Commun. Math. Phys. 98, 1–16 (1985)

Communications in Mathematical Physics

An Invariant Measure for the Equation $u_{tt} - u_{xx} + u^3 = 0$

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Abstract. Numerical studies of the initial boundary-value problem of the semilinear wave equation $u_{tt} - u_{xx} + u^3 = 0$ subject to periodic boundary conditions $u(t,0) = u(t,2\pi)$, $u_t(t,0) = u_t(t,2\pi)$ and initial conditions $u(0,x) = u_0(x)$, $u_t(0,x) = v_0(x)$, where $u_0(x)$ and $v_0(x)$ satisfy the same periodic conditions, suggest that solutions ultimately return to a neighborhood of the initial state $u_0(x)$, $v_0(x)$ after undergoing a possibly chaotic evolution. In this paper an appropriate abstract space is considered. In this space a finite measure is constructed. This measure is invariant under the flow generated by the Hamiltonian system which corresponds to the original equation. This enables one to verify the above "returning" property.

0. Introduction

During the Sixth I. G. Petrovskii memorial meeting of the Moscow Mathematical Society in January 1983 Professor V. E. Zakharov proposed the following problem. Numerical experiments demonstrated that the equation

$$u_{tt} - u_{xx} + u^3 = 0 \tag{0.1}$$

with periodic boundary conditions $u(t,0) = u(t,2\pi)$, $u_t(t,0) = u_t(t,2\pi)$ possesses the "returning" property, i.e. solutions appear to be very close to the initial state $u(0,x) = u_0(x)$, $u_t(0,x) = v_0(x)$, where the initial functions satisfy the above boundary conditions, after some time of rather chaotic evolution. The problem is to explain this phenomenon. According to the classical Poincaré theorem every flow which preserves a finite measure has the returning property modulo a set of measure zero. The aim of this paper is to build such a measure for the flow

$$\Phi(t)(u_0(x), v_0(x)) = (u(t,x), v(t,x)),$$

where $u(t,x)$ is the solution of (0.1), $v(t,x) = u_t(t,x)$, where the solution u satisfies the initial data $u(0,x) = u_0(x)$, $u_t(0,x) = v_0(x)$. The Eq. (0.1) can be rewritten as a Hamiltonian system

$$\left.\begin{aligned} u_t &= \delta H/\delta v \\ v_t &= -\delta H/\delta u \end{aligned}\right\} \tag{0.2}$$

with the Hamiltonian

$$H(u,v)=\int_0^{2\pi}(v^2/2+u_x^2/2+u^4/4)dx\,. \tag{0.3}$$

Our starting point is the desired formula

$$\int F(u,v)d\mu(u,v)=\int F(u,v)e^{-H(u,v)}\prod_{x\in S^1}du(x)dv(x) \tag{0.4}$$

for some class of "good" functionals F.

The right-hand side of (0.4) is the partition function. It can be determined by finite dimensional approximations (2.3). Roughly speaking the measure $d\mu$ is the "canonical symplectic measure" $\prod dudv$ multiplied by the function e^{-H} of the Hamiltonian and is invariant under the flow (0.2). However, the correct definition of the $d\mu$ involves some technical problems and the expression $\prod dudv$ does not have any meaning without the factor e^{-H}. The Hamiltonian H is the sum of

$$H_1(u)=\int_0^{2\pi}(u_x^2/2+u^4/4)dx \quad\text{and}\quad H_2(v)=\int_0^{2\pi}(v^2/2)dx\,,$$

so the measure $d\mu$ is the Cartesian product of the measures

$$d\mu_1=e^{-H_1(u)}\prod du(x) \quad\text{and}\quad d\mu_2=e^{-H_2(v)}\prod dv(x)\,.$$

The $d\mu_1$ is correctly defined by finite dimensional distributions $p(x_1,\dots,x_k;\xi_1,\dots,\xi_k)$:

$$d\mu_1\{u(x)\colon (u(x_1),\dots,u(x_k))\in M\}=\int_M p(x,\xi)d\xi$$

which are proportional to partition functions

$$\int_{\xi_j=u(x_j)} e^{-H_1(u)}\prod du\,, \tag{0.5}$$

which are calculated in Sect. 2. In order to formulate the result we introduce some notation. Let $x<y$ be two real numbers. $U(x,\xi;y,\eta;z)$ is the solution of the equation $U_{zz}=U^3$ in the segment $[x,y]$ with the boundary conditions $U(x)=\xi$, $U(y)=\eta$. Let

$$\begin{aligned}h_1(x,\xi;y,\eta)&=\int_x^y[U_z^2(x,\xi;y,\eta;z)/2+U^4(x,\xi;y,\eta;z)/4]dz\\&=\min\left\{\int_x^y(u_z^2/2+u^4/4)dz\,|\,u(x)=\xi,\ u(y)=\eta\right\},\end{aligned}$$

and let $D(x,\lambda;y,\eta)$ be regularized determinant of the operator, see [4],

$$-d^2/dz^2+3U^2(x,\xi;y,\eta;z)\,, \tag{0.6}$$

in the segment $[x,y]$ with the Dirichlet boundary conditions. The operator (0.6) is the operator of second variation of the functional

$$\int_x^y(u_z^2/2+u^4/4)dz\,;\qquad u(x)=\xi\,,\qquad u(y)=\eta$$

in the neighborhood of the extremum U. Then

$$p(x,\xi)=\frac{c(x)}{\sqrt{\prod D(x_j,\xi_j;x_{j+1},\xi_{j+1})}}\exp\{-\sum h_1(x_j,\xi_j;x_{j+1},\xi_{j+1})\}\,. \tag{0.7}$$

The function c is determined from the condition

$$\int p(x, \xi) d\xi = 1$$

and is equal to

$$\sigma(2\pi)^{-k/2} \prod_{j=1}^{k} (x_{j+1} - x_j)^{-1/2} \tag{0.8}$$

with some constant σ. The measure $d\mu_1$ is absolutely continuous with respect to the classical Wiener measure; so its support belongs to the space Lip^α, $0 < \alpha < 1/2$. After replacing the functional $H_1(u)$ with $\int (u_x^2/2) dx$ the construction will lead us exactly to the classical Wiener measure. The $d\mu_2$ is a realization of the abstract Wiener measure and it will be described in Sect. 3.

In Sect. 1 we investigate the determinant of the operator (0.6). In particular, we prove the formula

$$\det(\Delta_0 + F(x)) = \det(\Delta_0) \det(I + \Delta_0^{-1} F(x)), \tag{0.9}$$

where Δ_0 is the operator $-d^2/dx^2$ with the Dirichlet boundary conditions and $F(x)$ is a nonnegative smooth function. The determinants of $\Delta_0 + F(x)$ and Δ_0 are equal to $\exp(-\zeta'(0))$, where $\zeta(z)$ is the ζ-function of an operator; $\det(I + \Delta_0^{-1} F(x))$ is well defined because the operator $\Delta_0^{-1} F$ is nuclear, $\Delta_0^{-1} F \in \mathfrak{S}_1$. The formula (0.9) is not used in our constructions but we think it is interesting by itself. In Sect. 2 we calculate the partition function (0.5), in Sect. 3 we give the correct definition of the measure $d\mu$ and finally in Sect. 4 we prove the main result:

Theorem. *The measure $d\mu$ is invariant under the flow* (0.2).

