γ is the right endpoint of the interval $\{t \geq 0 : \varphi(t) < \infty\}$. We provide an example of this situation in Section 3.
Now we are ready to formulate the last result of the paper.

Theorem 1.5. *Let $a > 0$ and $\mathbb{P}\{X < 0\} > 0$.*

(a) Assume that $\mathbb{E}e^{a\tau(x)} < \infty$ for some (hence every) $x \geq 0$. Then $\mathbb{E}_\gamma S_\tau$ is positive and finite, and, as $x \to \infty$,

$$\mathbb{E}e^{a\tau(x)} \sim e^{\gamma x} \times \begin{cases} \frac{\mathbb{E}(e^{a\tau}-1)}{\gamma \mathbb{E}_\gamma S_\tau}, & \text{if the law of } X \text{ is non-lattice,} \\ \frac{\lambda \mathbb{E}(e^{a\tau}-1)}{(1-e^{-\lambda\gamma})\mathbb{E}_\gamma S_\tau}, & \text{if the law of } X \text{ is } \lambda\text{-lattice.} \end{cases} \tag{14}$$

(b) Assume that $\mathbb{E}e^{aN(x)} < \infty$ for some (hence every) $x \geq 0$. Then $\mathbb{E}_\gamma S_\tau$ is positive and finite, and, as $x \to \infty$,

$$\mathbb{E}e^{aN(x)} \sim e^{\gamma x} \times \begin{cases} \frac{e^{-a}\mathbb{E}_\gamma \int_0^{S_\tau} e^{\gamma y}\mathbb{E}[e^{aN(-y)}]\,\mathrm{d}y}{\mathbb{E}_\gamma S_\tau}, & \text{if the law of } X \text{ is non-lattice,} \\ \frac{\lambda e^{-a}\mathbb{E}_\gamma \sum_{k=1}^{S_\tau/\lambda} e^{\gamma\lambda k}\mathbb{E}[e^{aN(-\lambda k)}]}{\mathbb{E}_\gamma S_\tau}, & \text{if the law of } X \text{ is } \lambda\text{-lattice.} \end{cases} \tag{15}$$

(c) Assume that $\mathbb{E}e^{a\rho(x)} < \infty$ for some (hence every) $x \geq 0$. Then $M := \inf_{n\geq 1} S_n$ is positive with positive probability, and, as $x \to \infty$,

$$\mathbb{E}e^{a\rho(x)} \sim e^{\gamma x} \times \begin{cases} \frac{e^{-a}(1-\mathbb{E}e^{-\gamma M^+})}{\gamma \mathbb{E}Xe^{-\gamma X}}, & \text{if the law of } X \text{ is non-lattice,} \\ \frac{\lambda e^{-a}(1-\mathbb{E}e^{-\gamma M^+})}{(1-e^{-\lambda\gamma})\mathbb{E}Xe^{-\gamma X}}, & \text{if the law of } X \text{ is } \lambda\text{-lattice.} \end{cases} \tag{16}$$

In the λ-lattice case the limit is taken over $x \in \lambda\mathbb{N}$.

The rest of the paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1.2, 1.3 and 1.5. In Section 3 we provide three examples illustrating our main results.

2 Proofs of the main results

Proof of Theorem 1.2. (8) $\Rightarrow$ (5). Pick any $a \in (0,R]$ and let γ be as defined on p. 367. With this γ, the equality

$$Z_\gamma(A) := \sum_{n\geq 1} \frac{\mathbb{P}_\gamma\{S_n \in A\}}{n}$$

where $A \subset \mathbb{R}$ is a Borel set, defines a measure which is finite on bounded intervals. Furthermore, according to [1, Theorem 1.2], if $\mathbb{E}_\gamma X > 0$ then $Z_\gamma((-\infty, 0]) < \infty$, whereas if $\mathbb{E}_\gamma X = 0$ (this may only happen if $a = R$), then the function $x \mapsto Z_\gamma((-x, 0])$, $x > 0$, is of sublinear growth. Hence, for every $x \geq 0$,

$$\sum_{n\geq 1} \frac{e^{an}}{n} \mathbb{P}\{S_n \leq x\} \;=\; \sum_{n\geq 1} \frac{1}{n} \mathbb{E}_\gamma e^{\gamma S_n} \mathbb{1}_{\{S_n \leq x\}} \;=\; \int_{(-\infty,x]} e^{\gamma y}\, Z_\gamma(\mathrm{d}y) \;<\; \infty.$$

