GLOBAL BIFURCATION ANALYSIS OF AN ADAPTIVE CONTROL SYSTEM†

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Abstract. We present complete bifurcation analysis of a 2-parameter, 2-dimensional quadratic o.d.e. which arises in the study of Model Reference Adaptive Control (MRAC) systems. The 2-dimensional o.d.e. exhibits saddle-node, (subcritical) Hopf, and saddle-loop (a.k.a. homoclinic connection) bifurcations. We have been able to completely describe the bifurcation diagram by identifying all the bifurcation curves. All but the saddle-loop bifurcation curve have been explicitly characterized. The saddle-loop bifurcation curve, however, has been described qualitatively and its end-points as well as its relative position have been identified.

1. Introduction. This study concerns the global bifurcation analysis of a 2-dimensional quadratic ordinary differential equation (o.d.e.) which arises in the study of Model Reference Adaptive Control (MRAC) systems [11]. Specifically, the 2-dimensional o.d.e. captures the essential dynamics that a simple adaptive control system may generate in the presence of external disturbances. The dynamics of the equation were first characterized via digital computer simulations in [1] and were found to exhibit various types of bifurcations for a *specific set of parameters*. Due to the limitation of the computational accuracy of computers, it is essential to verify the bifurcations suggested by the (computer) simulations by analytical means. Moreover, analysis may provide explicit characterization of bifurcation curves and may subsequently lead to the construction of the complete global bifurcation diagram of the system analytically. Such analytical studies do provide support to computer simulations and could bring new insights into alternative choices of system parameters in the design stage of MRAC systems.

This paper is based on the conference version which appeared in [11]. It is organized as follows: In Section 1, we briefly describe the formulation in Model Reference Adaptive Control (MRAC) systems, then we specialize it to the prototype MRAC

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system and the essential 2-dimensional related equation. In Section 2, we determine the number of equilibria and sketch the various phase portraits of the system qualitatively. In Section 3, we study the stability of the equilibrium points in detail and explicitly determine the saddle-node bifurcation curve. We then prove the existence of the Hopf bifurcation and the saddle-loop bifurcation, and give an explicit expression for the Hopf bifurcation curve. As for the saddle-loop bifurcation curve, however, we only specify its end-points and give a qualitative description of its relative position to the other curves. In Section 4, we summarize our results in the form of a theorem.

2. The model reference adaptive control and the prototype adaptive system. Model Reference Adaptive Control (MRAC) systems refer to a control structure for systems modeled as linear time-invariant first order differential equations with unknown parameters. The structure has been constructed with the intent that the overall augmented system (i.e., the system and the associated control structure) may still be described via a set of first order linear equations. This set of equations may in the limit (i.e., as parameters update would have converged) be viewed as time-invariant. Consequently, analysis of MRAC systems has traditionally emphasized linear systems analysis and Laplace transform (frequency) methods.

The engineering idea of MRAC is that one chooses a reference model characterized by a known transfer function or a known set of linear differential equations. The reference model is (forced) by a (chosen) reference signal. Then the standard (feedback) control structure, which is constructed about the system (or plant), uses the same reference signal as its input. The control structure is fixed except for parameters to be specified or, more accurately, to be recursively updated. The parameters update is performed based on the error signals between the output of the system (or plant) and the (desired) output of the reference model. It may also use other accessible signals to achieve successful update. This general description, we hope, will become clear from the following succinct specialization to our prototype adaptive system.

We now describe the usual set-up of MRAC systems specialized to a first order plant with an unknown parameter. We delegate further details to Salam [2] and the references therein. The following defines the main components of the adaptive control system.

$$\text{Model : } \dot{x}_m = a_m x_m + r, \quad (1)$$

$$\text{Plant : } \dot{x} = ax + u + \lambda, \quad (2)$$

$$\text{Control : } u = f(x_m, x, k, r) = -kx + r, \quad (3)$$

$$\text{Adaptation law : } \dot{k} = g(x_m, x, k, r) = \gamma(x - x_m)x - \sigma k, \quad (4)$$

where $a_m < 0$ is a chosen parameter, r is a known reference signal, a and λ are unknown constants; γ and σ are control parameters to be chosen by the designer.

In control theory, a_m is referred to as the known stable pole (or eigenvalue) of the reference model, a is referred to as the unknown pole (or eigenvalue) of the plant, and λ represents an unknown external disturbance acting on the plant.

We note that the functions f and g are specified by the designer. The choice of f and g are crucial to the formulations of MRAC systems. The choice of f as in (3) is suggested by the fact that both plant and model are linear systems and by the well-known constant (state) feedback in linear system theory. Because one addresses only linear (time-invariant) plants and models, there is no obvious benefit from modifying f in (3).

The choice of (4) is a consequence of using a quadratic Liapunov function, then guaranteeing that its usual time-derivative along trajectories is less than or equal to zero (see [2] and the references therein).

Now combining (2) and (3), one obtains the representation of the plant augmented with the control; i.e.,

$$\dot{x} = (a - k)x + r + \lambda. \quad (5)$$

Equation (5) matches equation (1) provided that we choose k^* such that $a_m = (a - k^*)$. Note that this matching equality of parameters is facilitated by choosing the control in (3) as linear in x .