1. The Determinant of the Sturm-Liouville Operator with the Dirichlet Conditions

We investigate properties of the functional determinants by finite dimensional approximations. The key lemma is

Lemma 1. *Let $F(x) \in C^\varrho[0, a]$, $\varrho > 0$, and let Δ_0 be the operator $-d^2/dx^2$ with the Dirichlet conditions. Consider $(N-1) \times (N-1)$ matrices*

$$\delta_N = \frac{N^2}{a^2} \left| \begin{matrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 2 \end{matrix} \right| \quad \text{and} \quad f_N = \| f_{N;ij} \|,$$

where

$$f_{N;ij} = \begin{cases} \alpha F(j/N) + r_{jj}^{(N)} & \text{if} \quad i = j, \\ \beta F(j/N) + r_{j+1,j}^{(N)} & \text{if} \quad i = j+1, \\ \beta F((j-1)/N) = r_{j-1,j}^{(N)} & \text{if} \quad i = j-1 \\ 0 & \text{if} \quad |i-j| > 1, \end{cases}$$

$\alpha + 2\beta = 1$ *and*

$$\lim_{N \to \infty} \max_{i,j} |r_{ij}^{(N)}| = 0.$$

Then

$$\det(I+\Delta_0^{-1}F)=\lim_{N\to\infty}\det(I+\delta_N^{-1}f_N).$$

Proof. Consider the orthonormal basis $E_k(x)=\sqrt{2/a}\sin(\pi kx/a)$ of the eigenfunctions of the operator $\Delta_0:\Delta_0E_k=\lambda_kE_k$ with $\lambda_k=\pi^2k^2/a^2$, $k=1,2,\ldots$. Denote by H_*^s the scale of Sobolev spaces which are generated by $\Delta_0^{-1/2}:\|E_k(x)\|_s=\lambda_k^{s/2}$. The operator δ_N is defined on $\mathbb{C}^{N-1}$; its eigenvalues

$$\lambda_k^{(N)}=\frac{4N^2}{a^2}\sin^2\frac{\pi k}{2N},\qquad k=1,\ldots,N-1,$$

the corresponding eigenvectors

$$e_k^{(N)}=(e_{k1}^{(N)},\ldots,e_{kN-1}^{(N)})\quad\text{with } e_{ks}^{(N)}=\sqrt{2/a}\sin(\pi ks/N),\qquad k,s=1,\ldots,N-1.$$

We normalize $e_k^{(N)}$ by the condition

$$|e_k^{(N)}|^2=\frac{a}{N}\sum_{s=1}^{N-1}|e_{ks}^{(N)}|^2=1.$$

Let l_N^2 be the space $\mathbb{C}^{N-1}$ with the norm $|\cdot|$ and let h_N^s be the same space with the norm $|y|_s=|\delta_N^{s/2}y|$. Now we introduce the interpolation operator $i_N:l_N^2\to L^2[0,a]$ and the restriction operator $j_N:L^2[0,a]\to l_N^2$:

$$i_Ne_k^{(N)}=E_k(x),\qquad k=1,\ldots,N-1,$$

$$j_NE_k(x)=\begin{cases}e_k^{(N)} & \text{if } k=1,\ldots,N-1,\\ 0 & \text{if } k\geqq N.\end{cases}$$

We split the segment $[0,a]$ into N equal parts by the points $0=x_0<x_1<\ldots<x_{N-1}<x_N=a$; $x_j=ja/N$. The i_N is the operator of trigonometrical interpolation of the values at x_j; $j_N=r_NP_N$, where P_N is the ortho-projector onto the subspace spanned by $E_1,\ldots,E_{N-1}$ and

$$r_NG=(G(a/N),\ldots,G((N-1)a/N)).$$

First of all we notice that the norms of i_N and j_N as operators which map h_N^s into H_*^s and H_*^s onto h_N^s correspondingly are bounded by constants which do not depend on N because

$$1\leqq\lambda_k/\lambda_k^{(N)}=(\pi k/2N)^2/\sin^2(\pi k/2N)\leqq\pi^2/4;\qquad k=1,\ldots,N-1.$$

Consider the finite-dimensional operator

$$T_N=i_N\delta_N^{-1}f_Nj_N:L^2[0,a]\to L^2[0,a].$$

Clearly,

$$\det(I+T_N)=\det(I+\delta_n^{-1}f_N).$$

So the convergence of T_N to $T=\Delta_0^{-1}F$ in the space $\mathfrak{S}_1$ of nuclear operators implies the assertion of the lemma, see [5]. We split the proof of convergence into the following steps. The operators

(i) T_N are uniformly bounded in the space $\mathscr{L}(L^2,H_*^s)$ of linear operators $L^2\to H_*^s$.

(ii) $T_N \to T$ in the space $\mathscr{L}_s(L^2, L^2)$ with the strong topology. Let ϕ be a trigonometrical polynomial. Then

$$T_N\phi - T\phi = i_N\delta_N^{-1}[f_N j_N - j_N F(x)]\phi + i_N\delta_N^{-1} j_N(I-P_k)F(x)\phi$$
$$+ (i_N\delta_N^{-1} j_N - \Delta_0^{-1})P_k F(x)\phi - \Delta_0^{-1}(I-P_k)F(x)\phi\,. \qquad (1.1)$$

The second and the fourth terms on the right-hand side of (1.1) converge to 0 uniformly with respect to N when $k \to \infty$. Operators $(i_N\delta_N^{-1} j_N - \Delta_0^{-1})P_k$ have orthonormal basis of eigenfunctions $E_j(x)$. The corresponding eigenvalues are equal to

$$a^2/(4N^2\sin^2(\pi j/2N)) - a^2/(\pi^2 j^2) \xrightarrow{N\to\infty} 0 \quad \text{if} \quad j \leqq k-1 \quad \text{and}$$
$$0 \quad \text{if} \quad j \geqq k;$$

therefore, the third term in (1.1) converges to 0 when $N \to \infty$ and k is fixed. Let

$$[f_N r_N - r_N F(x)]\phi(x) = (y_1^{(N)}, \ldots, y_{N-1}^{(N)})\,.$$

Then

$$y_j^{(N)} = \beta F((j-1)a/N)\phi((j-1)a/N) + r_{j,j-1}^{(N)}\phi((j-1)a/N)$$
$$+ \alpha F(ja/N)\phi(ja/N) + r_{j,j}^{(N)}\phi(ja/N) + \beta F(ja/N)\phi((j+1)a/N)$$
$$+ r_{j,j+1}^{(N)}\phi((j+1)a/N) - F(ja/N)\phi(ja/N)$$

and $\lim_{N\to\infty} \max_j |y_j^{(N)}| = 0$. Thus $|(f_N r_N - r_N F)| \to 0$ in l_N^2. Further, $(r_N - j_N)F\phi \to 0$ when $N \to \infty$ and $r_N\phi = j_N\phi$ if N is sufficiently large. So the first term on the right-hand side of (1.1) converges to 0 when $N \to \infty$. Combining the results above we obtain that $T_N\phi \to T\phi$. The set of T_N is bounded and trigonometrical polynomials are dense in L^2; hence $T_N \to T$ in strong topology.