(5) $\Rightarrow$ (8). Suppose (5) holds for some $x = x_0 \geq 0$ and $a > R$. Pick $\varepsilon \in (0, a-R)$. Then $\sum_{n\geq 0} e^{(a-\varepsilon)n} \mathbb{P}\{S_n \leq x_0\} < \infty$ which is a contradiction to [12, Theorem 2.1(aiii)] (reproduced here as equivalence (9) $\Longleftrightarrow$ (11) of Theorem 1.3).
(5) $\Rightarrow$ (6). The argument given below will also be used in the proof of Theorem 1.5.
If (5) holds for some $x \geq 0$ then, according to the already proved equivalence (5) $\Longleftrightarrow$ (8), first, $a \leq R$ and, secondly, (5) holds for every $x \geq 0$. For $0 < a \leq R$ and $x \geq 0$, we have

$$\begin{aligned}\mathbb{E}e^{a\tau(x)} \;&=\; 1 + (e^a - 1)\sum_{n\geq 0} e^{an}\mathbb{P}\{\tau(x) > n\} \\ &=\; 1 + (e^a - 1)\sum_{n\geq 0} e^{an}\mathbb{P}\{M_n \leq x\} \end{aligned} \tag{17}$$

where $M_n := \max_{0\leq k\leq n} S_k$, $n \in \mathbb{N}_0$. According to [6, Formula (2.9)],

$$\sum_{n\geq 0} e^{an}\mathbb{P}\{M_n \leq x\} \;=\; \frac{\mathbb{E}e^{a\tau} - 1}{e^a - 1} \sum_{j\geq 0} e^{aj} \mathbb{P}\{L_j = j, S_j \leq x\} \tag{18}$$

where $L_j = \inf\{i \in \mathbb{N}_0 : S_i = M_j\}$, $j \in \mathbb{N}_0$. Since $a \leq R$, we can use the exponential measure transformation introduced in (12), which gives

$$e^{aj}\mathbb{P}\{L_j = j, S_j \leq x\} \;=\; \mathbb{E}_\gamma e^{\gamma S_j} \mathbb{1}_{\{L_j = j, S_j \leq x\}}.$$

Observe that $L_j = j$ holds iff $j = \sigma_k$ for some $k \in \mathbb{N}_0$ where σ_k ($\sigma_0 := 0$) denotes the kth strictly ascending ladder epoch of the random walk $(S_n)_{n\geq 0}$. Thus,

$$\begin{aligned}\sum_{j\geq 0} e^{aj}\mathbb{P}\{L_j = j, S_j \leq x\} \;&=\; \sum_{j\geq 0} \mathbb{E}_\gamma e^{\gamma S_j}\mathbb{1}_{\{L_j = j, S_j\leq x\}} \\ &=\; \sum_{j\geq 0}\mathbb{E}_\gamma \sum_{k\geq 0} e^{\gamma S_{\sigma_k}} \mathbb{1}_{\{\sigma_k = j, S_{\sigma_k}\leq x\}} \;=\; \mathbb{E}_\gamma \sum_{k\geq 0} e^{\gamma S_{\sigma_k}}\mathbb{1}_{\{S_{\sigma_k}\leq x\}} \\ &=\; e^{\gamma x}\int_{\mathbb{R}} e^{-\gamma(x-y)}\mathbb{1}_{[0,\infty)}(x-y)\, U_\gamma^{>}(\mathrm{d}y) \;=:\; e^{\gamma x} Z_\gamma^{>}(x)\end{aligned} \tag{19}$$

