NETWORK MODELS FOR SUPPLY CHAINS*

S. GÖTTLICH†, M. HERTY‡, AND A. KLAR§

Abstract. A mathematical model describing supply chains on a network is introduced. In particular, conditions on each vertex of the network are specified. Finally, this leads to a system of nonlinear conservation laws coupled to ordinary differential equations. To prove the existence of a solution we make use of the front tracking method. A comparison to another approach is given and numerical results are presented.

Key words. Supply chains, conservation laws, networks, front-tracking.

AMS subject classifications. 90B10, 65Mxx

1. Introduction

Supply chain modelling is characterized by different mathematical approaches: On the one hand there are discrete event simulations based on considerations of individual parts. On the other hand, continuous models like [1, 2, 3] using partial differential equations have been introduced.

In this paper we present a model for the behavior of supply chains on a network. We work with the conservation law presented in [1] which is derived there from a discrete time system.

More precisely, a supply chain with M suppliers is considered, where every supplier m is only linked with the previous supplier $m-1$. Further, every supplier consists of a processor characterized by its processing time $T(m)$, its maximal processing rate $\mu(m)$ and a queue in front, see [1]. The variable $\tau(m,n)$ denotes the arrival time of part n at supplier m. For computing the time evolution of every part the modelling of queues is essential. By assuming FIFO policy two cases of queue states can be distinguished: either the queue is empty or non-empty. If the queue is empty, part n is directly given into the processor m and is produced with time $T(m)$ (see equation (1.1a)). Otherwise the queue is non-empty, so part n has to wait. Its time of waiting is the inverse of the processing rate (see equation (1.1b)). After being produced in processor m the part n is given into the queue of processor $m+1$ (equation (1.2)).

$$\tau(m+1,n) = \tau(m,n) + T(m), \tag{1.1a}$$

$$\tau(m+1,n) = \tau(m+1,n-1) + \frac{1}{\mu(m)}. \tag{1.1b}$$

Summarizing these results yields, see again [1]:

$$\tau(m+1,n) = \max\{\tau(m,n) + T(m), \tau(m+1,n-1) + \frac{1}{\mu(m)}\},$$
$$m = 0,\ldots,M-1, \quad n \geq 0. \tag{1.2}$$

This time recursion can be investigated using the so called Nevell-curves (see [8], [14] for more details), which are curves of cumulative counts using the Heaviside function:

$$U(m,t) = \sum_{n=0}^{\infty} H(t - \tau(m,n)), \quad m = 0,\dots,M, \quad t > 0. \tag{1.3}$$

The focus in [1] is on the derivation of a conservation law based on the time recursion (1.2).

Mapping each supplier onto one gridpoint in space and performing an asymptotic analysis yields under certain assumptions a partial differential equation for the N-curve $\bar{u}(x,t)$ and the density $\bar{\rho}(x,t) = -\partial_x \bar{u}$:

$$\partial_t \bar{u} - \min\{-\frac{L}{T}\partial_x \bar{u}, \, \mu(x)\} = 0 \tag{1.4}$$

and

$$\partial_t \bar{\rho} + \partial_x \min\{\frac{L}{T}\bar{\rho}, \, \mu(x)\} = 0. \tag{1.5}$$

Due to discontinuities of the solution $\bar{u}$ of the first equation one obtains δ-distributions on the level of the solution of the equation for $\bar{\rho}$. These δ-distributions are natural, modelling the queues in the system, but do not allow for a simple theoretical treatment of the equation. In the following section we investigate a similar model. By modelling queues exactly in front of each supplier we obtain an equation for the density avoiding the above bottlenecks in the density. In section 3 an analysis of the new model is performed and the existence of the solutions is shown. Section 4 points out the differences and similarities of the two models. In particular, the above equation for the N-curve $\bar{u}$ from [1] is recovered. Numerical results are presented in section 5. A major advantage of the present approach is that it is easily adapted to more complicated networks with vertices with multiple entries and exits. This will be discussed shortly in the final section and in more detail in a forthcoming paper.

2. Modelling

In this section we introduce a new model for large queuing supply chain networks based on the work of Armbruster, Degond and Ringhofer (see [1]). The main new ingredients are the formulation as a PDE network problem and a separate modelling of the queues. One advantage of such a point of view is the easy accessibility to existence theory of the network problem. Moreover, in this framework situations with real networks having multiple incoming and outgoing arcs for each vertex are easily included. At first, we state the definition of a supply chain network and describe the connection between the network and the suppliers.

DEFINITION 2.1. *[Network definition] A supply chain network is a finite, connected directed, simple graph consisting of arcs $\mathcal{J} = \{1,\dots,N\}$ and vertices $\mathcal{V} = \{1,\dots,N-1\}$. Each supplier j is modelled by an arc j, which is again parameterized by an interval $[a_j,b_j]$. We use $a_1 = -\infty$ and $b_N = +\infty$ for the first respectively the last supplier in the supply chain.*

First, we consider the special case where each vertex is connected to exactly two arcs. For notational convenience we assume that $b_j = a_{j-1}$, c.f. Figure 2.1. As already mentioned, a supplier j is defined by a queue j and a processor j. Physically, the queue is located in front of each processor, i.e., at $x = a_j$. To avoid technical difficulties, we assume that the first supplier consists of a processor only.
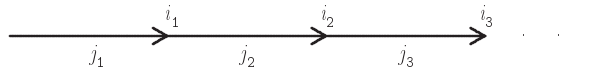
FIG. 2.1. *Example of a simple network structure*