(iii) $T_N \to T$ in the space $\mathscr{L}_s(L^2, H_*^2)$, by virtue of (i), (ii) and Banach-Steinhaus theorem.

(iv) $T_N \to T$ in the space $\mathscr{L}(H_*^s, H_*^2)$, $s > 0$, by virtue of (iii) and the compactness of the imbedding $H_*^s \hookrightarrow L^2$.

The space $\mathscr{L}(H_*^s, H_*^2)$ belongs to $\mathfrak{S}_1(H_*^s)$ when $s < 1$, see [6]. Hence $T_N \to T$ in $\mathfrak{S}_1(H_*^s)$; $0 < s < 1$. □

Lemma 2. *Let $F(x) \in C^2[0, a]$ and let $A(x)$ be the solution of the equation*

$$A''(x) = F(x)A(x)$$

with the boundary conditions

$$A(a) = 0\,, \qquad A'(a) = -\,\kappa/a\,,$$

where $A_\nu^{(N)}$, $\nu = 0, 1, \ldots, N$, $N = 2, 3, \ldots$, satisfies the difference equation

$$(N^2/a^2)(A_{\nu+1}^{(N)} - 2A_\nu^{(N)} + A_{\nu-1}^{(N)}) = F((N-\nu)a/N)$$

with

$$A_0^{(N)} = 0\,, \qquad A_1^{(N)} = \kappa/N\,.$$

Then

$$A(0) = \lim_{N\to\infty} A_N^{(N)}\,.$$

Proof. Let $R_\nu^{(N)} = A_\nu^{(N)} - A((N-\nu)a/N)$ and $C_\nu^{(N)} = R_\nu^{(N)} - R_{\nu-1}^{(N)}$. Then

$$(N^2/a^2)(R_{\nu+1}^{(N)} - 2R_\nu^{(N)} + R_{\nu-1}^{(N)}) = F((N-\nu)a/N)R_\nu^{(N)} + b_\nu^{(N)} \tag{1.2}$$

and

$$(N^2/a^2)(C_{\nu+1}^{(N)} - C_\nu^{(N)}) = F((N-\nu)a/N)\sum_{j=1}^{\nu} C_j^{(N)} + b_\nu^{(N)} \tag{1.2'}$$

with $R_0^{(N)} = 0$, $R_1^{(N)} = C_1^{(N)} + O(N^{-3})$ and $b_\nu^{(N)} = O(N^{-2})$ uniformly with respect to ν. Clearly, $C_\nu^{(N)}$ are bounded by the solutions of the equation of the type (1.2′) with $F((N-\nu)a/N)$, $b_\nu^{(N)}$ and $C_1^{(N)}$ replaced by $C_1 = \sup|F(x)|$, C_2/N^2 and C_3/N^3, respectively. Hence $R_\nu^{(N)}$ are bounded by the solution of the following difference equation,

$$(N^2/a^2)(r_{\nu+1}^{(N)} - r_\nu^{(N)} + r_{\nu-1}^{(N)}) = C_1 r^{(N)} + C_2/N^2\,; \qquad r_0^{(N)} = 0\,, \qquad r_1^{(N)} = C_3/N^2\,.$$

The general solution of this equation is

$$r_\nu^{(N)} = -C_2/(C_1N^2) + \alpha^{(N)}[\lambda_+^{(N)}]^\nu + \beta^{(N)}[\lambda_-^{(N)}]^\nu$$

with $\lambda_\pm^{(N)} = 1 \pm C_4/N + \dots$ and $\lambda_+^{(N)}\lambda_-^{(N)} = 1$. According to the initial conditions

$$\alpha^{(N)} + \beta^{(N)} = C_5/N^2\,, \qquad \alpha^{(N)}\lambda_+^{(N)} + \beta^{(N)}\lambda_-^{(N)} = C_3/N^2\,.$$

Hence

$$\alpha^{(N)} = ((C_3\lambda_+^{(N)})/N^3 - C_5/N^2)/(\lambda_+^{(N)2} - 1) = O(N^{-1})\,, \qquad \beta^{(N)} = O(N^{-1})\,.$$

Therefore,

$$r_N^{(N)} \leqq C_5/N^2 + C_6(I + C_7/N)^N/N \leqq C_8/N \quad \text{and} \quad R_N^{(N)} = O(N^{-1})\,. \quad \square$$

Theorem 1. *Let $F(x) \in C^2[0,a]$ and let $A(x)$ be the solution of the equation*

$$A''(x) = F(x)A(x)$$

with the boundary conditions $A(a) = 0$, $A'(a) = -I/a$. Then $\det(I + \Delta_0^{-1}F) = A(0)$.

Proof. Let $f_N = \operatorname{diag}(F(a/N), \dots, F((N-1)a/N))$ be the diagonal matrix. By Lemma 1

$$\begin{aligned}
\det(I + \Delta_0^{-1}F) &= \lim_{N\to\infty} \det(I + \delta_N^{-1} f_N) \\
&= \lim_{N\to\infty} \left[\det\begin{vmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 2 \end{vmatrix}\right]^{-1} \\
&\quad \cdot \det\begin{vmatrix} 2 + a^2F(a/N)/N^2 & -1 & \dots & 0 \\ -1 & & \dots & \\ \cdot & \cdot & \cdot & \cdot \\ 0 & & & -1 \\ 0 & \dots & -1 & 2 + a^2F((N-1)a/N)/N^2 \end{vmatrix} \\
&= \lim_{N\to\infty} N^{-1} \det D_N\,.
\end{aligned}$$

Above we have used the relation

$$\det\begin{vmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 2 \end{vmatrix} = N,$$

which can be proved easily.