As usual, one derives the error equations as follows. One defines the output (in this case, state) error as $e = x - x_m$. Then one obtains

$$\dot{e} = \dot{x} - \dot{x}_m = ae - ke + \lambda + (-a_m + a - k)x_m, \quad (6i)$$

where k has its dynamics defined by the adaptation law (4), which now reads

$$\dot{k} = \gamma e^2 - \sigma k + \gamma e x_m. \quad (6ii)$$

Equations (6) and equation (1) describe the dynamics of the overall (augmented) system. Note that the dynamics of the state of the reference model x_m (as specified by (1)) is decoupled from the other two states, namely, e and k . Note also that equations (6) are forced by the (independent) reference model state x_m .

Now suppose, for simplicity, that the reference signal $r = 0$. ($r = 0$ is the case referred to in control theory as the regulation problem and it is of interest in its own right [1, 2].) Then, the initial condition $x_m(0)$ only affects the transient behavior of equations (6); specifically, the steady state solutions are not changed. The qualitative dynamics of the overall system are captured by the dynamics on the plane defined by $x_m = 0$. We may thus choose the initial condition $x_m(0) = 0$ so that $x_m(t) = 0$ for all t . Consequently, the dynamics of the adaptive system are captured by the simpler 2-dimensional system

$$\dot{e} = ae - ke + \lambda, \quad (7i)$$

$$\dot{k} = \gamma e^2 - \sigma k. \quad (7ii)$$

Once the analysis of equations (7) are completed, the results can be extended to the case when the reference input r is a sufficiently small periodic function [12].

3. The equilibria and the phase portraits. To simplify the analysis, we first rescale system (7). Assuming that a and λ are nonzero, we define the transformations

$$x = \frac{a}{\lambda}e, \quad y = \frac{1}{a}k, \quad t' = at, \quad (8i)$$

$$\mu = \frac{\lambda^2}{a^4}\gamma, \quad \nu = -\frac{1}{a}\sigma. \quad (8ii)$$

We then get

$$\begin{aligned}\dot{x} &= x - xy + 1 =: F(x, y), \\ \dot{y} &= \mu x^2 + \nu y =: G(x, y),\end{aligned}\tag{9}$$

where the time derivatives are now with respect to t' . For convenience, however, we shall again use t to denote the time. Observe that system (9) has only two independent parameters ν and μ .

We divide the parameter space (ν, μ) by the lines $\nu = 0$, $\mu = 0$ and $\frac{\mu}{\nu} = -\frac{4}{27}$ into six regions as in Figure 1. The singular (equilibrium) points of system (9) are the intersections of the curves $y = 1 + \frac{1}{x}$ and $y = -\frac{\mu}{\nu}x^2$. Within each region and on each boundary, the number of singular points does not change. The x -coordinate of the singular points are the roots of the polynomial $\frac{\mu}{\nu}x^3 + x + 1 = 0$. Note that for a fixed ratio $\frac{\mu}{\nu}$, the roots of this polynomial are fixed. Correspondingly the position (and number) of the singular points of system (9) will be fixed. When $\frac{\mu}{\nu} = -\frac{4}{27}$, the polynomial has double roots $x = -\frac{3}{2}$ which give rise to the degenerate singular point $(x, y) = (-\frac{3}{2}, \frac{1}{3})$.

Corresponding to each region and to each boundary in the parameter space (ν, μ) , the two curves $y = 1 + \frac{1}{x}$ and $y = -\frac{\mu}{\nu}x^2$ divide the whole phase plane (x, y) into several regions. In each of these regions the signs of $\dot{x}$ and $\dot{y}$ are fixed. Consequently, one can sketch the trajectories in the whole (x, y) plane; see [3] and [4]. We have done so and the results are depicted in the various labeled cases of Figures 1 and 2. However, one can not ascertain the existence or otherwise of degeneracies, equilibria at infinity, periodic orbits, and/ or homoclinic orbits.

The sketched phase portraits in Figure 1 depict singular points labeled as P_1 , P_2 , and P_3 , respectively. P_1 is the label for the singular point residing in the first quadrant of the state space. Observe that its y -coordinate is always greater than 1 and its x -coordinate is always positive. P_2 and P_3 are the labels for the singular points residing in the second quadrant with their y -coordinates always less than 1 and their x -coordinates always negative. The y -coordinate of P_2 is always $\leq \frac{1}{3}$ and the y -coordinate of P_3 is always $\geq \frac{1}{3}$ (but is ≤ 1).