Each processor j is defined by a maximum processing capacity μ_j, its length L_j and the processing time T_j. The rate L_j/T_j describes the processing velocity. Further, ρ_j denotes the density of parts in the supply chain at point x and time t. The dynamics of each processor on an arc j are governed by an advection equation as in [1]:

$$\partial_t \rho_j(x,t) + \partial_x \min\{\mu_j, \frac{L_j}{T_j}\rho_j(x,t)\} = 0, \quad \forall x \in [a_j, b_j], t \in \mathbb{R}^+ \tag{2.1a}$$

$$\rho_j(x,0) = \rho_{j,0}(x), \quad \forall x \in [a_j, b_j]. \tag{2.1b}$$

Note that we use the flux functions derived in [1]

$$f : \mathbb{R}_0^+ \rightarrow [0,\mu], \quad f(\rho) = \min\{\mu, \frac{L}{T}\rho\}, \tag{2.2}$$

with a positive constant μ, i.e., the maximal rate for the processor. Clearly, f is Lipschitz with constant $L_f = \frac{L}{T}$.
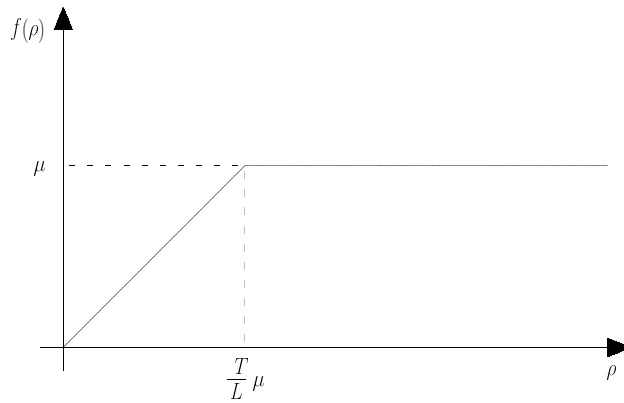

FIG. 2.2. *Relation between flow and density*

REMARK 2.2. Usually, an inflow profile $f_1(t)$ for the supply chain is given. In the above model, this profile can be translated into initial data $\rho_{1,0}(x) := \rho_{1,0}(b_1 - t) = f_1(t)$ on an (artificial) first arc, where in addition we assume $\mu_1 > \max f_1$, $L_1/T_1 = 1$.

In contrast to [1] we consider queues in front of each processor. Each queue is a time-dependent function $t \rightarrow q_j(t)$ and used to buffer demands for the processor j. If the capacity of processor $j-1$ and the demand of processor j are not equal, the queue q_j increases or decreases its buffer. Mathematically, we require each queue q_j to satisfy the following equation:

$$\partial_t q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1},t)) - f_j(\rho_j(a_j,t)), \quad j = 2,\dots,N \tag{2.3}$$

Due to the advection, there is freedom in defining the flux on the outgoing arc j. We use

$$f_j(\rho_j(a_j,t)) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1},t)),\mu_j\}, & q_j(t)=0 \\ \mu_j, & q_j(t)>0. \end{cases} \tag{2.4}$$

The flux $f_j(\rho_j(a_j,t))$ is dependent on the capacity of the queue. If the queue is empty, $f_j(\rho_j(a_j,t))$ is either $f_{j-1}(\rho_{j-1}(b_{j-1},t))$ or μ_j. Otherwise, if the queue is non-empty, the queue is reduced with rate μ_j. For the precise definition of a solution at the vertex, see Definition 3.5. Finally, we obtain the following coupled system of partial and ordinary differential equations (2.5) on a network given by Definition 2.1.

$$\partial_t \rho_j(x,t) = -\partial_x \min\{\mu_j, \frac{L_j}{T_j}\rho_j(x,t)\} \tag{2.5a}$$

$$\rho_j(x,0) = \rho_{j,0}(x) \tag{2.5b}$$

$$\partial_t q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1},t)) - f_j(\rho_j(a_j,t)) \tag{2.5c}$$

$$q_j(0) = q_{j,0} \tag{2.5d}$$

$$f_j(\rho_j(a_j,t)) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1},t)),\mu_j\}, & q_j(t)=0 \\ \mu_j, & q_j(t)>0. \end{cases} \tag{2.5e}$$

3. Theoretical Investigations

In this section we give precise statements of the definition of a solution to the supply chain model. We prove the existence of solutions for piecewise constant initial data up to any fixed positive time T. The construction of the solution and the proof is based on wave- or front-tracking method, see [5, 11].

First, we discuss the existence results for the Cauchy problem on a single arc without coupling conditions. We recall the definition of entropic solutions in sense of Kruzkov [13] and well-known existence results [5, 7, 11]. Then we define the solution at a single vertex. Finally, we combine the results to prove existence for the whole network.

We recall the definition of solutions for the general Cauchy problem

$$\partial_t \rho(x,t) + \partial_x f(\rho) = 0, \rho(x,0) = \rho_0(x) \tag{3.1}$$

in the sense of [13]:

DEFINITION 3.1. *A locally bounded and measurable function $\rho(x,t)$ on $\mathbb{R} \times \mathbb{R}_0^+$ is called an admissible weak solution to (3.1), if for any non-decreasing function $h(\rho)$ and any smooth non-negative function ϕ with compact support in $\mathbb{R} \times \mathbb{R}_0^+$,*

$$\int_0^\infty \int_{-\infty}^\infty (I(\rho)\phi_t + F(\rho)\phi_x)\,dx\,dt + \int_{-\infty}^\infty I(\rho_0)\phi(x,0)\,dx \geq 0 \tag{3.2}$$

where $I(\rho) = \int^\rho h(\xi)\,d\xi$ and $F(\rho) = \int^\rho h(\xi)\,df(\xi)$.

The following result is well-known.

THEOREM 3.2. *[Lemma 3.1 [7]] Assume f piecewise linear and Lipschitz continuous on $[m,M]$ and*

$$\rho_0(x) = \begin{cases} \rho_l & x \leq 0 \\ \rho_r & x \geq 0 \end{cases} \tag{3.3}$$

with constants $\rho_l, \rho_r \in [m, M]$. Then there exists an admissible weak solution of (3.1) which consists of a finite number of constant states separated by shocks centered at the origin.