By elementary transformations the matrix D_N can be transformed into

$$\begin{vmatrix} v_1 & -1 & 0 & \dots & 0 \\ 0 & v_2 & -1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & v_{N-1} \end{vmatrix}$$

with

$$v_j = 2 + a^2F(ja/N)/N^2 - 1/v_{j-1}, \qquad v_1 = 2 + a^2F(a/N)/N^2. \tag{1.3}$$

Our aim is to find $N^{-1}v_1 \cdots v_{N-1}$. Let

$$N^{-1}v_{N-\nu}\cdots v_{N-1} = A_\nu^{(N)}v_{N-\nu} + B_\nu^{(N)}; \qquad A_1^{(N)} = N^{-1}, \qquad B_1^{(N)} = 0.$$

It follows from (1.3) that

$$(N^2/a^2)(A_{\nu+1}^{(N)} - 2A_\nu^{(N)} + A_{\nu-1}^{(N)}) = F((N-\nu)a/N)A^{(N)},$$
$$A_0^{(N)} = 0, \qquad A_1^{(N)} = I/N.$$

The value

$$N^{-1}\det D_N = A_{N-1}^{(N)}v_1 + B_{N-1}^{(N)} = A_N^{(N)}$$

converges to $A(0)$ when $N\to\infty$ by Lemma 2. The theorem is proved. □

Now we shall prove the formula (0.9). Let us recall the definition of the determinant of a positive unbounded operator A. Assume that $A^{-\sigma}\in\mathfrak{S}_1$ for some positive σ. One can define the function $\zeta_A(z) = \operatorname{Tr}(A^{-z})$ which is regular in the half-plane $\operatorname{Re} z > \sigma$. In some cases (e.g. if A is a pseudodifferential operator) this function has the meromorphic continuation. It may happen that 0 is a regular point of this ζ-function. In this case we say that A has a determinant and $\det A = \exp(-\zeta_A'(0))$. This definition is a generalization of the finite-dimensional determinant.

Theorem 2. *Let $S \geqq c_0 > 0$ be a positive operator in a separable Hilbert space $\mathscr{H}$, let $S^{-\sigma}\in\mathfrak{S}_1$ for some σ, $0<\sigma<1$ and $\det S$ be defined. Let T be a bounded operator. Then there exists a constant C which depends upon c_0 and $\|T\|$ only, such that $\det A(\varepsilon) = \det(S+\varepsilon T)$ is defined when $|\varepsilon|<C$ and is equal to $\det S \det(I+\varepsilon S^{-1}T)$.*

Proof. One has the following integral representation on the strip $0<\operatorname{Re} z<1$, see [7]:

$$A^{-z}(\varepsilon) = \frac{\sin\pi z}{\pi}\int_0^\infty t^{-z}(tI + A(\varepsilon))^{-1}dt$$
$$= S^{-z} + \frac{\sin\pi z}{\pi}\int_0^\infty t^{-z}\sum_{k=1}^\infty (-1)^k\varepsilon^k[(tI+S)^{-1}T]^k(tI+S)^{-1}dt.$$

If $\varepsilon < c_0/\|T\|$ we can change the order of summation and integration:

$$A^{-z}(\varepsilon) - S^{-z} = \frac{\sin \pi z}{\pi} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \int_0^{\infty} t^{-z}[(tI+S)^{-1}T]^k (tI+S)^{-1} dt. \qquad (1.4)$$

Let us show that all terms on the right-hand side of (1.4) are nuclear operators and estimate their $\mathfrak{S}_1$-norms which will be denoted by $|||\cdot|||$. One has

$$\begin{aligned} &|||[(tI+S)^{-1}T]^k(tI+S)^{-1}||| \\ &\leqq |||S^{-\sigma}||| \cdot \|[(tI+S)^{-1}T]^k(tI+S)^{-1}S^{\sigma}\| \\ &\leqq |||S^{-\sigma}||| \cdot \|T\|^k (t+c_0)^{-k} \begin{cases} \sigma^{\sigma}(1-\sigma)^{1-\sigma} t^{\sigma-1} & \text{if} \quad t \geqq c_0(1-\sigma)/\sigma \\ c_0^{\sigma}/(t+c_0) & \text{if} \quad t < c_0(1-\sigma)/\sigma . \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} &\left|\left|\left| \int_0^{\infty} t^{-z}(tI+S)^{-1}T^k(tI+S)^{-1}dt \right|\right|\right| \\ &\quad \leqq |||S^{-\sigma}||| \cdot \|T\|^k \{(1-\sigma)\sigma^{-1}(1-\mathrm{Re}\, z)^{-1} c_0^{-k+\sigma+1-\mathrm{Re}\, z} \\ &\qquad + \sigma^{\sigma}(1-\sigma)^{1-\sigma}(\mathrm{Re}\, z + k - \sigma)^{-1} c_0^{-k+\sigma-\mathrm{Re}\, z}\}. \end{aligned}$$

Thus the series (1.4) is $\mathfrak{S}_1$-convergent when $\varepsilon < c_0/\|T\|$ and it defines the $\mathfrak{S}_1$-valued regular function on the strip $\sigma - 1 < \mathrm{Re}\, z < 1$. Hence $\zeta_{A(\varepsilon)}(z)$ has the meromorphic extension to the half-plane $\mathrm{Re}\, z > \sigma - 1$ and 0 is a regular point of this function:

$$\zeta'_{A(\varepsilon)}(0) - \zeta'_S(0) = \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \int_0^{\infty} \mathrm{Tr}\{[(tI+S)^{-1}T]^k(tI+S)^{-1}\} dt.$$

Note that

$$\frac{d}{dt}[(tI+S)^{-1}T]^k = -\sum_{i=0}^{k-1} [(tI+S)^{-1}T]^i (tI+S)^{-1}[(tI+S)^{-1}T]^{k-i}.$$

Hence

$$\mathrm{Tr}\{[(tI+S)^{-1}T]^k(tI+S)^{-1}\} = -\frac{1}{k}\frac{d}{dt}\mathrm{Tr}[(tI+S)^{-1}T]^k,$$

and

$$\begin{aligned} \zeta'_{A(\varepsilon)}(0) - \zeta'_S(0) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\varepsilon^k}{k} \int_0^{\infty} \frac{d}{dt} \mathrm{Tr}([(tI+S)^{-1}T]^k dt \\ &= \sum_{k=1}^{\infty} (-1)^k \varepsilon^k k^{-1} \mathrm{Tr}(S^{-1}T)^k = -\mathrm{Tr} \log(I + \varepsilon S^{-1}T). \end{aligned}$$

Thus

$$\begin{aligned} \det A(\varepsilon)/\det S &= \exp\{-[\zeta'_{A(\varepsilon)}(0) - \zeta'_S(0)]\} \\ &= \exp \mathrm{Tr} \log(I + \varepsilon S^{-1}T) = \det(I + \varepsilon S^{-1}T). \quad \square \end{aligned}$$

