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The KAM Theory of Systems with Short Range Interactions, I*

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Abstract. The existence of quasiperiodic trajectories for Hamiltonian systems consisting of long chains of nearly identical subsystems, with interactions which decay rapidly with increasing distance between the interacting components, is studied. Such models are of interest in statistical mechanics. It is shown that nonergodic motions persist for much larger perturbations than prior work indicated. If the number of degrees of freedom of the system is N, the allowed perturbation decreases only as an inverse power of N, as the number of degrees of freedom increases, rather than the inverse power of $N!$ which previous estimates yielded.

1. Introduction

The Kolmogorov, Arnol'd, Moser (KAM) theory [15, 1, 16] proves that "small" perturbations of integrable Hamiltonian systems possess "large" sets of initial conditions for which the trajectories remain quasiperiodic. In this paper we discuss how the "strength" of the allowed perturbation varies with the number of degrees of freedom, N, in the system. (We give precise meanings to the words in quotation marks below.) Classical estimates for a general analytic perturbation of strength ε_0 require

$$\varepsilon_0 < C(N!)^{-\alpha} \tag{1.1}$$

to ensure that the theory applies. Here C is a constant depending on all the parameters of the system except N, and [11] gives a value of $\alpha = 31$.

Recent numerical experiments [4, 6, 3, 10] indicate that at least in systems with short range interactions, perturbations much larger than those permitted by (1.1) still give rise to quasiperiodic motion. In the present paper we initiate a study of such

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systems and show that for a class of Hamiltonians with short range interactions the perturbation need only satisfy

$$\varepsilon_0 < C'N^{-\alpha'}, \tag{1.2}$$

to ensure the existence of quasiperiodic trajectories. We obtain a value of $\alpha' = 160$.

Nearly thirty years ago [8] it was pointed out that the existence of such trajectories is at variance with the commonly held belief that large systems behave ergodically. It seems that the rapid decay (with increasing N) of the estimate (1.1) ensuring applicability of the KAM theory has discouraged examination of its importance for statistical mechanics. Even the estimate (1.2) gives only a small region of quasiperiodic behavior as N becomes large, but the exponent of N in (1.2) is certainly not optimal, and we hope that the improvement in the range of applicability of the theory as compared to (1.1) will encourage further research, both to determine the optimal power of N in (1.2), and also to determine whether or not there may be a finite allowed perturbation as $N \to \infty$, for which the motions are not ergodic (although possibly not quasiperiodic either).

To state our results we introduce the following notation. Let V be the sphere of radius r in $\mathbb{R}^N$, (with respect to the Euclidean metric), and $T^N = [0, 2\pi]^N$, with opposite ends identified. A Hamiltonian in "action-angle" form is a function $H(\underline{I}, \underline{\phi})$: $V \times T^N \to \mathbb{R}$. We write the Hamiltonian as

$$H(\underline{I}, \underline{\phi}) = h^0(\underline{I}) + f^0(\underline{I}, \underline{\phi}). \tag{1.3}$$

with $f^0(\underline{I}, \underline{\phi})$ a perturbation of the integrable Hamiltonian $h^0(\underline{I})$. Since $f^0(\underline{I}, \underline{\phi})$ is periodic it may be expanded in a Fourier series,

$$f^0(\underline{I}, \underline{\phi}) = \sum_{\underline{v} \in \mathbb{Z}^N} f^0_{\underline{v}}(\underline{I}) e^{i\underline{v}\cdot\underline{\phi}}. \tag{1.4}$$