For our special flux function (2.2), the solution of Riemann problems for (3.1) and (2.2) is either given by (3.4) or by (3.5): Consider the initial data (3.3) with $\rho_l, \rho_r \in \mathbb{R}_0^+$. Let $\rho_l < \rho_r$, then the weak admissible solution to (3.1), (2.2) and (3.3) is given by

$$\rho(x,t) = \begin{cases} \rho_l, & -\infty < \frac{x}{t} \leq \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l}, \\ \rho_r, & \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l} < \frac{x}{t} < \infty. \end{cases} \tag{3.4}$$

In the case $\rho_r < \rho_l$ we distinguish three cases. If $\rho_l \leq \mu$ or if $\rho_r \geq \mu$ the solution is given by (3.4). In the remaining case $\rho_r < \mu < \rho_l$ we obtain the solution (3.5)

$$\rho(x,t) = \begin{cases} \rho_l, & -\infty < \frac{x}{t} \leq \frac{\mu - f(\rho_l)}{\mu - \rho_l}, \\ \mu, & \frac{\mu - f(\rho_l)}{\mu - \rho_l} < \frac{x}{t} \leq \frac{f(\rho_r) - \mu}{\rho_r - \mu}, \\ \rho_r, & \frac{f(\rho_r) - \mu}{\rho_r - \mu} < \frac{x}{t} < \infty. \end{cases} \tag{3.5}$$

In this case it holds $\frac{\mu - f(\rho_l)}{\mu - \rho_l} = 0$ and $\frac{f(\rho_r) - \mu}{\rho_r - \mu} = 1$.

The idea to construct a solution $\rho(x,t)$ for all times t is given by the front- or wave-tracking algorithm [5, 11]: We start with a step function $\rho_0(x)$ and solve at each point of a jump discontinuity a Riemann problem as described above. Then the solution $\rho(x,t), t > 0$ is again a step function with discontinuities travelling at constant speed (fronts). At some time $\bar{t} > 0$ one or more fronts may collide. As before we proceed by solving a Riemann problem with initial data $\rho(x, \bar{t})$. For all times t this procedure is well-defined and generates a solution $\rho(x,t)$ once we proved that the number of collisions is finite. The following lemma - which holds true in more general situations (Lemma 2.6 [11]) - can be proven rather elementary for the above flux function. This lemma is necessary to prove existence for a single conservation law as (3.1) as well as for a coupled system of equations.

LEMMA 3.3. *Consider the problem (3.1) and (2.2). Let M be any fixed positive number. For each fixed δ, such that $\mu/\delta \in \mathbb{N}$, consider the set of points $R := \{\rho^i\}$ given by $\rho^i = i\delta$ for $0 \leq i\delta \leq M$ and $i \in \mathbb{N}_0$. Let the initial data ρ_0 be a piecewise constant function with values in R.*

Then there exists only a finite number of interactions between discontinuities of the admissible weak solution to (3.1), (2.2) for each $t \in \mathbb{R}_0^+$.

Proof. We consider the case $M > \mu$. Note that by assumption $\exists i$ such that $\mu = \rho^i$ and therefore f has breakpoints in the set R. Consider the integer valued function $\phi(t)$ where $\phi(t)$ is the number of points of jump discontinuities of the solution $\rho(x,t)$ at time t. The function $N : \mathbb{R}_0^+ \to \mathbb{N}$ defined by $N(t) := \phi(t) + \phi(0)$ is strictly decreasing for each collision of discontinuities. Indeed, assume that $k \geq 2$ fronts given by the jumps given by $(\rho^1, \rho^2), (\rho^2, \rho^3), \dots, (\rho^k, \rho^{k+1})$ collide at time $\bar{t}$. By construction, the solution to the Riemann problem with initial data (ρ^i, ρ^{i+1}) is locally of the type (3.4) or (3.5). In the first case $j(t)$ decreases by $k-1$ and $N(t)$ decreases by at least one. The case (3.5) cannot happen due to the monotone flux function: Consider the case $k = 2$ and assume $\rho^1 > \mu > \rho^3$. For a collision of the two discontinuities we need that $f(\rho^1) - f(\rho^2) > 0$ which implies that $\rho^2 < \mu$. This is a contradiction to the speeds of the fronts. Finally, $N(t)$ is strictly decreasing for each collision and the construction

is well-defined. Furthermore, by (3.4) and (3.5) the total variation is non-increasing and bounded: $TV(\rho(\cdot,t)) \leq TV(\rho_0(\cdot))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Using the above lemma the proof of the following theorem is standard, see for example Theorem 2.13 [11].

THEOREM 3.4. *Let ρ_0 be a $L^1(\mathbb{R})$-function with bounded variation and let f be a Lipschitz function. Then there exists a unique admissible weak solution $\rho(x,t)$ to the initial value problem*

$$\partial_t\rho + \partial_x\min\{\mu,\rho\} = 0, \quad \rho(x,0) = \rho_0(x). \qquad (3.6)$$

We extend the results to the network case. For the remaining part we assume the following,

(A1) The supply chain can be modelled as in Definition 2.1. We assume a consecutive labelling of the processors, such that processor $j-1$ is connected at $x=b_{j-1}$ to processor j.

(A2) $L_j/T_j = 1$ for all $j \in \mathcal{J}$.

(A3) $\rho_{j,0}(x) \leq \mu_j$ a.e. $x \in [a_j,b_j]$ and all $j \in \mathcal{J}$

Recall that the density ρ_j in the processor $j \in \mathcal{J}$ of the supply chain satisfies

$$\partial_t\rho_j + \min\{\mu_j, \frac{L_j}{T_j}\rho_j\} = 0, \quad \forall(x,t) \in [a_j,b_j] \times \mathbb{R}_0^+ \qquad (3.7\text{a})$$

$$\rho_j(x,0) = \rho_{j,0}(x). \qquad (3.7\text{b})$$

We consider admissible weak solutions $\rho_j(x,t)$ in the sense of Definition 3.1 on each arc j. We define admissible solutions at a vertex $v \in \mathcal{V}$ as follows, see also Figure 3.1.