4. The complete global analysis.

4.1. On the nonexistence of periodic orbits surrounding the singular point P_2 of case 1⁽¹⁾ and the singular point P_1 of cases 7, 8, 9. *Case 1⁽¹⁾:* From case 1⁽¹⁾ in Figure 1, $\dot{y} > 0$ over all the region $y > 0$ in the (phase) plane. Also, $\dot{x} > 0$ over all the region $x > 0$, $y < 0$ in the (phase) plane. Consequently, there is no possibility of a periodic orbit existing in either one of the two regions. This conclusion extends to the (region) complement of $x < 0$, $y < 0$. Hence, if (9) has a periodic orbit, it must be entirely located within the simply connected region $x < 0$, $y < 0$. However, the divergence of system (9); i.e., $\text{div} (9) = 1 - y + \nu > 0$, is positive (and does not change sign) over any sub-region of region $x < 0$, $y < 0$. Consequently, by the Bendixson theorem [4], no periodic orbits are possible in this region either. Thus no periodic orbits are possible in this case.

Cases 7, 8, 9: Let $H(x, y) = x^{-1}$ be the Dulac function; then the quantity

$$\begin{aligned} \frac{\partial(FH)}{\partial x} + \frac{\partial(GH)}{\partial y} &= \frac{\partial F}{\partial x}H + F\frac{\partial H}{\partial x} + \frac{\partial G}{\partial y}H + G\frac{\partial H}{\partial y} \\ &= (1 - y)x^{-1} - (x - xy + 1)x^{-2} + \nu x^{-1} \\ &= \nu x^{-1} - x^{-2} < 0, \end{aligned}$$

when $x > 0$. Hence by Dulac's Theorem [4], there is no periodic orbit surrounding the focus P_1 in the simply connected region $x > 0$.

In all cases, except cases 7 and 8, the qualitative structure of (9) has been completely determined as depicted in the corresponding cases in Figures 1 and 2.

Next we address case 7, then case 8.

4.2. Singular points at infinity [4], [5], [6]. In order to determine the stability property of the singular point P_3 and in order to determine whether the stable and unstable manifolds of the saddle point P_2 produce a saddle-loop, we transform system (9) to be defined on the Poincare sphere.

Applying the Poincare transformation $x = \frac{1}{z}$, $y = \frac{u}{z}$ to (9) we get

$$\begin{aligned} \frac{du}{dt} &= -uzF\left(\frac{1}{z}, \frac{u}{z}\right) + zG\left(\frac{1}{z}, \frac{u}{z}\right) = \frac{\mu + (\nu - 1)uz + u^2 - uz^2}{z}, \\ \frac{dz}{dt} &= -z^2F\left(\frac{1}{z}, \frac{u}{z}\right) = \frac{uz - z^2 - z^3}{z}. \end{aligned} \quad (10)$$

Let $zdt' = dt$; then we obtain

$$\begin{aligned} \frac{du}{dt'} &= \mu + (\nu - 1)uz + u^2 - uz^2, \\ \frac{dz}{dt'} &= uz - z^2 - z^3. \end{aligned} \quad (11)$$

Singular points of (11) on the line $z = 0$ correspond to singular points at infinity of system (9). Since in case 7 the parameter $\mu > 0$, system (11) has no singular points on $z = 0$.

Now we are going to study the singular point at infinity in the direction parallel to the y -axis. Applying the different Poincare transformation $x = \frac{\nu}{z}$, $y = \frac{1}{z}$ to system (9) we get

$$\begin{aligned} \frac{d\nu}{dt} &= zP\left(\frac{\nu}{z}, \frac{1}{z}\right) - z\nu G\left(\frac{\nu}{z}, \frac{1}{z}\right) = \frac{-\nu + (1 - \nu)\nu z + z^2 - \mu\nu}{z}, \\ \frac{dz}{dt} &= -z^2G\left(\frac{\nu}{z}, \frac{1}{z}\right) = \frac{-\nu z^2 - \mu z\nu^2}{z}. \end{aligned} \quad (12)$$

Let $zdt' = dt$; then we obtain

$$\frac{d\nu}{dt'} = -\nu + (1 - \nu)\nu z + z^2 - \mu\nu^3, \quad (13i)$$

$$\frac{dz}{dt'} = -\nu z^2 - \mu z\nu^2. \quad (13ii)$$

Now let $z = 0$; then the right hand side of equation (13i) gives $-\nu - \mu\nu^3 = -\nu(1 + \mu\nu^2)$. Since $\mu > 0$, we only have the singular point $z = 0, \nu = 0$.

We are now going to determine the stability property of the singular point $(0, 0)$ of system (13).

Let $t' \rightarrow -t'$; then (13) becomes

$$\begin{aligned}\frac{dz}{dt'} &= F_2(z, \nu) = \nu z^2 + \mu z \nu^2, \\ \frac{d\nu}{dt'} &= \nu + G_2(z, \nu) = \nu - (1 - \nu)\nu z - z^2 + \mu\nu^3.\end{aligned}\tag{14}$$

Setting $\nu + G_2(z, \nu) = 0$, we obtain the following analytic expansion of ν in z : $\nu = z^2 + az^3 + \dots$, which when inserted into $F_2(z, \nu)$ gives $+\nu z^2 + \mu z^5 + \dots$. So the singular point $(0, 0)$ is a saddle-node [4, 6]. Observing that the denominator of system (12) is odd and that we have used the transformation $t' \rightarrow -t'$, the point $(0, 0)$ gives rise to the phase portrait depicted in Figure 3.