Corollary. *Let S be the same operator as in Theorem 2 and let T be a non-negative bounded operator. Then* $\det(S+T)$ *is defined and*

$$\det(S+T) = \det S \det(I + S^{-1}T).$$

Proof. Note that $S+\varepsilon T \geqq c_0$ for every $\varepsilon \geqq 0$. So we can apply Theorem 2 N times if N is sufficiently large and obtain

$$\det(S+T)=\det S \prod_{j=0}^{N-1} \det(I+N^{-1}(S+jN^{-1}T)^{-1}T).$$

The product $\prod$ in the last formula is equal to the $\det(I+S^{-1}T)$, as follows from the identity

$$\det(I+\varepsilon_1 S^{-1}T)\det(I+\varepsilon_2(S+\varepsilon_1 T)^{-1}T)=\det(I+(\varepsilon_1+\varepsilon_2)S^{-1}T). \tag{1.5}$$

In order to prove this identity we introduce $R=S^{-1}T$ and obtain

$$\begin{aligned}&(I+\varepsilon_1 S^{-1}T)(I+\varepsilon_2(S+\varepsilon_1 T)^{-1}T)\\ &\quad=I+\varepsilon_1 R+\varepsilon_2(I+\varepsilon_1 R)^{-1}R+\varepsilon_1\varepsilon_2 R(I+\varepsilon_1 R)^{-1}R\\ &\quad=I+\varepsilon_1 R+\varepsilon_2(I+\varepsilon_1 R)^{-1}R+\varepsilon_2(I+\varepsilon_1 R)(I+\varepsilon_1 R)^{-1}R-\varepsilon_2(I+\varepsilon_1 R)^{-1}R\\ &\quad=I+(\varepsilon_1+\varepsilon_2)R.\end{aligned}$$

Now (1.5) follows from the well known formula

$$\det(I+A_1)\det(I+A_2)=\det(I+A_1)(I+A_2)$$

with $A_1, A_2 \in \mathfrak{S}_1$, e.g. see [5]. □

Formula (0.9) follows from the Corollary. Note that

$$\zeta_{\Delta_0}(z)=(\pi/a)^{-2z}\zeta(2z) \quad \text{and} \quad \det\Delta_0=(\pi/a)e^{-2\zeta'(0)},$$

where $\zeta(z)$ is the Riemann ζ-function.

2. Calculation of the Partition Function

$$S(x,\xi;y,\eta)=\int_{\substack{u(x)=\xi\\ u(y)=\eta}} \exp\left\{-\int_x^y (u_z^2/2+u^4/4)dz\right\} \prod_{z\in[x,y]} du(z). \tag{2.1}$$

Let us split the interval $[x,y]$ into N equal parts $x=x_0<x_1<\ldots<x_N=y$. Consider the finite-dimensional approximation of S

$$S_N=\int \exp\left\{-\sum_{j=1}^{N} h(a/N,\xi_{j-1},\xi_j)\right\} d\xi_1 \ldots d\xi_{N-1}$$

with $a=y-x$, $\xi_0=\xi$, $\xi_N=\eta$; the definition of the function h is given in the introduction. The invariance of the equation $u_{zz}=u^3$ under the transformation $u(z) \to N^{-1}u(N^{-1}z)$ leads us to the homogeneity property

$$h(a/N,\xi_{j-1},\xi_j)=N^3 h(a,\xi_{j-1}/N,\xi_j/N). \tag{2.2}$$

Therefore,

$$\begin{aligned}S_N=N^{N-1}\int \exp\Bigg\{-N^3\Bigg[&h(a,\xi/N,\xi_1)+\sum_{j=2}^{N-2} h(a,\xi_{j-1},\xi_j)\\ &+h(a,\xi_{N-1},\eta/N)\Bigg]\Bigg\} d\xi_1 \ldots d\xi_{N-1}.\end{aligned} \tag{2.3}$$

We can apply the Laplace method to the integral in (2.3). The function $I(\xi_1, \ldots, \xi_{N-1})$ in the square brackets has the unique stationary point $(\xi_1^0, \ldots, \xi_{N-1}^0)$

$$\xi_j^0 = N^{-1} U(x, \xi; y, \eta; x + ja/N).$$

This point is the point of its strong minimum.

$$S_N = (2\pi)^{(N-1)/2} D_N^{-1/2} N^{-(N-1)/2} e^{-N^3 I(\xi_1^0, \ldots, \xi_{N-1}^0)}(1 + O(N^{-1})),$$

where

$$D_N = \det \|I''(\xi_1^0, \ldots, \xi_{N-1}^0)\|.$$

By the homogeneity property (2.2)

$$N^3 I(\xi_1^0, \ldots, \xi_{N-1}^0) = N^3 h(Na, \xi/N, \eta/N) = h(a, \xi, \eta)$$

and

$$D_N = N^{-(N-1)} L_N = N^{-(N-1)} \det \|J''(\xi_1^1, \ldots, \xi_{N-1}^1)\|$$

with

$$J = \sum_{j=1}^{N} h(a/N, \xi_{j-1}, \xi_j), \qquad \xi_j^1 = N \xi_j^0.$$

Finally,

$$S_N = (2\pi)^{(N-1)/2} L_N^{-1/2} e^{-h(a, \xi, \eta)}(1 + O(N^{-1})).$$

Proposition 1. *When* $N \to \infty$

$$L_N = (N^N / a^{N-1}) \det(I + 3\Delta_0^{-1} U^2(x, \xi; y, \eta; z))(1 + o(1)).$$

Corollary.

$$\lim_{N\to\infty} (2\pi a)^{(1-N)/2} N^{N/2} S_N = [\det(I + 3\Delta_0^{-1} U^2(x, \xi; y, \eta; z))]^{-1/2} e^{-h(y-x, \xi, \eta)}.$$

The expression on the right-hand side of the last formula will be called the partition function S.

Proof of Proposition 1. Let $L_{ij} = J''_{\xi_i \xi_j}(\xi_1^1, \ldots, \xi_{N-1}^1)$. From the definition of J it follows that

$$L_{jj} = \frac{\partial^2 h}{\partial \eta^2}(\tau, \xi_{j-1}^1, \xi_j^1) + \frac{\partial^2 h}{\partial \xi^2}(\tau, \xi_j^1, \xi_{j+1}^1), \qquad \tau = a/N;$$

$$L_{j,j+1} = L_{j+1,j} = \frac{\partial^2 h}{\partial \xi \partial \eta}(\tau, \xi_j^1, \xi_{j+1}^1),$$

$$L_{ij} = 0 \quad \text{when} \quad |i-j| > 1.$$

a) Calculation of L_{jj}. By the definition of the function h

$$L_{jj} = \frac{\partial^2}{\partial \xi^2} \int_{-\tau}^{\tau} \left[\frac{u_z'(\xi_{j-1}^1, \xi, \xi_{j+1}^1; z)^2}{2} + \frac{u^4}{4} \right] dz \bigg|_{\xi = \xi_j^1},$$

where u is the solution of the Euler-Lagrange equation $u'' = u^3$ for the energy functional, with the conditions $u(-\tau) = \xi_{j-1}^1$, $u(0) = \xi$, $u(\tau) = \xi_{j+1}^1$. By the formula for the second variation