DEFINITION 3.5. *Given functions $\rho_1(x,t),\rho_2(x,t)$ in $L^1([a_j,b_j] \times \mathbb{R}_0^+)$ and such that $\rho_j(\cdot,t)$ has bounded variation. Let $q(t) := q_2(t) \geq 0$ be a an absolutely continuous function on $[0,T]$ for T sufficiently large and let $f_j(\rho) := \min\{\mu_j,\rho\}$.*
Then we call (ρ_1,ρ_2,q) an admissible solution at the vertex for all times $0 \leq t \leq T$, if and only if

$$\frac{d}{dt}q(t) = f_1(\rho_1(b-,t)) - f_2(\rho_2(a+,t)) \quad a.e.\ t \qquad (3.8\text{a})$$

$$f_2(\rho_2(a+,t)) = \begin{cases} \mu_2, & q(t) > 0 \\ \min\{\mu_2, f_1(\rho_1(b-,t))\}, & q(t) = 0. \end{cases} \qquad (3.8\text{b})$$

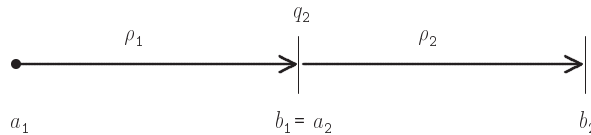Now, we study Riemann problems at the vertex. These solutions will be used to

FIG. 3.1. *Two suppliers linked by queue q_2*

construct solutions for the network problem by front-tracking. The first result gives existence of admissible solutions at the vertices for constant initial data.

THEOREM 3.6. *Let T be arbitrary fixed time, $a_j = 0$, $b_j = 1$ and let $\rho_{1,0} \leq \mu_1$, $\rho_{2,0} \leq \mu_2$ and $q_{2,0} \geq 0$ be constants.*

Then there exists a unique admissible solution (ρ_1,ρ_2,q) at the vertex and an admissible weak solution (ρ_1,ρ_2) on the arcs for all times $0 \leq t \leq T$ such that $\rho_j(x,0) = \rho_{j,0}$, $q_2(0) = q_{2,0}$ and the solution is given by

$$\rho_1(x,t) = \rho_{1,0} \tag{3.9a}$$

$$\rho_2(x,t) = \begin{cases} f_1(\rho_{1,0}) < \mu_2 \begin{cases} \rho_{1,0} & 0 \leq (x-t_0)/t < 1 = \frac{f_2(\mu_2)-f_2(\rho_{1,0})}{\mu_2-\rho_{1,0}} \\ \mu_2 & 1 \leq (x-t_0)/t \text{ and } x/t < 1 \\ \rho_{2,0} & 1 \leq x/t < \infty \end{cases} \\ f_1(\rho_{1,0}) \geq \mu_2 \begin{cases} \mu_2 & 0 \leq x/t < 1 = \frac{f_2(\mu_2)-f_2(\rho_{2,0})}{\mu_2-\rho_{2,0}} \\ \rho_{2,0} & 1 \leq x/t < \infty \end{cases} \end{cases} \tag{3.9b}$$

$$q_2(t) = q_{2,0} + \int_0^t f_1(\rho_{1,0}) - f_2(\rho_2(a+,\tau))d\tau \tag{3.9c}$$

wherein $t_0 = q_{2,0}/(\mu_2 - f_1(\rho_{1,0}))$.

REMARK 3.7. We refer to Figure 3.2 for a sketch of the solution $\rho_2(x,t)$ in the $x-t-$plane in the case $q_{2,0} > 0$ and $f_1(\rho_{1,0}) < \mu_2$.

In the case $q_{2,0} = 0$ we obtain $\rho_2(x,t) = \rho_{1,0}$ for $0 \leq x/t < 1$ and $\rho_2(x,t) = \rho_{2,0}$ for $1 \leq x/t < \infty$. t_0 denotes the first time at which the queue is empty.

Obviously, the discussion is similar removing assumption (A2).

Proof. The solution (3.9a-3.9c) satisfies (3.8a)-(3.8b). Further, it is an admissible weak solution in all cases. In the interesting case $f_1(\rho_{1,0}) < \mu_2$ the solution $\rho_2(x,t)$ is the restriction of the solution to the Cauchy problem (3.7) with initial data $\rho_0(x) = \rho_{1,0}$ for $x \leq -t_0$, $\rho_0(x) = \mu_2$ for $-t_0 < x \leq 0$ and $\rho_0(x) = \rho_{2,0}$ for $x > 0$.

For the uniqueness we notice that since $f_j' \geq 0$ the solution $\rho_1(x,t)$ has to be (3.9a). The map $\rho \to f_j'(\rho)$ is invertible for $0 \leq \rho \leq \mu_j$. If $f_1(\rho_{1,0}) \geq \mu_2$ then (3.8b) implies (3.9b).
We discuss the case $f_1(\rho_{1,0}) < \mu_2$ and $q_{2,0} > 0$ with the remaining case being similar. There exists $t_0 > 0$ such that $q(t_0) = 0$ and $\rho_2(a+,t) = \mu_2$ for $t \leq t_0$. t_0 is given by $0 = q_{2,0} + \int_0^{t_0} f_1(\rho_{1,0}) - \mu_2\, d\tau = q_{2,0} - t_0(\mu_2 - f_1(\rho_{1,0}))$. For $t \geq t_0$ we obtain $\rho_2(a+,t) = \rho_{1,0}$ and $q(t) = 0$. Hence the solution is given by (3.9b).   □
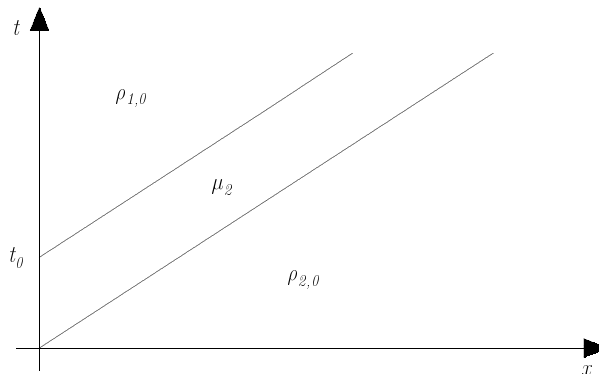