Now we are going to prove that the stable manifold of P_2 in region $y > 0$ can not tend to ∞ in a direction parallel to the y -axis. In region $\dot{x} > 0, \dot{y} < 0$, the stable manifold can be expressed as a function, say $y(x)$. If $\lim_{x \rightarrow x_L} y(x) = \infty$, where $x_L \leq x_{P_2} < 0$, then we will have

$$\infty = \lim_{x \rightarrow x_L} \frac{\mu \frac{x^2}{y(x)} + \nu}{\frac{x}{y(x)} - x + \frac{1}{y(x)}} = -\frac{\nu}{x_L}.$$

Thus, the contradiction proves our assertion.

As a consequence, we now have three qualitative possibilities for the stable manifold of P_2 in region $y > 0$. These possibilities are depicted on the Poincaré semi-sphere in Figure 4.

4.3. Hopf bifurcation at $P_3(x_0, y_0)$. Let the equilibrium point P_3 have the coordinates (x_0, y_0) . Let $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$. System (9) then becomes

$$\begin{aligned}\frac{dx}{dt} &= (1 - y_0)x - x_0y - xy, \\ \frac{dy}{dt} &= 2x_0\mu x + \nu y + \mu x^2,\end{aligned}\tag{15}$$

where for simplicity we again write (x, y) instead of $(\bar{x}, \bar{y})$. If Hopf bifurcation occurs, (x_0, y_0) must satisfy $(\text{div}(15)) := \text{divergence of system (15)}$

$$\text{div}(15) = (1 - y_0) - y + \nu|_{(0,0)} = 1 - y_0 + \nu = 0.$$

That is, $y_0 = 1 + \nu$. And since $y_0 = 1 + \frac{1}{x_0}$, then $x_0 = \frac{1}{\nu}$. Now substituting into (15) and letting $t \rightarrow -t$, we get

$$\begin{aligned}\frac{dx}{dt} &= \nu x + \frac{1}{\nu}y + xy, \\ \frac{dy}{dt} &= -\frac{2\mu}{\nu}x - \nu y - \mu x^2;\end{aligned}\tag{16}$$

$\frac{2\mu}{\nu^2} - \nu^2 > 0$ is the necessary condition for $(0,0)$ to be a center or a focus of (16). Let $\sigma = \sqrt{\frac{2\mu}{\nu^2} - \nu^2}$, $x = \frac{1}{c}\zeta - \frac{a}{c\sigma}\eta$, $y = -\frac{1}{\sigma}\eta$, and $t' = \sigma t$, where $a = \nu$, $c = -2\frac{\mu}{\nu}$. System (16) is now transformed to

$$\begin{aligned}\frac{d\zeta}{dt} &= -\eta - \frac{\nu^3}{4\mu\sigma}\zeta^2 - \frac{1}{\sigma^2}\left(1 + \frac{\nu^4}{2\mu}\right)\zeta\eta - \frac{\nu}{\sigma^3}\left(1 + \frac{\nu^4}{4\mu}\right)\eta^2, \\ \frac{d\eta}{dt} &= \zeta - \frac{\nu^2}{4\mu}\zeta^2 - \frac{\nu^3}{2\mu\sigma}\zeta\eta - \frac{\nu^4}{4\mu\sigma^2}\eta^2.\end{aligned}\tag{17}$$

Denote

$$\begin{aligned}a_{20} &= -\frac{\nu^3}{4\mu\sigma}, & a_{11} &= -\frac{1}{\sigma^2}\left(1 + \frac{\nu^4}{2\mu}\right), & a_{02} &= -\frac{\nu}{\sigma^3}\left(1 + \frac{\nu^4}{4\mu}\right), \\ b_{20} &= -\frac{\nu^2}{4\mu}, & b_{11} &= -\frac{\nu^3}{2\mu\sigma}, & b_{02} &= -\frac{\nu^4}{4\mu\sigma^2}, \\ A &= a_{20} + a_{02} = -\frac{\nu^3}{4\mu\sigma} - \frac{\nu}{\sigma^3}\left(1 + \frac{\nu^4}{4\mu}\right), \\ B &= b_{20} + b_{02} = -\frac{\nu^2}{4\mu} - \frac{\nu^4}{4\mu\sigma^2}, \\ \alpha &= a_{11} + 2b_{02} = -\frac{1}{\sigma^2} - \frac{\nu^4}{\mu\sigma^2}, \\ \beta &= b_{11} + 2a_{20} = -\frac{\nu^3}{\mu\sigma}.\end{aligned}$$

Since $\nu^8 - 4\mu\nu^4 < 0$, then

$$\frac{1}{\sigma^4} - \frac{\nu^4}{4\mu^2} = \nu^4\left(\frac{1}{4\mu^2 + \nu^8 - 4\mu\nu^4} - \frac{1}{4\mu^2}\right) > 0.$$

For $\nu < 0$, $\mu > 0$ we get

$$W_1 = A\alpha - B\beta = \frac{\nu}{\sigma}\left(\frac{5\nu^4}{4\mu\sigma^4} + \frac{\nu^2}{4\mu\sigma^2} + \frac{1}{\sigma^4} + \frac{\nu^8}{4\mu^2\sigma^4} - \frac{\nu^4}{4\mu^2}\right) < 0.$$

By the results in [7], and considering the change of the sign of t in (16), $(0,0)$ is an unstable weak focus of order one; it means that under the Hopf bifurcation only one unstable limit cycle will be produced near the singular point $(0,0)$. One may directly use analogous formulae based on the theory of Hopf bifurcation, as was reported in [8].