$$L_{jj} = \int_{-\tau}^{\tau} [v'^2 + 3u_0^2 v^2] dz\,,$$

where $u_0 = u(\xi_{j-1}^1, \xi_j^1, \xi_{j+1}^1; z)$, v is the solution of the equation

$$v'' = 3u_0^2 v\,, \qquad v(\tau) = v(-\tau) = 0\,, \qquad v(0) = 1\,.$$

Integrating by parts and taking into account the relation $u_0'' = u_0^3$, we obtain

$$L_{jj} = v'(-0) - v'(+0) = -[v'](0)\,.$$

Let us split v into the sum of v_0 and w:

$$\begin{aligned} &v_0'' = 3(\xi_j^1)^2 v_0\,;\\ &w'' - 3u_0^2 w = 3[u_0^2 - (\xi_j^1)^2] v_0\,;\\ &v_0(\tau) = v_0(-\tau) = w(-\tau) = w(0) = w(\tau) = 0\,, \qquad v_0(0) = 1\,. \end{aligned} \tag{2.5}$$

The first equation in (2.5) has the solution

$$v_0(z) = \operatorname{sh}\alpha(\tau - |z|)/\operatorname{sh}\alpha\tau, \qquad \alpha = \sqrt{3}\,|\xi_j^1|\,.$$

The solution of the second equation in (2.5) has the representation

$$w(z) = 3 \sum_{j=1}^{\infty} (-1)^{j+1} (3Ku_0^2)^j K[u_0^2 - (\xi_j^1)^2] v_0\,, \tag{2.6}$$

where K is the inverse to $-d^2/dx^2$ with zero conditions at the points $\pm\tau$ and 0. It is an integral operator with the kernel

$$K(x,y) = \begin{cases} |x|(\tau - |y|)/\tau & \text{if } |x| \leqq |y|\,, \quad \operatorname{sign} x = \operatorname{sign} y\,,\\ |y|(\tau - |x|)/\tau & \text{if } |x| > |y|\,, \quad \operatorname{sign} x = \operatorname{sign} y\,,\\ 0 & \text{if } \operatorname{sign} x \neq \operatorname{sign} y\,. \end{cases}$$

The series (2.6) is asymptotic with respect to $\tau \to 0$ because K is of order τ. Hence

$$\begin{aligned} -[w'](0) &\sim (3K(u_0^2 - (\xi_j^1)^2) v_0)'\\ &= -3 \int_0^{\tau} \frac{(\tau - z)\operatorname{sh}\alpha(\tau - z)}{\operatorname{sh}\alpha\tau} (u_0^2(z) - u_0^2(-z)) dz = O(\tau^3)\,. \end{aligned}$$

Further,

$$-[v'](0) = 2\alpha \operatorname{cth}\alpha\tau = \frac{2}{\tau} + \frac{2}{3}\alpha^2 \tau + O(\tau^3)\,.$$

Finally,

$$L_{jj} = \frac{2}{\tau} + 2(\xi_j^1)^2 \tau + O(\tau^3)\,. \tag{2.7}$$

b) Calculation of $L_{j,j+1}$. By definition

$$L_{j,j+1} = \frac{\partial^2}{\partial\xi\partial\eta} \int_0^{\tau} \left[\frac{u_z'(\xi,\eta;z)^2}{2} + \frac{u^4}{4} \right] dz \Bigg|_{\xi = \xi_j^1,\, \eta = \xi_{j+1}^1}$$

with $u(\xi,\eta;z)=U(0,\xi;\tau,\eta;z)$. As above one can easily check that $L_{j,j+1}=v'(\tau)$, where $v(\tau)$ is the solution of the equation $v''=3u_0^2v$ with the boundary conditions $v(0)=1$, $v(\tau)=0$; $u_0=u(\xi_j^1,\xi_{j+1}^1;z)$. Splitting v into the sum of $v_0(z)=\operatorname{sh}\alpha(\tau-|z|)/\operatorname{sh}\alpha\tau$ and $w(z)$ we obtain that

$$v_0'(\tau)=-\frac{1}{\tau}+\frac{\alpha^2\tau}{6}+O(\tau^3),$$

$$w'(\tau)\sim 3\int_0^\tau \frac{z}{\tau}(u^2-(\xi_j^1)^2)v_0dz=O(\tau^2),$$

and finally,

$$L_{j,j+1}=-\frac{1}{\tau}+\frac{1}{2}(\xi_j^1)^2\tau+O(\tau^2). \tag{2.8}$$

Now it remains to apply Lemma 1 with

$$F(z)=3U^2(x,\xi;y,\eta;x+z),\quad \alpha=2/3 \text{ and } \beta=1/3. \quad \square$$

3. The Measure $d\mu$

Let us fix points $x_1<x_2<\ldots<x_k<x_1+2\pi$ on the circle. Consider the function

$$S(x,\xi)=S(x_1,\xi_1;x_2,\xi_2)S(x_2,\xi_2;x_3,\xi_3)\ldots S(x_k,\xi_k;x_1+2\pi,\xi_1).$$

Proposition 2. *Let $x_j'=(x_1,\ldots,\hat{x}_j,\ldots,x_k)$, $\xi_j'=(\xi_1,\ldots,\hat{\xi}_j,\ldots,\xi_k)$ (the sign ^ means that the corresponding variable is omitted). Then*

$$\int S(x,\xi)d\xi_j=(2\pi)^{1/2}\sqrt{\frac{(x_{j+1}-x_j)(x_j-x_{j-1})}{x_{j+1}-x_{j-1}}}\,S(x_j',\xi_j'). \tag{3.1}$$

We assume that $x_0=x_k-2\pi$, $x_{k+1}=x_1+2\pi$, $\xi_0=\xi_k$, $\xi_{k+1}=\xi_1$.

Proof. Let all ratios $(x_{m+1}-x_m)/(x_{n+1}-x_n)$ be rational; $x_{m+1}-x_m=N_m\tau$. By Proposition 1

$$\int S(x,\xi)d\xi_j=\lim_{m\to\infty}(2\pi)^{k/2}(2\pi)^{(-m/2)\Sigma N_i}(m/\tau)^{(m/2)\Sigma N_i}$$
$$\cdot\prod_{\nu=1}^{k}(x_{\nu+1}-x_\nu)^{1/2}\int\exp\{-\Sigma h(\tau/m,\xi_\nu^{(m)},\xi_{\nu+1}^{(m)})\}\frac{d\xi^{(m)}}{d\xi_j'},$$

where $x_1=x_1^{(m)}<x_2^{(m)}<\ldots$ is the partition of the circle into equal segments of length τ/m. On the other hand,

$$S(x_j',\xi_j')=\lim_{m\to\infty}(2\pi)^{(k-1)/2}(2\pi)^{-(m/2)\Sigma N_i}\prod_{\nu=1}^{k}(x_{\nu+1}-x_\nu)^{1/2}$$
$$\cdot\sqrt{\frac{x_{j+1}-x_{j-1}}{(x_{j+1}-x_j)(x_j-x_{j-1})}}\exp\{-\Sigma h(\tau/m,\xi_\nu^{(m)},\xi_{\nu+1}^{(m)})\}\frac{d\xi^{(m)}}{d\xi_j'}.$$