FIG. 3.2. *Admissible solution at the vertex in the subcase $q_{2,0} > 0$ and $f_1(\rho_{1,0}) < \mu_2$ in the $x-t-$plane*

Next, we define solutions for the network problem.

DEFINITION 3.8. *Assume a network geometry as in assumption (A1) with consecutive labelling of the processors according to (2.5). Let $T>0$, values $q_{j,0} \geq 0$, $j = 2,\dots,N$ and functions $\rho_{j,0} : [a_j,b_j] \to \mathbb{R}$ in L^1 and with bounded variation for all $j=1,\dots,N$ be given. The supply chain problem then reads with $f_j = \min\{\mu_j, \rho_j\}$ and $\forall j=1,\dots,N \, \forall (x,t) \in (a_j,b_j) \times (0,T), \, \forall j=2,\dots,N$*

$$\partial_t \rho_j + \partial_x f_j(\rho_j) = 0, \quad \rho_j(x,0) = \rho_{j,0}(x), \tag{3.10a}$$
$$\partial_t q_j(t) = f_{j-1}(\rho_{j-1}(b-,t)) - f_j(\rho_j(a+,t)). \tag{3.10b}$$

We call a family $\rho_j : [a_j,b_j] \times [0,T]$ of L^1-functions with bounded variation and functions q_j absolutely continuous an admissible solution to the network problem, if for each vertex $(\rho_j, q_j)_j$ is an admissible solution at the vertex in the sense of Definition 3.5, if for all $j=2,\dots,N$, $q_j(0) = q_{j,0}$ and if for all j ρ_j is an admissible weak solution for the processor in the sense of Definition 3.1.

REMARK 3.9. In contrast to the notation of solutions proposed in traffic flow theory, see for example [6, 12], the above defined solution is **not** a solution in the sense that

$$\sum_j \int_{a_j}^{b_j} \int_0^\infty \left(\partial_t \phi_j \rho_j + \partial_x \phi_j f_j(\rho_j) dx \, dt \right) = 0 \tag{3.11}$$

where $\{\phi_j\}_{j=1}^N$ is a family of smooth test-functions and where each ϕ_j has compact support in $[a_j,b_j] \times \mathbb{R}_0^+$ and is smooth across a vertex, i.e., $\phi_j(b,t) = \phi_{j+1}(a,t)$ if arcs j and $j+1$ are connected. Especially (3.11) implies that for sufficiently regular solutions

$$f_j(\rho_j(b-,t)) = \min\{\mu_j, \rho_j(b-,t)\}$$
$$= \min\{\mu_{j+1}, \rho_{j+1}(a+,t)\} = f_{j+1}(\rho_{j+1}(a+,t)).$$

This condition is replaced by $\partial_t q_{j+1}(t) + f_{j+1}(\rho_j(b-,t)) = f_j(\rho_j(a+,t))$ with the physical interpretation that the incoming flux is distributed among the outgoing processor and its buffering queue.

Using the Front–Tracking idea we show for piece-wise constant initial data, we can define a solution to the network problem (Definition 3.8) up to any positive time T. We introduce a discretization of the flux functions. Let $\bar{\mu} := \max\{\mu_j : j=1,\dots,N\}$ and introduce a equi-distant grid $(i\delta)_{i=0}^{N_x}$ such that $0 \leq i\delta \leq \bar{\mu}$ and such that $\forall j \, \exists i_j \, i\delta = \mu_j$. Due to assumption (A3) $\rho_{j,0}(x) \leq \mu_j$ we can approximate the initial data by a step function $\rho_{j,0}^\delta$ taking values in the set $\{i\delta : i=0,\dots,N_x\}$. Further, this construction ensures that the solution at the vertex only takes values in this set, see (3.9a-3.9c). For the Riemann problems with data $\rho_{j,0}^\delta$ on each arc j we can find solutions ρ_j^δ using Lemma 3.3. Then the solutions define a set of discontinuities moving along the intervals $[a_j,b_j]$. Clearly, ρ^δ can be defined until the first discontinuities collide. Either the collision can be resolved by solving a new Riemann problem on the arc (which is a well-defined process due to Theorem 3.4) or we obtain a collision with a vertex discussed in Lemma 3.10. In both cases we obtain new discontinuities which can be propagated until the next collisions. This can be repeated and below we show that indeed ρ^δ can be defined up to any given time T.

From now we consider a step function $\rho_j^\delta(x,t)$, $j=1,\dots,N$, defined by a number of constant states $\rho_{j,i}^\delta$ for $i=1,\dots,N-1$. For notational convenience we assume that $\rho_j^\delta(a_i,\cdot) = \rho_{j,1}^\delta$ and so forth.

LEMMA 3.10. *Given a vertex with ingoing arc $j=1$ and outgoing arc $j=2$, assume states $\rho_{1,0}, \rho_{2,0}$ and $q_{2,0}$ given with $\rho_{j,0} \leq \mu_j$. Consider a discontinuity colliding with the vertex at time $\bar{t}$.*

Then the admissible solution to the network problem in sense of Definition 3.8 is again a step function and the integer valued function $4\phi_1(t) + \phi_2(t)$ is strictly decreasing after the collision, where $\phi_j(t)$ denotes the number of possible jump discontinuities on arc j at time t and is defined by (3.12).