4.4. On the existence of at most one limit cycle when $1 - y_0 + \nu \leq 0$. We rewrite system (15) in the form

$$\frac{dx}{dt} = f_0(x) - f_1(x)y,\tag{18i}$$

$$\frac{dy}{dt} = g_0(x) + g_1(x)y,\tag{18ii}$$

where $f_0(x) = (1 - y_0)x$, $f_1(x) = x_0 + x$, $g_0(x) = 2x_0\mu x + \mu x^2$, and $g_1(x) = \nu$. Now let $\eta = f_0(x) - f_1(x)y$. Then (18) becomes

$$\begin{aligned}\frac{dx}{dt} &= \eta, \\ \frac{d\eta}{dt} &= -\psi_0(x) - \psi_1(x)\eta - \psi_2(x)\eta^2,\end{aligned}\tag{19}$$

where

$$\begin{aligned}\psi_0(x) &= f_1(x)g_0(x) + f_0(x)g_1(x) = \mu x^3 + 3x_0\mu x^2 + (2x_0^2\mu + \nu(1 - y_0))x, \\ \psi_1(x) &= \frac{f_1'(x)}{f_1(x)}f_0(x) - f_0'(x) - g_1(x) = \frac{-(1 - y_0 + \mu)x_0 - \nu x}{x_0 + x}, \\ \psi_2(x) &= -\frac{f_1''(x)}{f_1(x)} = -\frac{1}{x_0 + x}.\end{aligned}$$

Let $\eta = u \exp\left(-\int_0^x \psi_2(z) dz\right)$, then (19) becomes

$$\begin{aligned}\frac{dx}{dt} &= u \exp\left(-\int_0^x \psi_2(z) dz\right), \\ \frac{du}{dt} &= -\psi_0(x) \exp\left(\int_0^x \psi_2(z) dz\right) - \psi_1(x)u.\end{aligned}\tag{20}$$

Let $-\nu = u + \int_0^x \psi_1(s) \exp\left(\int_0^s \psi_2(z) dz\right) ds$, and define the new time transformation

$$\frac{dt}{dt'} = \exp\left(\int_0^x \psi_2(z) dz\right).$$

Equation (20) gives the Lienard system

$$\begin{aligned}\frac{dx}{dt'} &= -\nu - F(x) \\ \frac{d\nu}{dt'} &= g(x),\end{aligned}\tag{21}$$

where

$$\begin{aligned}F(x) &= \int_0^x \psi_1(z) \exp\left(\int_0^z \psi_2(s) ds\right) dz \\ &= \int_0^x \frac{-\nu z - (1 - y_0 + \nu)x_0}{x_0 + z} \exp\left(\int_0^z \frac{-1}{x_0 + s} ds\right) dz, \\ g(x) &= x(\mu x^2 + 3x_0\mu x + 2x_0^2\mu + \nu(1 - y_0)) \exp\left(-2 \int_0^x \frac{1}{x_0 + s} ds\right).\end{aligned}$$

Define

$$a = \frac{-3x_0\mu - \sqrt{x_0^2\mu^2 - 4\mu\nu(1 - y_0)}}{2\mu},$$

and

$$b = \frac{-3x_0\mu + \sqrt{x_0^2\mu^2 - 4\mu\nu(1-y_0)}}{2\mu}.$$

Since $-8x_0^2\mu^2 - 4\mu\nu(1-y_0) < 0$ is the necessary condition for the eigenvalues of (18) to be complex, this implies that $0 < a < b$ and $xg(x) > 0$, whenever $x < a$ and $x \neq 0$. So the limit cycles of (18) must be located in the region $x < a$.

Now we are going to prove that the quantity $(\frac{f(x)}{g(x)})' < 0$ whenever $x < a$, where

$$f(x) := F'(x) = \psi_1(x) \exp\left(\int_0^x \psi_2(s) ds\right).$$

Then by a theorem in [9], system (21), and hence (18), has at most one limit cycle. If this limit cycle exists, it will be unstable.