Defining $\underline{z}^{\underline{v}} = \prod_{j=1}^{N} z_j^{v_j} = \prod_{j=1}^{N} e^{iv_j\phi_j}$ we can regard $f^0(\underline{I}, \underline{\phi})$ as a function on $\mathbb{C}^{2N}$, which we write as $f^0(\underline{I}, \underline{z})$. We demand that $H(\underline{I}, \underline{z})$ be analytic on

$$W(\rho_0, \xi_0; V) \equiv \bigcup_{\underline{I} \in V} \{(\underline{I}', \underline{z}) | (\underline{I}', \underline{z}) \in \mathbb{C}^{2N}, e^{-\xi_0} < |z_j| < e^{\xi_0}$$

$$|I'_j - I_j| < \rho_0;\ \forall j = 1, \ldots, N\}. \tag{1.5}$$

Given a Hamiltonian define E_0, ε_0, by

$$\sup |\partial h^0/\partial \underline{I}| \leqq E_0, \tag{1.6}$$

$$\sup \left\{ \left| \frac{\partial f^0}{\partial \underline{I}}(\underline{I}, \underline{z}) \right| + \rho_0^{-1} \left| \frac{\partial f^0}{\partial \underline{\phi}}(\underline{I}, \underline{z}) \right| \right\} \leqq \varepsilon_0, \tag{1.7}$$

where the suprema run over $W(\rho_0, \xi_0; V)$. In (1.7), $\partial/\partial\underline{\phi} \equiv i(z_1(\partial/\partial z_1), \ldots, z_N(\partial/\partial z_N))$. Also, by $\partial^2 f/\partial\phi_i \partial\phi_j$ we mean $-z_i(\partial/\partial z_i)(z_j(\partial f/\partial z_j))$, the only possible confusion coming when $i = j$. For any $\underline{v} \in \mathbb{C}^N$ define $|\underline{v}| = \sum_{j=1}^{N} |v_j|$. For a matrix M, $|M| \equiv \sum_{i,j=1}^{N} |M_{ij}|$. The factor of ρ_0^{-1} in the second term of (1.7) is to keep the dimensions of the two terms the same, since as was pointed out in

[11] this greatly simplifies the resulting estimates. Also, we will assume for convenience that $\rho_0 < E_0$.

The Hamiltonians we consider consist of almost independent almost identical subsystems, lying along a line, with interactions which decrease rapidly in strength as the distance between the points of interaction increases. (Note that keeping r, the radius of the set V, fixed corresponds to decreasing the average energy in each subsystem, which is why the volume of phase space does not grow as $(\text{const})^N$ as one would expect for N identical systems.) As a prototype consider

$$H(\underline{I}, \underline{\phi}) = \tfrac{1}{2}\underline{I}\cdot\underline{I} + \varepsilon \sum_{i=1}^{N-1} \cos(\phi_{i+1} - \phi_i). \tag{1.8}$$

We note that the constant ε is not the "strength of the perturbation," ε_0, as defined in (1.7), but differs from it by a constant. (Comparison with (1.7) shows that one should have, $(\frac{1}{2})\varepsilon N e^{2\xi_0} \leqq \varepsilon_0$.)

More generally we require:

(a) nearly independent, nearly identical subsystems:

Define $\underline{\omega}^0(\underline{I}) = (\partial h^0/\partial \underline{I})(\underline{I})$. We require

$$\sup\left|\frac{\partial \omega_i^0}{\partial I_j}\right| \leqq \varepsilon_0 \rho_0^{-1}(\varepsilon_0 \rho_0^{-1})^{|i-j|}; \quad i \neq j, \tag{1.9}$$

and

$$\frac{\partial \omega_i^0}{\partial I_i}(\underline{I}) = 1 + \chi_i^0(\underline{I}), \tag{1.10}$$

with $\sup_{i,\underline{I}} |\chi_i^0(\underline{I})| \leqq B_1 N^{-1}$, and B_1 some universal constant, say 2^{-3}. These conditions are satisfied for (1.8).

(b) weak, short range interactions:

The Fourier coefficients of $f^0(\underline{I}, \underline{z})$ must satisfy

$$\sup |f_{\underline{v}}^0(\underline{I})| \leqq \varepsilon_0 \rho_0 (\varepsilon_0 \rho_0^{-1})^{d(\operatorname{supp} \underline{v})} e^{-\xi_0 |\underline{v}|}, \tag{1.11}$$

where $d(\operatorname{supp} \underline{v})$ = distance between the two most widely separated points in $\operatorname{supp} \underline{v}$ (regarding $\underline{v}$ as a function on $[1, N] \cap \mathbb{Z}$). We also require that if l is the point in $\operatorname{supp} \underline{v}$ farthest form j

$$\sup\left|\frac{\partial}{\partial I_j} f_{\underline{v}}^0(\underline{I})\right| \leqq \varepsilon_0 (\varepsilon_0 \rho_0^{-1})^{|l-j|} e^{-\xi_0 |\underline{v}|}, \tag{1.12}$$

and finally

$$\sup\left|\frac{\partial^2 f_{\underline{v}}^0}{\partial I_i \partial I_j}(I)\right| \leqq \varepsilon_0 \rho_0^{-1} (\varepsilon_0 \rho_0^{-1})^{|i-j|} e^{-\xi_0 |\underline{v}|}, \tag{1.13}$$

In each of the inequalities (1.9)–(1.13), the supremum is taken over $W(\rho_0, \xi_0; V)$. Also note that (1.8) obeys (1.7) and (1.11)–(1.13) if we take $\rho_0 = 1$, and $\varepsilon < \min((2\varepsilon_0/N)e^{-2\xi_0}, 2\varepsilon_0^2 e^{-2\xi_0})$. We mention in passing that one could choose ε to be the "strength of the perturbation" in the Hamiltonian (1.8), rather than using the more general definition (1.7), and the statement of Theorem 1.1 would be unchanged, save that ε would replace ε_0 on the left-hand side of (1.15) and the precise values of the constants B, β, γ, and σ on the right-hand side of that inequality would change.

We require a great deal of analyticity in the angular variables $\underline{z}$, enough to set

$$\xi_0 = B_2 \ln N, \tag{1.14}$$

with B_2 a universal constant. (Take $B_2 = 12\delta + 4$, where δ is the constant defined in Sect. 2.) This is again satisfied for (1.8), for any N, but this is one restriction on our theorem which it would be very nice to weaken—at least to remove the N dependence.

Under assumptions (1.6)–(1.14) we obtain

Theorem 1.1. *There exist constants $B, \beta, \gamma, \sigma > 0$ such that when*

$$\varepsilon_0 < \rho_0 B \lambda^{\beta} (E_0 \rho_0^{-1})^{-\gamma} N^{-\sigma}, \tag{1.15}$$

for some $\lambda \in (0,1)$, there exists an open set $\Gamma \subset V$, and constants $\hat{\rho}_0, \hat{\xi}_0 > 0$ and a change of variables, $C: (\underline{I}', \underline{z}') \to (\underline{I}, \underline{z})$, defined and analytic on $W(\hat{\rho}_0, \hat{\xi}_0; \Gamma)$. Furthermore the image of $W(\hat{\rho}_0, \hat{\xi}_0; \Gamma)$ under C is contained in $W(\rho_0, \xi_0; V)$, so define

$$\hat{H}(\underline{I}', \underline{z}') \equiv H \circ C(\underline{I}', \underline{z}') = \hat{h}^0(\underline{I}') + \hat{f}^0(\underline{I}', \underline{z}').$$

Then

$$\sup \left| \frac{\partial \hat{h}^0}{\partial \underline{I}'} \right| \leqq \hat{E}_0 \leqq 2E_0, \tag{1.16}$$

$$\sup \left\{ \left| \frac{\partial \hat{f}^0}{\partial \underline{I}'} \right| + \hat{\rho}_0^{-1} \left| \frac{\partial \hat{f}^0}{\partial \underline{\phi}'} \right| \right\} \leqq \hat{\varepsilon}_0 \leqq \rho_0 (\varepsilon_0 \rho_0^{-1})^N, \tag{1.17}$$

$$\frac{\partial^2 \hat{h}^0}{\partial I_i'^2}(\underline{I}') = 1 + \hat{\chi}_i(\underline{I}'), \tag{1.18}$$

with

$$\sup_{i, \underline{I}'} |\hat{\chi}_i(\underline{I}')| \leqq 2^{-2} N^{-1}, \tag{1.19}$$

and

$$\sup \left| \frac{\partial^2 \hat{h}^0}{\partial I_i' \partial I_j'}(\underline{I}') \right| \leqq \varepsilon_0 \rho_0^{-1} (\varepsilon_0 \rho_0^{-1})^{(1/8)|i-j|}; \quad \text{if } i \neq j. \tag{1.20}$$