The relation (3.1) follows from the last two formulas. In the general case, it is valid because of the continuity of both sides. $\square$

Corollary 1.

$$\int S(x,\xi)d\xi=\sigma^{-1}(2\pi)^{k/2}\prod_{\nu=1}^{k}(x_{\nu+1}-x_\nu)^{1/2} \tag{3.2}$$

with some constant σ. Actually,

$$\int S(x,\xi)d\xi_1' = (2\pi)^{(k-2)/2} \prod_{\nu=1}^{k} (x_{\nu+1}-x_\nu)^{1/2} S(0,\xi_1;2\pi,\xi_1).$$

Simple estimates show that

$$\sigma^{-1} = \frac{1}{2\pi}\int S(0,\xi;2\pi,\xi)d\xi < \infty .$$

Corollary 2. *The functions*

$$p(x,\xi) = \sigma(2\pi)^{-k/2}\prod(x_{\nu+1}-x_\nu)^{-1/2}S(x,\xi) \tag{3.3}$$

are finite-dimensional densities of a probability measure $d\mu_1$. Indeed, they are continuous and satisfy the agreement and the normalization conditions.

Let dw be a conditional Wiener measure, see [8], in the space of continuous functions which vanish at some fixed point x_0 on the circle: $\delta(f) = f(x_0) = 0$, and

$$d\tilde{w} = dw \times (2\pi)^{-1/2}\exp(-\delta^2/2)d\delta$$

is the measure in the space of all continuous functions.

Proposition 3. *$d\mu_1$ is absolutely continuous with respect to dw and*

$$\frac{d\mu_1}{d\tilde{w}}(f) = \sigma(2\pi)^{-1/2}\exp\left\{-\frac{1}{4}\int f^4(x)dx + \frac{1}{2}f^2(x_0)\right\}. \tag{3.4}$$

Proof. Let us choose a function f, a partition $x_0 < x_1 < \dots < x_k < x_0 + 2\pi$ of the circle and a set

$$M \subset \prod_{j=0}^{k} (f(x_j)-\varepsilon, f(x_j)+\varepsilon).$$

We assume that $|x_{j+1}-x_j| < \varepsilon$, $j = 0, \dots, k$. By (3.3)

$$\begin{aligned} d\mu_1\{\mathscr{M}\} &= d\mu_1\{u\colon (u(x_0), \dots, u(x_k)) \in M\} \\ &= \sigma(2\pi)^{-(k+1)/2}\sum_{\nu=0}^{k}(x_{\nu+1}-x_\nu)^{-1/2}\int_M S(x,\xi)d\xi . \end{aligned}$$

Using the definition of h and Theorem 1 we can obtain after simple computations that

$$S(x,\xi) = \exp\left\{-\sum_{j=0}^{k}(\xi_{j+1}-\xi_j)^2/2(x_{j+1}-x_j) - \frac{1}{4}\int f^4 dx\right\}(1+o(1))$$

when $\varepsilon \to 0$. Thus

$$d\mu_1\{\mathscr{M}\} = \sigma(2\pi)^{-1/2}\exp\left\{\frac{1}{2}f^2(x_0) - \frac{1}{4}\int f^4 dx\right\} d\tilde{w}(\mathscr{M})(1+o(1)).$$

Corollary. *The measure $d\mu_1$ has a support in the space* Lip^α, $\alpha < 1/2$.

For the definition of the $d\mu_2$ we consider functionals A_j and B_ν:

$$v = A_0 + \sum(A_j \cos jy + B_\nu \sin \nu y).$$

Let $M \subset R^{2N+1}$. Then by definition

$$d\mu_2\{v\colon (A_0, \ldots, A_N; B_1, \ldots, B_N) \in M\}$$
$$= 2^{-N} \int_M \exp\left\{-\pi A_0^2 - (\pi/2) \sum_{j=1}^{N} (A_j^2 + B_j^2)\right\} dA dB.$$

The $d\mu_2$ is a realization of the abstract Wiener measure. It has a support in the space of generalized functions

$$\mathrm{Lip}^{-1/2-\varepsilon} = \mathrm{Const} + \frac{d}{dx} \mathrm{Lip}^{1/2-\varepsilon}, \qquad \varepsilon > 0.$$

4. Invariance of the $d\mu$

Let $\Phi(t)$ be the flow defined by (0.1). First of all we intend to prove its continuity.

Lemma 3. *$\Phi(t)$ maps continuously the space* $\mathrm{Lip}^{\alpha}(S^1) \times \mathrm{Lip}^{\alpha-1}(S^1)$ *into itself*, $0 < \alpha < 1/2$.

Proof. Consider two Cauchy problems

$$\begin{cases} u_{tt} - u_{xx} + u^3 = 0, \\ u|_{t=0} = u_0(x) \in \mathrm{Lip}^{\alpha}, \quad u_t|_{t=0} = v_0(x) \in \mathrm{Lip}^{\alpha-1} \end{cases}$$

and

$$w_{tt} - w_{xx} = 0, \qquad w|_{t=0} = u_0, \qquad w_t|_{t=0} = v_0.$$

If $0 < t < \pi$,

$$w(t,x) = \frac{u_0(x+t) + u_0(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy.$$

Clearly, $w \in \mathrm{Lip}^{\alpha}$, $w_t \in \mathrm{Lip}^{\alpha-1}$, and (w, w_t) depends continuously on (u_0, v_0). Let $r(t,x) = u - v$. Then

$$r_{tt} - r_{xx} + (r+w)^3 = 0, \qquad r|_{t=0} = r_t|_{t=0} = 0,$$

and according to the Duhamel principle

$$r(t,x) = -\int_0^t d\tau \int \frac{\theta(x-y+t-\tau) - \theta(x-y-t+\tau)}{2} [r(y,\tau) + w(y,\tau)]^3 dy, \quad (4.1)$$

where θ is the Heaviside function. The expression on the right-hand side of (4.1) is a contraction operator in a ball in $C([0,t], \mathrm{Lip}^{\alpha})$ when t is sufficiently small. Therefore, $(r, r_t) \in \mathrm{Lip}^{\alpha} \times \mathrm{Lip}^{\alpha-1}$ for sufficiently small t, and hence $(u, u_t) \in \mathrm{Lip}^{\alpha} \times \mathrm{Lip}^{\alpha-1}$. Now the assertion of the lemma follows from the group property of $\Phi(t)$ and its invariance under the transformation $t \mapsto -t$. □