Thus, we calculate

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{\psi_0(x)\psi_1'(x) - \psi_0(x)\psi_1(x)\psi_2(x) - \psi_1(x)\psi_0'(x)}{\psi_0^2(x)} \exp\left(-\int_0^x \psi_2(s) ds\right),$$

where

$$\begin{aligned} & \psi_0(x)\psi_1'(x) - \psi_0(x)\psi_1(x)\psi_2(x) - \psi_1(x)\psi_0'(x) \\ &= \frac{1}{x_0 + x} \left\{ -\nu x[\mu x^2 + 3x_0\mu x + 2x_0^2\mu + \nu(1-y_0)] \right. \\ & \quad \left. + [(1-y_0+\nu)x_0 + \nu x][3\mu x^2 + 6x_0\mu x + 2x_0^2\mu + \nu(1-y_0)] \right\} \\ &= \frac{1}{x_0 + x} [2\nu\mu x^3 + 3\mu x_0(1-y_0+2\nu)x^2 \\ & \quad + 6\mu x_0^2(1-y_0+\nu)x + x_0\nu(1-3y_0)(1-y_0+\nu)]. \end{aligned}$$

Since

$$x_0 + a = \frac{-x_0\mu - \sqrt{x_0^2\mu^2 - 4\mu\nu(1-y_0)}}{2\mu} < 0,$$

then $x_0 + x < 0$, when $x < a$. Limit cycles can only be located in the region $x < c = \min(a, -x_0 - 1)$; and $\psi_0(x)\psi_1'(x) - \psi_0(x)\psi_1(x)\psi_2(x) - \psi_1(x)\psi_0'(x) < 0$, whenever $x < -x_0 - 1$. In the above calculations, we have used the inequalities $-1 < \nu < 0$ and $0 < \mu < -\nu^3(1+\nu)$, which will be proved in 4.5. Thus we conclude that $(\frac{f(x)}{g(x)})' < 0$, whenever $x < c$.

4.5. On the existence of the saddle-loop curve ℓ_3 and its relative position to the Hopf bifurcation curve $\ell_1 : 1 - y_0 + \nu = 0$ and the curve $\ell_2 : 1 - y_0^* + \nu = 0$. Let the label $P_2(x_0^*, y_0^*)$ denote the equilibrium point P_2 and its coordinates (x_0^*, y_0^*) . Let the label $P_3(x_0, y_0)$ denote the equilibrium point P_3 and its coordinates (x_0, y_0) . We define the Hopf bifurcation curve ℓ_1 to be the curve in the parameter space onto which the equilibrium point $P_3(x_0, y_0)$ is critical and the divergence of system (9) equals $1 - y_0 + \nu = 0$. Analogously, we define the zero-divergence curve ℓ_2 to be the curve in the parameter space onto

which the equilibrium point $P_2(x_0^*, y_0^*)$ renders the divergence of system (9) equal to $1 - y_0^* + \nu = 0$.

At the point $P_2(x_0^*, y_0^*)$, we have the curve $\ell_2 : 1 - y_0^* + \nu = 0$. When $\mu \rightarrow 0$, $y_0^* \rightarrow 0$, so $\nu \rightarrow -1$. When $\frac{\mu}{\nu} \rightarrow -\frac{4}{27}$, $y_* \rightarrow \frac{1}{3}$, so $\nu \rightarrow -\frac{2}{3}$, $\mu \rightarrow \frac{8}{81}$. That is, the end points of ℓ_2 are $B(-1, 0)$ and $A(-\frac{2}{3}, \frac{8}{81})$.

At the point $P_3(x_0, y_0)$, we have the Hopf bifurcation curve $\ell_1 : 1 - y_0 + \nu = 0$. When $\mu \rightarrow 0$, $y_0 \rightarrow 1$, and so $\nu \rightarrow 0$. When $\frac{\mu}{\nu} \rightarrow -\frac{4}{27}$, the double root of $\frac{4}{27}x^3 + x + 1 = 0$ is $x_0 = -\frac{3}{2}$ and so $y_0 = 1 + \frac{1}{x_0} = \frac{1}{3}$. Consequently, $\nu \rightarrow -\frac{2}{3}$ and $\mu \rightarrow \frac{8}{81}$. That is, the end points of ℓ_1 are $(0, 0)$ and $A(-\frac{2}{3}, \frac{8}{81})$.

At every singular point $P(x_0, y_0)$, we have

$$x_0 - x_0 y_0 + 1 = 0, \quad \mu x_0^2 + \mu y_0 = 0. \quad (22)$$

From these two equations we get

$$\frac{\mu + \nu y_0(y_0 - 1)^2}{(y_0 - 1)^2} = 0 \quad \text{or} \quad \mu + \nu y_0(y_0 - 1)^2 = 0.$$

Consequently,

$$\ell_1 : \mu = -\nu^3(1 + \nu) > 0, \quad -\frac{2}{3} < \nu < 0, \quad \lim_{\nu \rightarrow 0} \frac{d\mu}{d\nu} = 0, \quad \lim_{\nu \rightarrow -\frac{2}{3}} \frac{d\mu}{d\nu} = \frac{4}{-27}. \quad (23)$$

Similarly we have

$$\begin{aligned} \ell_2 : \mu = -\nu^3(1 + \nu) > 0, \quad -1 < \nu < -\frac{2}{3}, \\ \frac{d\mu}{d\nu} = -\nu^2(3 + 4\nu) \begin{cases} < 0, & -\frac{3}{4} < \nu < -\frac{2}{3}, \\ > 0, & -1 < \nu < -\frac{3}{4}, \end{cases} \\ \lim_{\nu \rightarrow -1} \frac{d\mu}{d\nu} = 1, \quad \lim_{\nu \rightarrow -\frac{2}{3}} \frac{d\mu}{d\nu} = \frac{-4}{27}. \end{aligned} \quad (24)$$

The maximum value of μ occurs at $\nu = -\frac{3}{4}$; it equals $\frac{27}{256}$. The curves ℓ_1 and ℓ_2 are depicted in Figure 2.