Proof. The colliding discontinuities can arrive from the ingoing arc $j=1$ only, since $f_j'(\cdot) \geq 0$. Denote by $(\bar{\rho}_1, \rho_{1,0})$ the colliding discontinuity with $\bar{\rho}_1 \leq \mu_1$.

By Theorem 3.6 we obtain that $\rho_1(x,t) = \bar{\rho}_1$ for $t \geq \bar{t}$ and $\rho_2(x,t)$ is a step function with at most three steps. We define ϕ_j before we discuss the possible solutions ρ_2 arising due to the collision of $(\bar{\rho}_1, \rho_{1,0})$ at time $\bar{t}$.

$$\phi_j(t) := \varphi_j(t) + 2\psi_j(t) \tag{3.12}$$

Here φ_j denotes the number of discontinuities in the solution ρ_j at time t. The function ψ_j indicates possible discontinuities arising due to the queue q_j. To be more precise we define

$$\psi_j(t) := \begin{cases} 0, & \text{if } \partial_t q_j(t) \geq 0 \\ 1, & \text{if } \partial_t q_j(t) < 0 \end{cases}, \tag{3.13}$$

and $\psi_j(0) = 1$, if $q_j \neq 0$. If (the integer–valued function) $\phi_j(t) = 0$ at some time $t > 0$, then ρ_j is a constant. The collision yields a reduction of φ_1 by one. For the outgoing arc $j=2$ we distinguish the following cases and remember that the solution ρ_2 is given by (3.9b). First assume $\bar{\rho}_1 > \mu_2$, then ϕ_j increases at most by three, since both φ_2 and ψ_2 might increase by at most one. Second, assume that $\bar{\rho}_1 < \mu_2$. Then φ_j increase by at most one at time $\bar{t}$ and possibly again by at most one at time $\bar{t} + t_0$, where t_0 is as in Theorem 3.6. If at time $\bar{t} + t_0$ φ_j increases, then ψ_j decreases by exactly one, since the queue $\partial_t q_2(\bar{t} + t_0) = 0$. Summarizing, ϕ_2 increases by at most three at time $\bar{t}$ and is possibly decreasing at time $\bar{t} + t_0$ by at least one. This yields the assertion. □

REMARK 3.11. Note that due to the positive velocity of the travelling discontinuities (or equivalently due to the monotonicity of the flux functions f_j) an estimate on the number of travelling discontinuities is also an estimate on the number of possible interactions (i.e. collisions).

REMARK 3.12. Due to Lemma 3.3 we already know that for each single arc the function $\phi_j(t) + \phi_j(0)$ is strictly decreasing for each collision. Generalizing this to a network fulfilling the assumptions $(A1 - A3)$ with N arcs, we see that the function

$$N(t) := \sum_{j=1}^{N} (\phi_j(t) + \phi_j(0)) + \sum_{j=1}^{N} 4^{N-j} \phi_j(t)$$

is strictly decreasing for each collision. Hence, the total number of collisions is bounded by $N(0)$ and the construction outlined above is well-defined.

In the setting of Lemma 3.10 we cannot expect that the total variation is non-increasing. Indeed it may increase due to a collision with the vertex. Assume $\rho_{2,0} = \mu_2$, $q_2 > 0$, $f_1(\rho_{1,0}) < \mu_2$ and $\bar{\rho}_1$ arbitrary. Further let $\bar{t} < q_2/(\mu_2 - f(\rho_{1,0}))$ be the collision time of the discontinuity $(\bar{\rho}_1, \rho_{1,0})$ with the vertex. Then for $t \geq \bar{t}: TV(\rho_2(\cdot,t)) \geq |\bar{\rho}_1 - \mu_2|$ which cannot be bounded by $|\bar{\rho}_1 - \rho_{1,0}|$ and $TV(\rho_2(\cdot,0))$.

Therefore, the total variation might increase due to collisions with the vertices and there is no bound uniformly in δ.

THEOREM 3.13. *Assume (A1-A3) and a network of N processors. Consider the problem (3.1) and (2.2). Assume that the initial data $(\rho_{1,0}(x),\ldots,\rho_{N,0}(x))$ are step functions.*

Then the problem (3.1), (2.2) has a weak admissible solution constructed by admissible network solutions in the sense of Definition 3.8.

4. Comparison with the approach by Armbruster, Degond, Ringhofer [1]

In this section we compare the present model to the model developed in [1]. In [1] a PDE for the N-curve $\bar{u}(x,t)$ is derived. We use the notation $\bar{u}$ and $\bar{\rho}$ to denote the solutions from [1]. The density $\bar{\rho}(x,t)$ is determined as the negative space derivative of $\bar{u}(x,t)$. As we will see, although the concept of solution of the network problem is different, both models give the same equation for the N-curves.

Assume $L = T = 1$ and two connected processors as in Figure 3.1. Then the equation for the N-curve $\bar{u}(x,t)$ in [1] on the interval $x \in [a_1,b_1] \cup [a_2,b_2]$ with $b_1 = a_2$ is given by

$$\partial_t \bar{u}(x,t) = \underbrace{\min(\mu(x), -\partial_x \bar{u}(x,t))}_{=:f(x,\bar{\rho})}, \tag{4.1a}$$

$$\bar{u}(a_1,t) := \int_0^t f(a_1, \bar{\rho}(a_1,t'))\, dt'. \tag{4.1b}$$

By the discussion in the introduction we have $\bar{\rho} = -\partial_x \bar{u}$. Further, for $x \le b_1 : \mu(x) = \mu_1$ and $f(x,\bar{\rho}) \equiv f_1(\bar{\rho})$ and for $x \ge a_2 : \mu(x) = \mu_2, f(x,\bar{\rho}) \equiv f_2(\bar{\rho})$.