where $U_\gamma^{>}$ denotes the renewal function of the random walk $(S_{\sigma_k})_{k\geq 0}$ under $\mathbb{P}_\gamma$, that is, $U_\gamma^{>}(\cdot) = \sum_{k\geq 0}\mathbb{P}_\gamma\{S_{\sigma_k} \in \cdot\}$. Thus, $Z_\gamma^{>}(x)$ is finite for all $x \geq 0$ since it is the integral of a directly Riemann integrable function with respect to $U_\gamma^{>}$.
(6) $\Rightarrow$ (5) and (7) $\Rightarrow$ (5). Since $\tau(y) \leq N(y) + 1$, $y \geq 0$, it suffices to prove the first implication. To this end, let

$$K(a) \;:=\; \sum_{n\geq 1} \frac{e^{an}}{n}\mathbb{P}\{S_n \leq 0\}.$$

By a generalization of Spitzer's formula [6, Formula (2.6)], the assumption $\mathbb{E}e^{a\tau} < \infty$ immediately entails the finiteness of $K(a)$:

$$\infty \ > \ \mathbb{E}e^{a\tau} \ = \ 1 + (e^a - 1)\sum_{n\geq 0} e^{an}\mathbb{P}\{M_n = 0\} \ = \ 1 + (e^a - 1)e^{K(a)}.$$

We already know that if the series in (5) converges for $x = 0$, *i.e.*, if $K(a) < \infty$, then it converges for every $x \geq 0$.
(6) $\Rightarrow$ (7). By the equivalence (5) $\Longleftrightarrow$ (6), $\mathbb{E}e^{a\tau(x)} < \infty$ for every $x \geq 0$. According to [13, Formula (3.54)],

$$\mathbb{P}\{N = k\} \ = \ \mathbb{P}\{\inf_{n\geq 1} S_n > 0\}\mathbb{P}\{\tau > k\}, \ \ k \in \mathbb{N}_0,$$

where $\mathbb{P}\{\inf_{n\geq 1} S_n > 0\} > 0$, since, under the present assumptions, $(S_n)_{n\geq 0}$ drifts to $+\infty$ a.s. Hence, $\mathbb{E}e^{aN} < \infty$. Further, for $y \in \mathbb{R}$,

$$\widehat{N}(x, y) \ := \sum_{n>\tau(x)} \mathbb{1}_{\{S_n - S_{\tau(x)} \leq y\}} \tag{20}$$

is a copy of $N(y)$ that is independent of $(\tau(x), S_{\tau(x)})$. We have

$$N(x) \ = \ \tau(x) - 1 + \widehat{N}(x, x - S_{\tau(x)}) \ \leq \ \tau(x) + \widehat{N}(x, 0) \tag{21}$$

Hence, $\mathbb{E}e^{aN(x)} < \infty$, for every $x \geq 0$. The proof is complete. $\square$

Proof of Theorem 1.3. The equivalence (9) $\Longleftrightarrow$ (11) has been proved in [12, Theorem 2.1].
(9) $\Rightarrow$ (10). According to the just mentioned equivalence, if (9) holds for some $x \geq 0$ it holds for every $x \geq 0$. It remains to note that for $x \geq 0$

$$\mathbb{P}\{\rho(x) = n\} \ = \int\limits_{(-\infty,x]} \mathbb{P}\{\inf_{k\geq 1} S_k > x - y\}\, \mathbb{P}\{S_n \in \mathrm{d}y\} \ \leq \ \mathbb{P}\{S_n \leq x\}. \tag{22}$$