On ℓ_1 , the relative positions of the stable and unstable manifolds of the saddle point $P_2(x_0^*, y_0^*)$ can only be as depicted in Figure 4, case (b). This follows from the fact that on the Hopf bifurcation curve ℓ_1 the quantity $\text{div}(9) |_{P_2} = 1 - y_0^* + \nu > 0$ and also P_3 is unstable. Note that if cases (a) and (c) would occur, there would be at least one-sided stable periodic orbit surrounding P_3 . However, that is impossible as was shown earlier in sub-section 3.4.

On the other hand, suppose $\text{div}(9) |_{P_2} = 1 - y_0^* + \nu \leq 0$. Observe that $y > y_0^*$ at every point on the whole stable and unstable manifolds of $P_2(x_0^*, y_0^*)$. Thus, $1 - y + \nu < 0$ and consequently, $\int \int_G (1 - y + \nu) dx dy < 0$, where the region G is bounded by $S = P_2 N_1 \cup \overline{N_1 N_2} \cup N_2 P_2$ as depicted in Figure 2 - case 7⁽¹⁾.

$\overline{N_1 N_2}$ is a segment on the outer-normal of $N_2 P_2$ at Point N_2 . By Green's Theorem [4]:

$$\int \int_G (1 - y + \nu) dx dy = \oint_S X_n(x, y) dx = \int_{\overline{N_1 N_2}} X_n(x, y) ds < 0,$$

where $X_n(x, y)$ denotes the projection of the vector field $((F(x, y), G(x, y)))$ on the outer-normal of S . Thus the relative position of the stable and unstable manifolds of $P_2(x_0^*, y_0^*)$ must be as in Figure 4, case (a). By continuity, there must exist a saddle-loop line ℓ_3 between ℓ_1 and ℓ_2 .

One end point of ℓ_3 is $A(-\frac{2}{3}, \frac{8}{81})$ and the other is $C(\nu_0, 0)$ with $-1 \leq \nu_0 \leq 0$. It is obvious that $\nu_0 \neq -1$. If $\nu_0 = 0$, then when $\mu \rightarrow 0$ system (11) is transformed into

$$\begin{aligned}\frac{du}{dt'} &= -uz + u^2 - uz^2, \\ \frac{dz}{dt'} &= uz - z^2 - z^3.\end{aligned}\tag{25}$$

The corresponding homogeneous system is

$$\frac{du}{dt'} = u, \quad \frac{dz}{dt'} = z.$$

Hence, as shown in Figure 3 on the Poincare sphere, the singular point $E(\tilde{E})$ is a critical node; i.e., along every ray there is a trajectory going into $E(\tilde{E})$. On the other hand, as $(\nu, \mu) \rightarrow (\nu_0, 0)$ along ℓ_3 , the saddle-loop becomes larger and larger. Thus, it will contradict the continuity of the vector field near E and so $-1 < \nu_0 < 0$.

In the region bounded by ℓ_1 and the line $\frac{\mu}{\nu} = -\frac{4}{27}$, system (9) correspondingly has no saddle-loop. Suppose not; then there exists a saddle-loop L^* for some (ν^*, μ^*) . Now P_3 is unstable and observe also that L^* is repelling from the inside. Consequently, there is at least one stable limit cycle L engulfed by L^* . In this region, P_3 is always unstable, so L must have bifurcated from the saddle-loop L^* . From $\text{div}(9)|_{P_2} = 1 - y_0^* + \nu > 0$, we calculate the integral around L^* , namely,

$$\oint_{L^*} (1 - y + \nu^*) dt > 0.$$

Consequently,

$$\oint_L (1 - y + \nu^*) dt > 0$$

whenever L is sufficiently near L^* . That is, L is unstable. Thus the contradiction proves the nonexistence of L^* .

By the arguments in sub-section 3.4, the saddle-loop and the limit cycle do not coexist. Hence, as one varies (μ, ν) to consecutively traverse the regions (and curves) from $7^{(1)}$, ℓ_1 , $7^{(2)}$, ℓ_3 , and $7^{(3)}$, we obtain the qualitative phase portraits as correspondingly depicted in Figure 2.

4.6. Case 8 ($\frac{\mu}{\nu} = -\frac{4}{27}$ and $\nu < 0$). The eigenfunction of system (15) is

$$\lambda^2 - (1 - y_0 + \nu)\lambda + (1 - 3y_0)\nu = 0.$$

On the line $\frac{\mu}{\nu} = -\frac{4}{27}$, we have $P_2(-\frac{2}{3}, \frac{1}{3})$ and hence $(1 - 3y_0)\nu = 0$; i.e., one of the eigenvalues equals zero. When $\nu < -\frac{2}{3}$, respectively $\nu > -\frac{2}{3}$, the other eigenvalue is less than, respectively greater than, zero. So the singular point P_2 is a sink-saddle, respectively a source-saddle. At the point $A(-\frac{2}{3}, \frac{8}{81})$, we have $(1 - y_0 + \nu) = 0$ and

$(1 - 3y_0)\nu = 0$. Hence both eigenvalues equal zero and the equilibrium point P_2 is degenerate.