Also, the transformation C is 1–1 and canonical on $\Gamma \times T^N$, and $\operatorname{vol} \Gamma \geqq (1-\lambda) \operatorname{vol} V$. *In (1.16)–(1.20) the suprema run over $W(\hat{\rho}_0, \hat{\xi}_0; \Gamma)$. (When we say C is canonical on $\Gamma \times T^N$, we are identifying T^N, with the set of $\underline{z} \in \mathbb{C}^N$ such that $|z_i| = 1$, $i = 1, \ldots, N$, and then C is canonical with respect to the variables $(\underline{I}, \underline{\phi})$, $\underline{I} \in \Gamma$ and $\underline{\phi}$ defined by $\underline{z} = e^{i\underline{\phi}}$.)*

It is possible to compute the values of the constants β, γ, and σ. The best values I have obtained are $\beta = 12$, $\gamma = 18$, and $\sigma = 160$. We do not present those calculations here—the interested reader may refer to [20].

Note that Theorem 1.1 is not a KAM type theorem since $\hat{H}(\underline{I}, \underline{z})$ is not an integrable system. To complete the proof we need:

Theorem 1.2. *Given the Hamiltonian $\hat{H}(\underline{I}, \underline{z})$ constructed in Theorem 1.1, analytic on $W(\hat{\rho}_0, \hat{\xi}_0; \Gamma)$ and satisfying (1.16)–(1.20), there are constants $\hat{B}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\sigma}$, $\hat{\kappa}$, $\hat{\zeta}$ and $\hat{\mu} > 0$ such that if*

$$\hat{\varepsilon}_0 < \hat{\rho}_0 \hat{B} \hat{\lambda}^{\hat{\beta}} (1-\lambda)^{\hat{\zeta}} (\hat{\rho}_0^{-1} \hat{E}_0)^{-\hat{\gamma}} (\rho_0^{-1} E_0)^{\hat{\kappa}} N^{-\hat{\sigma} N} e^{-\hat{\mu} N}, \tag{1.21}$$

for $\hat{\lambda} \in (0,1)$, and λ the parameter in Theorem 1.1, then there exists $\hat{\Gamma} \subset \Gamma$ and $2N$, $C^\infty(\Gamma \times T^N)$ functions $I'_1, \ldots, I'_N, z'_1, \ldots, z'_N$ such that $I'_1, \ldots, I'_N$ are first integrals for the perturbed motions starting in $\hat{\Gamma} \times T^N$. Furthermore, the change of variables $\hat{C}: (\underline{I}', \underline{z}') \to (\underline{I}, \underline{z})$ is 1–1 and canonical on $\hat{\Gamma} \times T^N$, and for $\underline{I}', \underline{z}'$ in the preimage (with respect to $\hat{C}$) of $\Gamma \times T^N$, there is a function h, depending only on $\underline{I}'$ such that,

$$\hat{H} \circ \hat{C}(\underline{I}', \underline{z}') = h(\underline{I}'). \tag{1.22}$$

Finally, $\operatorname{vol} \hat{\Gamma} \geqq (1-\hat{\lambda}) \operatorname{vol} \Gamma$.

We note that $\hat{\Gamma}$ will turn out to be a Cantor set. To define derivatives of the functions $\underline{I}', \underline{z}'$ on $\hat{\Gamma} \times T^N$, we take the restrictions of their derivatives on $\Gamma \times T^N$ to $\hat{\Gamma} \times T^N$. In fact, the functions are constructed by just the opposite procedure—one constructs a *family* of functions on $\hat{\Gamma} \times T^N$ which are shown to obey the formal relations expected of the derivatives of $(\underline{I}', \underline{z}')$, and these are then extended to give $C^\infty(\Gamma \times T^N)$ functions by the theorem of Whitney [21]. (See [5, 17] for more details.) Once more a long calculation gives $\hat{\beta} = 17$, $\hat{\zeta} = 17$, $\hat{\gamma} = 2$, $\hat{\kappa} = 16$, $\hat{\sigma} = 14$ and $\hat{\mu} = 185$. The desired KAM theorem is