(10) $\Rightarrow$ (11). Suppose $\mathbb{E}e^{a\rho(x_0)} < \infty$ for some $x_0 \geq 0$ and $a > 0$. Since $\mathbb{E}e^{a\rho(x)}$ is increasing in x, we have $\mathbb{E}e^{a\rho} < \infty$. Condition $a \leq R$ must hold in view of (2) and implication (6) $\Rightarrow$ (8) of Theorem 1.2. If $a < R$, we are done. In the case $a = R$ it remains to show that

$$\mathbb{E}Xe^{-\gamma_0 X} > 0. \tag{23}$$

Define the measure V by

$$V(A) \ := \sum_{n\geq 0} e^{Rn}\mathbb{P}\{S_n \in A\}, \tag{24}$$

for Borel sets $A \subset \mathbb{R}$. Then from (22) we infer that

$$\infty \ > \ \mathbb{E}e^{R\rho} \ = \int\limits_{(-\infty,0]} \mathbb{P}\{\inf_{n\geq 1} S_n > -y\}\, V(\mathrm{d}y). \tag{25}$$

Under the present assumptions, the random walk $(S_n)_{n\geq 0}$ drifts to $+\infty$ a.s. Therefore, $\mathbb{P}\{\inf_{n\geq 1} S_n > \varepsilon\} > 0$ for some $\varepsilon > 0$. With such an ε,

$$\infty \ > \int\limits_{(-\varepsilon,0]} \mathbb{P}\{\inf_{n\geq 1} S_n > -y\}\, V(\mathrm{d}y) \ \geq \ \mathbb{P}\{\inf_{n\geq 1} S_n > \varepsilon\}V((-\varepsilon, 0]).$$

Thus,

$$\infty \ > \ V((-\varepsilon,0]) \ = \ \sum_{n=0}^{\infty}\mathbb{E}_{\gamma_0}e^{\gamma_0 S_n}\mathbb{1}_{\{S_n\in(-\varepsilon,0]\}} \ \geq \ e^{-\gamma_0\varepsilon}\sum_{n=0}^{\infty}\mathbb{P}_{\gamma_0}\{-\varepsilon<S_n\leq 0\}.$$

Hence $(S_n)_{n\geq 0}$ must be transient under $\mathbb{P}_{\gamma_0}$, which yields the validity of (11) in view of (13) and $\mathbb{E}_{\gamma_0}S_1 = e^R\mathbb{E}Xe^{-\gamma_0 X}$. The proof is complete. □

Proof of Theorem 1.5. (a) In view of (17), (18) and (19), in order to find the asymptotics of $\mathbb{E}e^{a\tau(x)}$, it suffices to determine the asymptotic behaviour of $Z_\gamma^>(x)$ defined in (19). By the key renewal theorem on the positive half-line,

$$Z_\gamma^>(x) \underset{x\to\infty}{\to} \begin{cases} \frac{1}{\gamma\mathbb{E}_\gamma S_\tau} & \text{if the law of } X \text{ is non-lattice,} \\ \frac{\lambda}{(1-e^{-\lambda\gamma})\mathbb{E}_\gamma S_\tau} & \text{if the law of } X \text{ is } \lambda\text{-lattice} \end{cases} \tag{26}$$

where the limit $x\to\infty$ is taken over $x\in\lambda\mathbb{N}$ when the law of X is lattice with span $\lambda>0$.
It remains to check that $\mathbb{E}_\gamma S_\tau$ is finite. As pointed out in (13), either $\mathbb{E}_\gamma X\in(0,\infty)$ or $\mathbb{E}_\gamma X=0$. In the first case, $S_n\to\infty$ a.s. under $\mathbb{P}_\gamma$ and, therefore, $\mathbb{E}_\gamma\tau<\infty$, see, for instance, [4, Theorem 2, p. 151], which yields $\mathbb{E}_\gamma S_\tau<\infty$ by virtue of Wald's identity. If, on the other hand, $\mathbb{E}_\gamma X=0$, then $\mathbb{E}_\gamma\tau=\infty$ and we cannot argue as above. But in this case, by [5, Formula (4a)], $\mathbb{E}_\gamma(S_1^+)^2<\infty$ is sufficient for $\mathbb{E}_\gamma S_\tau<\infty$ to hold. Now the finiteness of