Now we shall build the finite-dimensional approximation of $\Phi(t)$. Let us divide the circle into $2N+1$ equal parts by the points $y_j = 2\pi j/(2N+1), j = 0, \ldots, 2N$. Let $\xi_j, \eta_j, j = 0, \ldots, 2N$, be some real numbers. We denote by $u_N(\xi, x)$ the solution of the equation $u_{xx} = u^3$ which satisfies the conditions $u_N(\xi, y_j) = \xi_j$;

$$v_N(\eta, x) = A_0 + \sum_{j=1}^{N} (A_j \cos jy + B_j \sin jy)$$

is an interpolation trigonometrical polynomial, that is

$$\eta_j = A_0 + \sum_{\nu=1}^{N} (A_\nu \cos(2\pi j/(2N+1)) + B_\nu \sin(2\pi j/(2N+1))). \tag{4.2}$$

Clearly,

$$\sum_{j=0}^{N} \eta_j^2 = (2N+1)A_0^2 + (2N+1)/2 \sum_{j=1}^{N} (A_j^2 + B_j^2). \tag{4.3}$$

Let

$$H_N(\xi, \eta) = \frac{1}{2} \sum_{j=0}^{2N} \eta_j^2 + \frac{2N+1}{2} \int \left[\frac{u_{Nx}^2(\xi, x)}{2} + \frac{u_N^4}{4} \right] dx .$$

By $\Phi_N(t)$ we denote the Hamiltonian flow with the Hamiltonian H_N:

$$\dot{\xi}_j = \partial H_N / \partial \eta_j = \eta_j ; \qquad \dot{\eta}_j = -\partial H_N / \partial \xi_j . \tag{4.4}$$

Let $u(x) \in \mathrm{Lip}^\alpha$, $v(x) \in \mathrm{Lip}^{\alpha-1}$, $0 < \alpha < 1/2$. For a finite-dimensional approximation of these functions we take vectors

$$\xi^N(u(x)) = (u(y_0), \ldots, u(y_{2N})) \quad \text{and} \quad \eta^N(v(x)) = (\eta_0, \ldots, \eta_{2N}),$$

with η_j defined by (4.2); A and B are Fourier coefficients of v. By r_N we denote the restriction operator $r_N(u, v) = (\xi^N(u), \eta^N(v))$; i_N is the interpolation operator, $i_N(\xi, \eta) = (u_N(\xi, x), v_N(\eta, x))$.

Lemma 4. *Let $u(x) \in C^2$ and $v(x) \in C^1$. Then*

$$i_N \Phi_N(t) r_N(u, v) \to \Phi(t)(u, v) \quad \textit{when} \quad N \to \infty$$

in the space $C^1 \oplus C$.

Proof. Using the formula of the variation of a functional with a free end, we obtain that

$$\partial H_N / \partial \xi_j = -\frac{2N+1}{2\pi} [u_N'(\xi, y_j)] .$$

The function u satisfies the equation $u_N'' = u^3 = \xi_j^3 + O(1/N)$. Solving this equation without the term $O(1/N)$ and estimating the remainder, we can easily obtain that

$$\frac{\partial H_N}{\partial \xi_j} = -\frac{\xi_{j+1} - 2\xi_j + \xi_{j-1}}{(2\pi/(2N+1))^2} + \xi_j^3 + O(1/N) .$$

Thus (4.4) can be rewritten in the form

$$\dot{\xi}_j = \eta_j , \qquad \dot{\eta}_j = \frac{\xi_{j+1} - 2\xi_j + \xi_{j-1}}{(2\pi/(2N+1))^2} - \xi_j^3 + O(1/N) . \tag{4.4'}$$

The initial conditions are $\xi_j(0) = u(y_j)$ and $\eta_j(0) = v^{(N)}(y_j)$, where $v^{(N)}$ is the partial sum of the Fourier series of v. We have that $v^{(N)}(y_j) - v(y) = O(N^{-1+\varepsilon})$, $\varepsilon > 0$, uniformly with respect to j because $v \in C^1$. The system (4.4') with such initial conditions is a difference approximation for the problem

$$u_{tt} - u_{xx} + u^3 = 0 , \qquad u(0, x) = u(x) , \qquad u_t(0, x) = v(x) .$$

To finish the proof, we must apply a standard technique, in order to prove the convergence of the solutions of the difference equation to the solution of the differential equation. □

Consider a continuous non-linear functional $\mathscr{F}$ on $H^\alpha \oplus H^{\alpha-1}$ (H is the Sobolev space) such that $|\mathscr{F}(u,v)| \leqq 1$. Then

$$\int \mathscr{F}(u,v)d\mu = \lim_{N\to\infty} d_N \int \mathscr{F}[u_N(\xi,x), v_N(\eta,x)] \exp(-2\pi H_N/(2N+1)) d\xi dA dB .$$

The coordinates (A,B) and η are linearly dependent, therefore, $dAdB = c_N d\eta$. From the invariance of the measure $d\xi d\eta$ under the flow (4.4) it follows that

$$\begin{aligned} &d_N \int \mathscr{F}[\Phi_N(t)(u_N(\xi), v_N(\eta))] \exp(-2\pi H_N/(2N+1)) d\xi dA dB \\ &\quad = d_N \int \mathscr{F}[(u_N(\xi), v_N(\eta)] \exp(-2\pi H_N/(2N+1)) d\xi dA dB . \end{aligned} \tag{4.5}$$

The expression on the right-hand side of (4.5) converges to $\int \mathscr{F}(u,v)d\mu$ when $N\to\infty$. By the same technique as in Lemmas 1 and 3 (the spaces h^α and the Duhamel formula) it is easy to verify that $i_N \Phi_N(t) r_N$ are uniformly continuous with respect to N as operators from $\mathrm{Lip}^\alpha \oplus \mathrm{Lip}^{\alpha-1}$ into $H^\alpha \oplus H^{\alpha-1}$, $0<\alpha<1/2$. Taking into account Lemma 4,

$$\mathscr{F}[\Phi_N(t) r_N(u,v)] \to \mathscr{F}[\Phi(t)(u,v)], \qquad (u,v) \in \mathrm{Lip}^\alpha \oplus \mathrm{Lip}^{\alpha-1} .$$

By the Lebesque theorem, the left-hand side in (4.5) converges to $\int \mathscr{F}[\Phi(t)(u,v)]d\mu$ when $N\to\infty$. Therefore, $\int \mathscr{F}(u,v)d\mu = \int \mathscr{F}[\Phi(t)(u,v)]d\mu$. The last formula means the invariance of $d\mu$ under $\Phi(t)$.

Note. After sending the paper to the publisher, we discovered that the main results of Sect. 1 – Theorem 1 and the formula (0.9) – were proved simultaneously, independently and by different methods by Wodzicki [9].

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Communicated by J. Mather

Received October 13, 1983; in revised form August 6, 1984