Corollary 1.3. *Let $H(\underline{I}, \underline{z})$ be analytic on $W(\rho_0, \xi_0; V)$ and satisfy (1.6)–(1.14). Then there exists $B' > 0$ such that if*

$$\varepsilon_0 < \rho_0 B' (E_0 \rho_0^{-1})^{-\gamma'} \lambda^{\zeta'} (1-\lambda)^{\zeta''} N^{-\sigma'}, \tag{1.23}$$

there exists a set $\tilde{\Gamma} \subset V$ and $2N$, $C^\infty(V \times T^N)$ functions $I'_1, \ldots, I'_N, z'_1, \ldots, z'_N$ such that $I'_1, \ldots, I'_N$ are first integrals for motions of the perturbed system starting in $\tilde{\Gamma}$, and the change of variables $\tilde{C}: (\underline{I}', \underline{z}') \to (\underline{I}, \underline{z})$ is one to one and canonical on $\tilde{\Gamma} \times T^N$. Furthermore $H \circ \tilde{C}(\underline{I}', \underline{z}') = h(\underline{I}')$, on the preimage of $\tilde{\Gamma} \times T^N$ and $\operatorname{vol} \tilde{\Gamma} \geqq (1-2\lambda) \operatorname{vol} V$. *We may take $\gamma' = 18$, $\zeta' = 8$, $\zeta'' = 2$ and $\sigma' = 160$.*

Proof. The corollary follows immediately if we take $\lambda = \hat{\lambda}$ and $\tilde{C} = C \circ \hat{C}$, where C and $\hat{C}$ are the two canonical transformations constructed in Theorems 1.1 and 1.2, since (1.23) implies that both (1.15) and (1.22) are satisfied, using the definition of $\hat{\rho}_0$ that emerges from Sect. 2, and the fact that $\hat{\xi}_0$ can be chosen to be one (which also comes from Sect. 2).

We close this section with some comments of a general nature. First we note that Theorem 1.2 is almost identical to the theorems of Chierchia and Gallavotti [5] and Pöschel [17] except that instead of assuming a bound on the anisochrony parameter $\eta_0 = \sup |(\partial h^0 / \partial \underline{I}\, \partial \underline{I})^{-1}|$, we will derive a bound on this quantity from (1.16)–(1.20). This gives better control over the constant $\hat{\gamma}$.

Our second comment concerns (1.10). This condition looks very restrictive but we point out two respects in which it is somewhat less so than it may at first appear. First is the trivial observation that we may replace the "one" on the right-hand side of (1.10) by any positive constant, just by changing the scale of I, and possibly altering the constant B_1. Secondly suppose $h^0(\underline{I}) = \sum_{i=1}^{N} \tilde{h}^0(I_i)$, with $\tilde{h}^0(I_i)$ analytic in a ball of radius $\tilde{r}$ in $\mathbb{C}$. Then if $\tilde{h}^{0\prime\prime}(0) > 0$, we may write

$$\frac{\partial \omega_i^0}{\partial I_i}(I_i) = \tilde{h}^{0\prime\prime}(0) + \tilde{\chi}_i(I_i), \tag{1.24}$$

and $\tilde{\chi}_i(I_i)$ will obey the bound (1.11) provided $\tilde{r}$ is sufficiently small. One might also wonder how the restrictions (1.11)–(1.13) arise. It would seem more general perhaps to drop the factors of $\varepsilon_0 \rho_0$, ε_0, and $\varepsilon_0 \rho_0^{-1}$ in front of (1.11)–(1.13). In fact the proof goes through with very few modifications if one makes this change, and the choices used in (1.11)–(1.13) are just one convenient choice of the many possible. Note further, that if one chose to make the change suggested above one could recover the present form by a slight redefinition of ε_0 and ρ_0.