$$\mathbb{E}_\gamma e^{\gamma S_1} \ = \ \varphi(\gamma)^{-1} \ < \ \infty,$$

implies the finiteness of $\mathbb{E}_\gamma(S_1^+)^2$, and the proof of part (a) is complete.
(b) We only consider the case when the law of X is non-lattice since the lattice case can be treated similarly. Denote by $R_x := S_{\tau(x)}-x$ the overshoot. Since $\mathbb{E}e^{a\tau(x)} = \mathbb{E}_\gamma e^{\gamma S_{\tau(x)}}$, we have in view of the already proved part (a)

$$\lim_{x\to\infty}\mathbb{E}_\gamma e^{\gamma R_x} \ = \ \frac{\mathbb{E}e^{a\tau}-1}{\gamma\mathbb{E}_\gamma S_\tau}. \tag{27}$$

By Theorem 1.2, if $\mathbb{E}e^{aN(x)}<\infty$, then $\mathbb{E}e^{a\tau(x)}<\infty$. Therefore, according to part (a), we have $0<\mathbb{E}_\gamma S_\tau<\infty$. This implies (see, for instance, [10, Theorem 10.3 on p. 103]) that, as $x\to\infty$, R_x converges in distribution to a random variable R_∞ satisfying

$$\mathbb{P}_\gamma\{R_\infty\leq x\} \ = \ \frac{1}{\mathbb{E}_\gamma S_\tau}\int_0^x \mathbb{P}_\gamma\{S_\tau>y\}\,\mathrm{d}y, \quad x\geq 0.$$

In particular, under $\mathbb{P}_\gamma$, $e^{\gamma R_x}$ converges in distribution to $e^{\gamma R_\infty}$. Further,

$$\mathbb{E}_\gamma e^{\gamma R_\infty} \ = \ \frac{1}{\mathbb{E}_\gamma S_\tau}\int_0^\infty e^{\gamma y}\mathbb{P}_\gamma\{S_\tau>y\}\,\mathrm{d}y \ = \ \frac{\mathbb{E}_\gamma e^{\gamma S_\tau}-1}{\gamma\mathbb{E}_\gamma S_\tau} \ = \ \frac{\mathbb{E}e^{a\tau}-1}{\gamma\mathbb{E}_\gamma S_\tau}.$$

Therefore, (27) can be rewritten as follows:

$$\lim_{x\to\infty}\mathbb{E}_\gamma e^{\gamma R_x} \ = \ \mathbb{E}_\gamma e^{\gamma R_\infty}. \tag{28}$$

Now we invoke a variant of Fatou's lemma sometimes called Pratt's lemma [14, Theorem 1]. To this end, note that, by a standard coupling argument, we can assume w.l.o.g. that $R_x\to R_\infty$ $\mathbb{P}_\gamma$-a.s. From (21) we infer that for $f(y) := \mathbb{E}e^{aN(y)}$, $y\in\mathbb{R}$ we have

$$f(x) \ = \ \mathbb{E}e^{aN(x)} \ = \ e^{-a}\mathbb{E}e^{a\tau(x)}f(-R_x) \ = \ e^{\gamma x}e^{-a}\mathbb{E}_\gamma e^{\gamma R_x}f(-R_x).$$

f is an increasing function and, therefore, has only countably many discontinuities. Hence $e^{\gamma R_x} f(-R_x)$ converges $\mathbb{P}_\gamma$-a.s. to $e^{\gamma R_\infty} f(-R_\infty)$. Further,

$$e^{\gamma R_x} f(-R_x) \;\le\; e^{\gamma R_x} f(0)$$

and $e^{\gamma R_x} f(0)$ converges $\mathbb{P}_\gamma$-a.s. to $e^{\gamma R_\infty} f(0)$. Finally,

$$\lim_{x\to\infty} \mathbb{E}_\gamma e^{\gamma R_x} f(0) \;=\; \mathbb{E}_\gamma e^{\gamma R_\infty} f(0).$$