Consider now the present model and the network solution $\rho(x,t)$ defined in this paper and $u(a_1,t) \equiv \bar{u}(a_1,t)$ as above, we set

$$u(x,t) := u(a_1,t) - \int_{a_1}^x \rho(x',t)\, dx' - q(t)\chi_{[a_2,x]}. \tag{4.2}$$

Formally, we calculate

$$\partial_t u(x,t) = \partial_t u(a_1,t) - \int_{a_1}^x \partial_t \rho(x',t)\, dx' - \partial_t q(t)\chi_{[a_2,x]}$$

and one obtains for $x \in [a_1,b_1] : \partial_t u(x,t) = f_1(\rho_1(x,t)) = f(x,\rho)$ and for $x \in [a_2,b_2]$:

$$\begin{aligned}
\partial_t u(x,t) &= \partial_t u(a_1,t) + \int_{a_1}^{b_1} \partial_x f(\rho(x',t))\, dx' + \int_{a_2}^x \partial_x f(\rho(x',t))\, dx' - \partial_t q(t) \\
&= \partial_t u(a_1,t) + f_1(\rho_1(b_1,t)) - f_1(\rho_1(a_1,t)) + f_2(\rho_2(x,t)) \\
&\quad - f_2(\rho_2(a_2,t)) - \big(f_1(\rho_1(b_1,t)) - f_2(\rho_2(a_2,t))\big) \\
&= f_2(\rho_2(x,t)) = f(x,\rho).
\end{aligned}$$

I.e. we obtain the same equation as in [1] and further, $\bar{u} = u$.

In other words the definition of the N-curve $u(x,t)$ given above coincides with the N-curve $\bar{u}(x,t)$ in [1]. However, the definition of the density $\rho(x,t)$ is different. Here, the points of discontinuity of $\bar{u}(x,t)$, i.e. the queues, are modelled by a separate function $q(t)$. Thus, the remaining part of $u(x,t)$ gives a well defined density $\rho(x,t)$ which can be treated by the usual front tracking algorithm avoiding the issue of distributional solutions of the equation for $\bar{\rho}(x,t)$ in [1].

5. Numerical Results

We present numerical results for our network model explained in the sections before. The example we consider is similar to the one in [1]: We consider a supply chain with $N=4$ consisting of three processors with queues characterized by L_j, μ_j and T_j for $j=1,\ldots,4$ and an inflow arc. We use the labelling according to our network Definition 2.1. According to the discussion above the numerical results for the **N-curve** $u(x,t)$ **are equivalent** to those obtained in [1]. This in particular true due to the same Upwind discretization given below and in [1].

Numerically, we discretize the system (2.5) using an Upwind-scheme for the advection equation and an explicit Euler-scheme for the queues. Therein, each arc j could have different space increments, namely $\Delta x_j = \frac{L_j}{N_j}$, where N_j is the number of space discretization points. For simplicity, the time steps Δt are constant and satisfy the CFL condition on each arc.

To compare the results with [1] we use the data in Table 5.1. Initial values for

Processor j	N_j	μ_j	T_j	L_j
1	10	25	1	1
2	10	15	1	0.2
3	30	10	3	0.6
4	10	15	1	0.2

TABLE 5.1. *Parameters of the example problem*

the first arc are fixed by the inflow profile $f_1(t)$, see the discussion in Remark 2.2. All other initial values are zero, i.e., $\rho_{j,0}=0$ and $q_{j,0}=0$ for $j=2,3,4$. The initial profile is such that it exceeds the maximum capacity of the processors, see Figure 5.1.
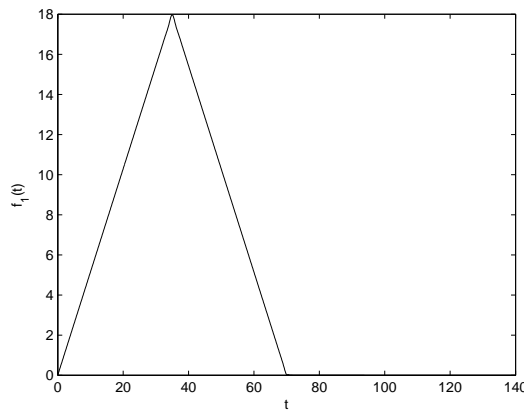

FIG. 5.1. *Inflow profile $f_1(t)$ prescribed as initial data on arc one.*

In Figure 5.2 we present the numerical solution to the supply chain model. At the top we find plots of the queues located at the beginning of the processors $j=2,3,4$. The queue q_4 remains empty, since the maximal capacity $\mu_4 > \mu_3$. In the queues q_2 and q_3 we observe the buffering of an exceeding demand. At the bottom, we find

a plot of the final density. The density ρ_2 of processor two corresponds to the strip $0 \le x \le 10, t > 0$, ρ_3 for processor three to $10 \le x \le 40, t > 0$ and ρ_4 to the remaining part of the plot. Since the initial data exceeds for some time the maximum capacity of the processors two and three, respectively, we observe functions that are plateau shaped. This can also be seen in Figure 5.3 showing the contour plot of the corresponding fluxes $f_j(\rho_j)$.
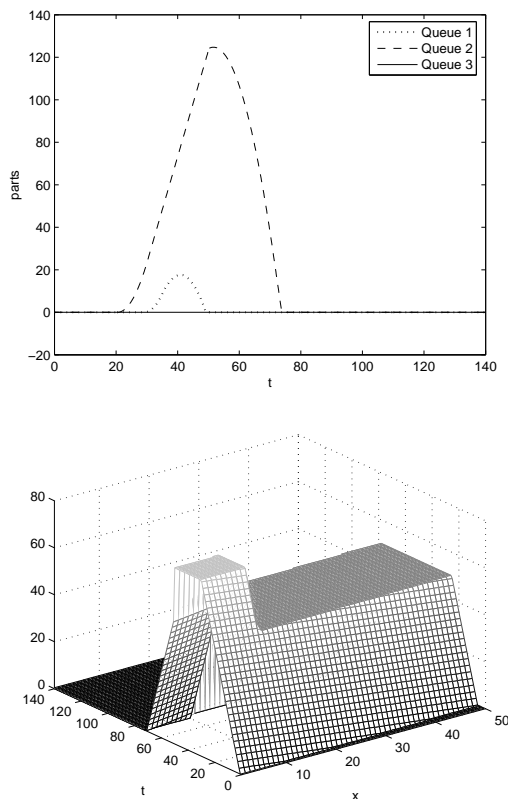