Our third comment concerns the numerical experiments mentioned earlier [4, 6, 3, 10]. These seem to show that nonergodic behavior persists in the system for a finite perturbation even when N becomes very large. Even more surprising, the maximum perturbation per mode (ε_0/N in our notation) for which nonergodic trajectories survive seems to be almost independent of N for N larger than about ten. While the exponent of N which we quote in (1.23) is certainly not optimal, and might be reduced by more careful estimates, it is by no means clear how the present methods could yield an N-independent estimate for the allowed perturbation. Thus a very interesting unresolved question becomes can one extend the KAM theory to explain the experimental results? If not, can one perhaps show that there is some nonquasiperiodic motion, but also nonergodic motion in these systems? If the answer to either of these questions is yes, one should also ask what implications such motions have for statistical mechanics.

As we mentioned, the proof of Theorem 1.2 is a straightforward application of the KAM machinery which we present in [19]. The proof of Theorem 1.1, on the other hand, proceeds by making a finite number of canonical transformations which successively reduce the strength of the interaction while including the effect of interactions on longer and longer distance scales. This multiple length scale analysis appears trivial in the case of (1.8) which has only nearest neighbor interactions. However, the first change of variables produces interactions of arbitrarily long range, making the multiple length scales necessary. To study the decay of the interactions we generate, we borrow an idea from statistical mechanics and field theory [17, 2] which is introduced in [19].

The KAM theory is constructed so that the sequence of canonical transformations may be repeated indefinitely, but the transformations used in Theorem 1.1 permit only a finite number of iterations. We iterate until the characteristic length scale is of order N (the size of the system), and then appeal to Theorem 1.2. This procedure is reminiscent of the renormalization group where one starts with a system in the critical region and iterates the renormalization group transformation a finite number of times, until one is far enough from the critical region to treat the system by other means (e.g., perturbation theory). Because of this similarity we shall often refer to the transformed Hamiltonian as the renormalized Hamiltonian, and to the interactions generated by this procedure as the renormalized interactions. The similarity between KAM theory and the renormalization group has been commented on previously by Doveil and Escande [7], Gallavotti [11] and Kadanoff [14]. Also, that part of our analysis which studies the effects of small denominators on longer and longer length scales is reminiscent of the work of Fröhlich and Spencer [9] on electron localization.

At a number of points we bound derivatives by what we refer to as dimensional estimates. When the derivative so estimated is a derivative with respect to $\underline{I}$, this is just the observation that if $f(\underline{I})$ is an analytic function on the domain D, and r is such that the set of points $\{\underline{I} \mid |I_j - I_{0j}| < r\}$ is contained in D, then $|\partial f/\partial I_j(\underline{I}_0)| \leqq \sup_D |f(\underline{I})|/r$. The case of derivatives with respect to $\underline{\phi}$ is somewhat more delicate and we refer the reader to Appendix 2 for an explanation. Also, a number of constants $c_1, c_2, \ldots, c', c'', \ldots$ appear throughout the paper. These are generic constants of magnitude less than one. They may represent different constants in different contexts.

We close this section with an outline of the remainder of the paper. In Sect. two we present an inductive lemma leading to a proof of Theorem 1.1. Section three defines the sequence of canonical transformations used to prove this lemma and shows that the resulting changes of variables are well defined. Section four proves that the interactions in our renormalized Hamiltonian decay nearly as fast as those in the original Hamiltonian, and finally in Sect. five we estimate the amount of phase space lost in this procedure.

2. An Outline of the Proof of Theorem 1.1

Theorem 1.1 is proved by constructing a sequence of Hamiltonians $H^k(\underline{I}, \underline{z})$, each of which will obey a short range condition similar to (1.11) but with the additional requirement that the size (in the sense of (1.7)) of the nonintegrable part has been reduced with respect to $H^{k-1}(\underline{I}, \underline{z})$. Specifically, if $k_0 = 1 +$ integer part of $[(\ln N)/(\ln 3/2)^{-1}]$, one has:

Lemma 2.1. *There exist positive constants B, β, γ, σ such that if*

$$\varepsilon_0 < \rho_0 B \lambda^\beta (E_0 \rho_0^{-1})^{-\gamma} N^{-\sigma}, \tag{2.1}$$

for some $\lambda \in (0, 1)$, there exists a decreasing sequence of regions $V \supset V_0 \supset \cdots \supset V_{k_0 - 1}$ such that $\mathrm{vol}(V_k \backslash V_{k+1}) \leqq (1/4)\lambda e^{-(1/2)(3/2)^{k+1}}$ (vol V), (and vol$(V \backslash V_0) \leqq (1/4)\lambda e^{-3/4}$ vol V) and transformations C^k, $\tilde{C}^k$ which are 1–1 and canonical on $V_k \times T^N$. (In each case, here and below, $0 \leqq k < k_0$.) These transformations are analytic on $W(4\rho_{k+1}, \xi_k - 2\delta; V_k)$ and

$$C^k: W(2\rho_{k+1}, \xi_k - 3\delta; V_k) \to W(4\rho_{k+1}, \xi_k - 2\delta; V_k),$$
$$\tilde{C}^k: W(2\rho_{k+1}, \xi_k - 3\delta; V_k) \to W(4\rho_{k+1}, \xi_k - 2\delta; V_k), \tag{2.2}$$

with $W(4\rho_{k+1}, \xi_k - 2\delta; V_k) \subset W(\rho_k, \xi_k; V_{k-1})$. (We define $V_{-1} = V$.) On their common domain of definition $C^k \circ \tilde{C}^k = \tilde{C}^k \circ C^k =$ identity.

A sequence of Hamiltonians is defined by

$$H^{k+1}(\underline{I}, \underline{z}) = H^k \circ C^k(\underline{I}, \underline{z}) \equiv h^{k+1}(\underline{I}) + f^{k+1}(\underline{I}, \underline{z}), \tag{2.3}$$

which satisfy

$$\sup\left\{\left|\frac{\partial f^k}{\partial \underline{I}}\right| + \rho_k^{-1}\left|\frac{\partial f^k}{\partial \underline{\phi}}\right|\right\} \leqq \varepsilon_k, \quad \sup\left|\frac{\partial h^k}{\partial I}\right| \leqq E_k, \tag{2.4}$$

with the supremum running over $W(\rho_k, \xi_k; V_{k-1})$. (We begin the induction by setting $H^0(\underline{I}, \underline{z}) = H(\underline{I}, \underline{z})$, our initial Hamiltonian.) Defining $\underline{\omega}^k(\underline{I}) = (\partial h^k/\partial \underline{I})$ one has

$$\frac{\partial \omega_i^k}{\partial I_i} = 1 + \chi_i^k(\underline{I}), \tag{2.5}$$

with $\sup|\chi_i^k(\underline{I})| \leqq B_1 N^{-1} + \sum_{j=0}^{k-1} 2N\varepsilon_j\rho_j^{-1}$ for $i = 1, \ldots, N$, and for $i \neq j$

$$\sup\left|\frac{\partial \omega_i^k}{\partial I_j}\right| \leqq \theta(k; i, j) \equiv (\varepsilon_0\rho_0^{-1})\left\{(\varepsilon_0\rho_0^{-1})^{|i-j|} + \sum_{m=0}^{k-1} (\varepsilon_0\rho_0^{-1})^{(1-\eta_m)|i-j|}\right\}. \tag{2.6}$$

Finally, each of these Hamiltonians obeys a short range condition, namely, if $f_{\underline{\nu}}^k(\underline{I})$ are the coefficients in the Laurent expansion for $f^k(\underline{I}, \underline{z})$,

$$\sup|f_{\underline{\nu}}^k(\underline{I})| \leqq \varepsilon_0\rho_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)d(\operatorname{supp}\underline{\nu})}e^{-\xi_k|\underline{\nu}|},$$

$$\sup\left|\frac{\partial f_{\underline{\nu}}^k}{\partial I_j}(\underline{I})\right| \leqq \varepsilon_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)|l-j|}e^{-\xi_k|\underline{\nu}|}, \tag{2.7}$$

and

$$\sup\left|\frac{\partial f_{\underline{\nu}}^k}{\partial I_i \partial I_j}(\underline{I})\right| \leqq \varepsilon_0\rho_0^{-1}(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)|i-j|}e^{-\xi_k|\underline{\nu}|}.$$

Here, the l on the right-hand side of the second inequality is the point in $\operatorname{supp}\underline{\nu}$ farthest from j. (Again, all suprema are over $W(\rho_k, \xi_k; V_{k-1})$.)