Therefore the assumptions of Pratt's lemma are fulfilled and an application of the lemma yields

$$\begin{aligned}
\lim_{x\to\infty} e^{-\gamma x} f(x) &= e^{-a} \lim_{x\to\infty} \mathbb{E}_\gamma e^{\gamma R_x} f(-R_x) \;=\; e^{-a}\mathbb{E}_\gamma e^{\gamma R_\infty} f(-R_\infty)\\
&= \frac{e^{-a}}{\mathbb{E}_\gamma S_\tau} \int_0^\infty e^{\gamma y} f(-y) \mathbb{P}_\gamma\{S_\tau > y\}\, \mathrm{d}y\\
&= \frac{e^{-a}\mathbb{E}_\gamma \int_0^{S_\tau} e^{\gamma y} f(-y)\,\mathrm{d}y}{\mathbb{E}_\gamma S_\tau}.
\end{aligned}$$

(c) >From (22) and (24) (with R replaced by a and $M = \inf_{k\ge 1} S_k$), we infer

$$\begin{aligned}
\mathbb{E}e^{a\rho(x)} &= \int\limits_{(-\infty,x]} \mathbb{P}\{M > x - y\}\, V(\mathrm{d}y)\\
&= V(x)\mathbb{P}\{M > 0\} - \int\limits_{(0,\infty)} V(x-y)\,\mathbb{P}\{M \in \mathrm{d}y\}, \quad x \ge 0.
\end{aligned}$$

Assume that the law of X is non-lattice and set $D_1 := \frac{e^{-a}}{\gamma \mathbb{E} X e^{-\gamma X}}$. It follows from (11) that $D_1 \in (0,\infty)$ and from [12, Theorem 2.2] that

$$V(x) \;\sim\; D_1 e^{\gamma x}, \;\; x \to \infty. \tag{29}$$

The latter implies that for any $\varepsilon > 0$ there exists an $x_0 > 0$ such that

$$(D_1 - \varepsilon) e^{\gamma y} \;\le\; V(y) \;\le\; (D_1 + \varepsilon) e^{\gamma y}$$

for all $y \ge x_0$. Fix one such x_0. Then for all $x \ge x_0$,

$$\begin{aligned}
(D_1 - \varepsilon) e^{\gamma x} \int\limits_{(0,x-x_0]} e^{-\gamma y}\, \mathbb{P}\{M \in \mathrm{d}y\} \;&\le\; \int\limits_{(0,x-x_0]} V(x-y)\,\mathbb{P}\{M \in \mathrm{d}y\}\\
&\le\; (D_1 + \varepsilon) e^{\gamma x} \int\limits_{(0,x-x_0]} e^{-\gamma y}\, \mathbb{P}\{M \in \mathrm{d}y\},
\end{aligned}$$

and $\int_{(x-x_0,\infty)} V(x-y)\mathbb{P}\{M \in \mathrm{d}y\} \in [0, V(x_0)]$. Letting first $x \to \infty$ and then $\varepsilon \to 0$ we conclude that

$$\lim_{x\to\infty} e^{-\gamma x} \int\limits_{(0,\infty)} V(x-y)\,\mathbb{P}\{M \in \mathrm{d}y\} \;=\; D_1 \mathbb{E} e^{-\gamma M} \mathbb{1}_{\{M>0\}}.$$

Together with (29) the latter yields

$$\begin{aligned}\mathbb{E}e^{a\rho(x)} &\sim D_1\big(\mathbb{P}\{M>0\}-\mathbb{E}e^{-\gamma M}\mathbb{1}_{\{M>0\}}\big)e^{\gamma x}\\ &= D_1\big(1-\mathbb{E}e^{-\gamma M^+}\big)e^{\gamma x},\ \ x\to\infty.\end{aligned}$$