FIG. 5.2. *Queue and density of the network model.*

To compare the model with the existing results we also give a plot of the antiderivative $u(x,t)$ obtained by the calculations in the preceding section, see equation (4.2) and Figure 5.4. This plot is similar to the results proposed in [1]. For further comparisons, we formally calculate $\bar{\rho} = -\partial_x u$. A plot is given in Figure 5.5. As expected, one observes the appearance of sharp peaks approximating $\delta-$distributions, see remarks in [1]. Those peaks do not occur in the new proposed network model (Figure 5.2), due to the introduced buffering by queues.

6. Conclusions

1. Using the reformulation of the model in [1] which is presented here allows for an existence theory directly for the density ρ. The new model consists of queues and processors modelled as a coupled system of partial and ordinary equations and it allows a formulation as a network of processing units.
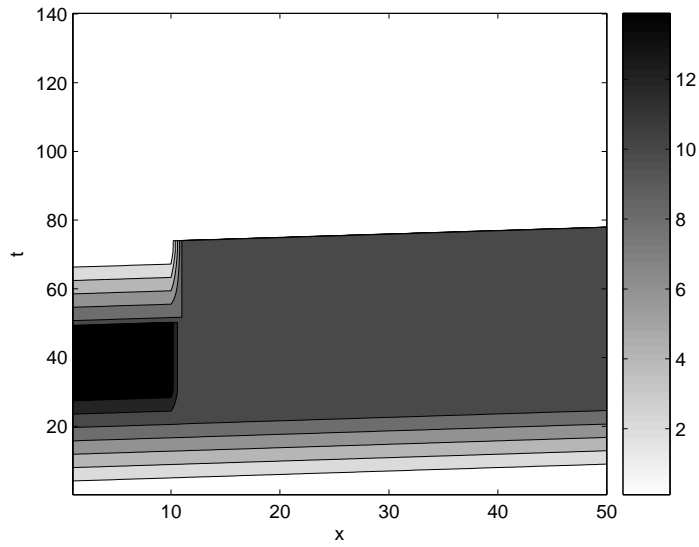
FIG. 5.3. *Contour lines of flux*
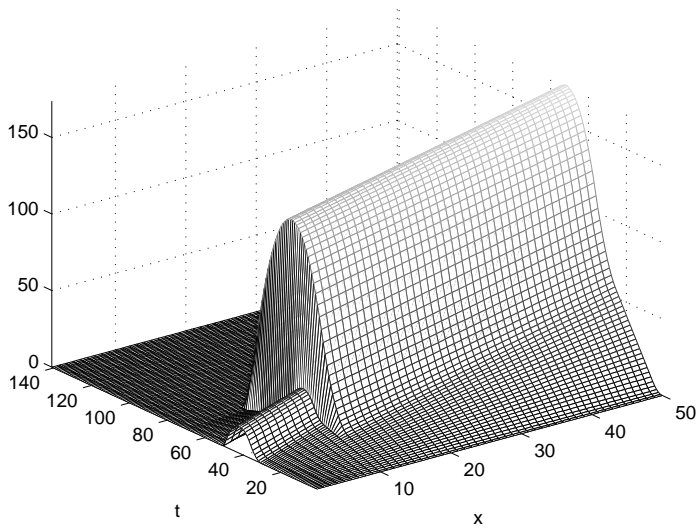
FIG. 5.4. *Antiderivative of the density ρ*

2. The model is easy to adapt to networks with multiple entries and exits at the vertices corresponding to more complex supply chain geometries. The main difference to the existing approach is the definition of appropriate conditions for the queues. This is currently under investigation and will be studied in a
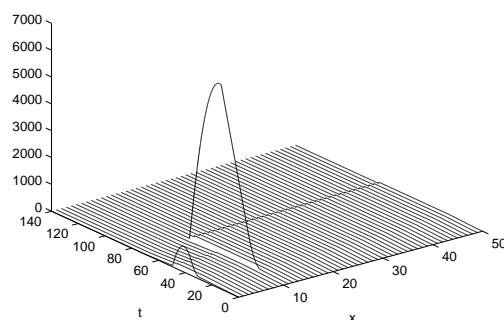
FIG. 5.5. *Density with δ-functions concentrations computed by the model in [1]*

forthcoming paper. As an example we consider a vertex with two incoming arcs and only one outgoing arc. Obviously, the ordinary differential equation of the queue q_j is composed of the two incoming fluxes f_{j-2}, f_{j-1} and the outgoing flux f_j.

$$\partial_t q_j(t) = f_{j-2}(\rho_{j-2}(b_{j-2},t)) + f_{j-1}(\rho_{j-1}(b_{j-1},t)) - f_j(\rho_j(a_j,t)) \qquad (6.1)$$

where

$$f_j(\rho_j(a_j,t)) = \begin{cases} \min\{f_{j-2}(\rho_{j-2}(b_{j-2},t)) + f_{j-1}(\rho_{j-1}(b_{j-1},t)), \mu_j\} & q_j(t) = 0 \\ \mu_j & q_j(t) > 0. \end{cases}$$

Similarly, other structures can be described in a straightforward way.

3. Also larger networks and optimization approaches are currently under investigation using simplification methods similar to those used in traffic flow theory, see [10].

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