Summary of the results. We first relate our system to a system in Chapter 7 of [10]. At the point $A(-\frac{2}{3}, \frac{8}{81})$ in the parameter space (ν, μ) we can transform the original system (9) about the singular point $x = -\frac{3}{2}$, $y = \frac{1}{3}$ to the system

$$\begin{aligned}\frac{dx}{dt} &= y + o(\epsilon^2), \\ \frac{dy}{dt} &= \nu_1 + \epsilon\nu_2 y + \epsilon xy + x^2 + o(\epsilon^2),\end{aligned}\tag{26}$$

where $\nu_1, \nu_2 \in o(1)$. System (26) is identically equation (7.3.12) in [10]. Thus, the results obtained for equation (7.3.12) in [10] also apply to system (9), near the singular point $(-\frac{3}{2}, \frac{1}{3})$. Observe, however, that the results in [10] are only local. In contrast, the results in this paper are global.

We now summarize our results in the form of a theorem.

Theorem.

1. For system (9) :
 - a) the parametric space (ν, μ) is divided into eight regions by the curves: $\nu = 0$, $\mu = 0$, $\frac{\mu}{\nu} = -\frac{4}{27}$, the Hopf bifurcation curve $\ell_1 : \mu = -\nu^3(1 + \nu)$, $-\frac{2}{3} < \nu < 0$, and the saddle connection curve ℓ_3 .
 - b) The end points of ℓ_3 are $A(-\frac{2}{3}, \frac{8}{81})$ and $C(\nu_0, 0)$, where $-1 < \nu_0 < 0$.
 - c) ℓ_1 and ℓ_3 are tangent to the curve $\frac{\mu}{\nu} = -\frac{4}{27}$ at the point $A(-\frac{2}{3}, \frac{8}{81})$.
2. Except for an analytic expression for the saddle connection curve ℓ_3 , the global bifurcation analysis of system (9) is completely determined.
3. System (9) has 13 different topological structures in the above regions and boundaries, namely:

I.	$9, 1^{(1)}$	one focus,
II.	$6^{(1)}, 6^{(2)}, 4$	one saddle,
III.	$5^{(1)}, 5^{(2)}$	no singular points,
IV.	$1^{(2)}$	one degenerate node,
V.	2	two saddle, one node,
VI.	3	one saddle, one saddle-node,
VII.	$8^{(1)}$	one focus, one degenerate singular point,
VIII.	$8^{(2)}$	one focus, one source-saddle,
IX.	$8^{(3)}$	one focus, one sink-saddle,
X.	$7^{(1)}$ on ℓ_1	two foci, one saddle,
XI.	$7^{(2)}$	two foci, one saddle, one limit cycle generated via Hopf bifurcation,
XII.	on ℓ_3	saddle-connection, two foci, one saddle, the limit cycle vanishes, the stable and unstable manifolds of saddle point P_2 become coincident
XIII.	$7^{(3)}$	two foci, one saddle saddle connection vanishes.

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1(1): P_2 -focus, $v > 0$, $\frac{\mu}{v} > 0$

1(2): P_2 -node ($v \neq 1$), degenerate node ($v = 1$), $v > 0$, $\mu = 0$

2: P_2 -node, P_1, P_3 -saddle,

$$v > 0, \quad -\frac{4}{27} < \frac{\mu}{v} < 0$$

3: P_1 -saddle, P_2 -saddle node,

$$v > 0, \quad -\frac{4}{27} = \frac{\mu}{v}$$

4: P_1 -saddle,
 $v > 0, -\frac{4}{27} > \frac{\mu}{v}$

5⁽¹⁾: $\mu > 0, v = 0$

 $5^{(2)}: v=0, \mu < 0$

 $6^{(1)}: P_2\text{-saddle,}$

$$v < 0, \frac{\mu}{v} > 0$$

6⁽²⁾ : P_2 -saddle,
 $v < 0, \mu = 0$

7: P_2 -saddle, P_1, P_3 -focus,
 $v < 0, -\frac{4}{27} < \frac{\mu}{v}$

8(1): P_1 -focus, P_2 -degenerate singular point,

$$v = -\frac{2}{3}, \frac{\mu}{v} = -\frac{4}{27}$$

8(2): P_1 -focus, P_2 -source-saddle,

$$-\frac{2}{3} < v < 0, \frac{\mu}{v} = -\frac{4}{27}$$

8⁽³⁾: P_1 -focus, P_2 -sink-saddle,
 $v < -\frac{2}{3}$, $\frac{\mu}{v} = -\frac{4}{27}$

9: P_1 -focus, $v < 0$, $\mu \leq -\frac{4}{27}$

Figure 2

Figure 3

Figure 4