Under the present assumptions, the random walk $(S_n)_{n\geq 0}$ drifts to $+\infty$ a.s. Therefore, $\mathbb{P}\{M>0\}>0$ which implies that $1-\mathbb{E}e^{-\gamma M^+}>0$ and completes the proof in the non-lattice case.
The proof in the lattice case is based on the lattice version of [12, Theorem 2.2] and follows the same path. □

3 Examples

In this section, retaining the notation of Section 1, we illustrate the results of Theorem 1.2 and Theorem 1.3 by three examples.

Example 3.1 (Simple random walk). Let $1/2<p<1$ and $\mathbb{P}\{X=1\}=p=1-\mathbb{P}\{X=-1\}=:1-q$. Then the Laplace transform φ of X is given by $\varphi(t)=pe^{-t}+qe^{t}$ and $R=-\log(2\sqrt{pq})$. According to [8, Formula (3.7) on p. 272] and [7, Example 1], respectively,

$$\mathbb{P}\{\tau=2n-1\}\ =\ \frac{1}{2q}\frac{\binom{2n}{n}}{2^{2n}(2n-1)}(2\sqrt{pq})^{2n},\ \mathbb{P}\{\tau=2n\}=0,\quad n\in\mathbb{N};$$

$$\mathbb{P}\{\rho=2n\}\ =\ (p-q)\binom{2n}{n}(pq)^n,\ \mathbb{P}\{\rho=2n+1\}=0,\quad n\in\mathbb{N}_0.$$

Stirling's formula yields

$$\frac{\binom{2n}{n}}{2^{2n}}\sim\frac{1}{\sqrt{\pi n}},\ \ n\to\infty, \tag{30}$$

which implies that

$$\mathbb{E}e^{R\tau}<\infty\quad\text{and}\quad\mathbb{E}e^{R\rho}=\infty.$$

Example 3.2. Let $X\overset{d}{=}Y_1-Y_2$ where Y_1 and Y_2 are independent r.v.'s with exponential distributions with parameters α and κ, respectively, $0<\alpha<\kappa$. Then $\varphi(t)=\mathbb{E}e^{-tX}=\frac{\alpha\kappa}{(\alpha+t)(\kappa-t)}$ and $R=-\log(\frac{4\alpha\kappa}{(\alpha+\kappa)^2})$. According to [9, Formula (8.4) on p. 193], for $a\in(0,R]$,

$$\mathbb{E}e^{a\tau}\ =\ (2\alpha)^{-1}(\alpha+\kappa-\sqrt{(\alpha+\kappa)^2-4\alpha\kappa e^a})\ <\ \infty.$$

Further, for $n\in\mathbb{N}_0$,

$$\begin{aligned}\mathbb{P}\{\rho=n\} &= \int_{(-\infty,0]}\mathbb{P}\{\inf_{k\geq 1}S_k>-x\}\,\mathbb{P}\{S_n\in\mathrm{d}x\}\\ &= \int_{(-\infty,0]}\int_{(-x,\infty)}\mathbb{P}\{\inf_{k\geq 0}S_k>-x-y\}\,\mathbb{P}\{S_1\in\mathrm{d}y\}\,\mathbb{P}\{S_n\in\mathrm{d}x\}.\end{aligned}$$

According to [9, Formula (5.9) on p. 410],

$$\mathbb{P}\{\inf_{k\geq 0}S_k>-x-y\}\ =\ \mathbb{P}\{\sup_{k\geq 0}(-S_k)<x+y\}\ =\ 1-\frac{\alpha}{\kappa}e^{-(\kappa-\alpha)(x+y)}.$$

Note that S_n has the same law as the difference of two independent random variables with gamma distribution with parameters (n,α) and (n,κ), respectively, which particularly implies that, for $x>0$, the density of S_1 takes the form $\frac{\alpha\kappa e^{-\alpha x}}{\alpha+\kappa}$. Thus[2], for $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}\{\rho = n\} &= \int\limits_{(-\infty,0]} \int_{-x}^{\infty} \left(1 - \frac{\alpha}{\kappa} e^{-(\kappa-\alpha)(x+y)}\right) \frac{\alpha\kappa e^{-\alpha y}}{\alpha+\kappa}\, \mathrm{d}y\, \mathbb{P}\{S_n \in \mathrm{d}x\} \\
&= \frac{\kappa-\alpha}{\kappa} \int\limits_{(-\infty,0]} e^{\alpha x}\, \mathbb{P}\{S_n \in \mathrm{d}x\} \\
&= \frac{\kappa-\alpha}{\kappa} \int_0^{\infty}\int_0^{t} e^{\alpha(s-t)} \frac{\alpha^n s^{n-1} e^{-\alpha s}}{(n-1)!} \frac{\kappa^n t^{n-1} e^{-\kappa t}}{(n-1)!} \mathrm{d}s\, \mathrm{d}t \\
&= \frac{\kappa-\alpha}{\kappa} \frac{\alpha^n\kappa^n}{n!(n-1)!} \int_0^{\infty} t^{2n-1} e^{-(\alpha+\kappa)t} \mathrm{d}t \\
&= \frac{\kappa-\alpha}{(\kappa+\alpha)^{2n}} \alpha^n \kappa^{n-1} \binom{2n-1}{n},
\end{aligned}$$

and

$$\mathbb{P}\{\rho = 0\} = \frac{\kappa-\alpha}{\kappa}.$$

Hence,

$$\mathbb{E}e^{R\rho} = \frac{\kappa-\alpha}{\kappa}\left(1 + \sum_{n\geq 1} 4^{-n}\binom{2n-1}{n}\right) = \infty,$$

since relation (30) implies that the summands are of order $1/\sqrt{n}$, as $n \to \infty$.

Finally, we point out an explicit form of distribution of X for which $\mathbb{E}e^{R\rho(x)} < \infty$ for every $x \geq 0$.

Example 3.3. Fix $h > 0$ and take any probability law μ_1 on $\mathbb{R}$ such that the Laplace-Stieltjes transform

$$\psi(t) := \int_{\mathbb{R}} e^{-tx} \mu_1(\mathrm{d}x), \quad t \geq 0,$$

is finite for $0 \leq t \leq h$ and infinite for $t > h$, and the left derivative of ψ at h, $\psi'(h)$, is finite and positive. For instance, one can take

$$\mu_1(\mathrm{d}x) := c e^{-h|x|}/(1+|x|^r)\mathrm{d}x, \quad x \in \mathbb{R}$$

where $r > 2$ and $c := \left(\int_{\mathbb{R}} e^{-h|x|}(1+|x|^r)^{-1}\mathrm{d}x\right)^{-1} > 0$.
Now choose s sufficiently large such that $\psi'(h) < s\psi(h)$. Then $\varphi(t) = e^{-st}\psi(t)$ is the Laplace-Stieltjes transform of the distribution $\mu := \delta_s * \mu_1$. Let X be a random variable with distribution μ. Plainly, $\varphi(t)$ is finite for $0 \leq t \leq h$ but infinite for $t > h$. Furthermore,

$$\varphi'(t) = e^{-st}(\psi'(t) - s\psi(t)), \quad |t| \leq h.$$

In particular, $\varphi'(h) < 0$ which, among other things, implies that $R = -\log\varphi(h)$ and that $\gamma_0 = h$. Therefore, $\mathbb{E}Xe^{-\gamma_0 X} = -\varphi'(h) > 0$, and by Theorem 1.2, $\mathbb{E}e^{R\rho(x)} < \infty$ for all $x \geq 0$.

[2] We do not claim that this formula is new, but we have not been able to locate it in the literature.

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