The constants in this lemma are:

$$C = 2^{11}(\rho_0\lambda)^{-1}N^2,$$

$$\varepsilon_k = \rho_0(\varepsilon_0\rho_0^{-1})^{(3/2)^k},$$

$$E_{k+1} = E_k + \varepsilon_k,$$

$$L_k = 8(3/2)^k,$$

$$\xi_{k+1} = \xi_k - 4\delta,$$

for δ some (fairly large) constant which must be chosen inversely proportional to c_2—the constant in the definition of β_k, below.

$$M_k = \delta^{-1}|\ln \varepsilon_k\rho_0^{-1}|, \tag{2.8}$$

$$\rho_{k+1} = \rho_k \cdot 2^{-6}(E_k CN)^{-1}\exp(-2M_k - 2L_k), \quad \text{and} \quad \eta_k = 2\sum_{j=0}^{k-1}\beta_j + c_1 \cdot k/k_0,$$

where $\beta_k = c_2(3/2)^{-k} + c_3/k_0$. Here c_1, c_2, and c_3 are non-zero constants chosen so that $\eta_{k_0} \leqq 3/4$, and in each of the definitions in (2.8), $0 \leqq k \leqq k_0$.

We first remark that the conditions of Sect. 1 insure that our initial Hamiltonian satisfies (2.4)–(2.7), for the case $k=0$.

Second note that this lemma yields Theorem 1.1 if we set $\Gamma = V_{k_0-1}$, $\hat{H}(\underline{I},\underline{z}) = H^{k_0}(\underline{I},\underline{z})$, $C = C^0, \ldots, C^{k_0-1}$, and let $\hat{\rho}_0$ and $\hat{\xi}_0$ equal ρ_{k_0} and ξ_{k_0} respectively. Then on $W(\hat{\rho}_0, \hat{\xi}_0; \Gamma)$ we have

$$\sup\left|\frac{\partial \hat{h}_0}{\partial \underline{I}}\right| \leqq E_{k_0} = E_0 + \sum_{0}^{k_0-1} \varepsilon_k \leqq 2E_0 .$$

Similarly,

$$\sup\left\{\left|\frac{\partial \hat{f}^0}{\partial \underline{I}}\right| + \hat{\rho}_0^{-1}\left|\frac{\partial \hat{f}^0}{\partial \underline{\phi}}\right|\right\} \leqq \varepsilon_{k_0} = \rho_0(\varepsilon_0\rho_0^{-1})^{(3/2)^{k_0}} \leqq \rho_0(\varepsilon_0\rho_0^{-1})^N,$$

and

$$\sup\left|\frac{\partial^2 \hat{h}^0}{\partial I_i \partial I_j}\right| \simeq \theta(k_0; i, j) \leqq \varepsilon_0\rho_0^{-1}(\varepsilon_0\rho_0^{-1})^{1/8|i-j|}$$

(since $1-\eta_k > 1/8$). Finally,

$$\operatorname{vol}\Gamma = \operatorname{vol} V_{k_0-1} \geqq \left\{1 - \sum_{k=0}^{\infty} (\lambda/4)e^{-(1/2)(3/2)^k}\right\} \operatorname{vol} V \geqq (1-\lambda)\operatorname{vol} V.$$

Thus, we have reduced the proof of Theorem 1.1 to proving Lemma 2.1.

We close by noting that we lose a fixed amount of analyticity in the z variables each time we iterate this procedure. Thus, it must terminate after finitely many steps. This is why Theorem 1.1 must be coupled with Theorem 1.2.

3. The Inductive Step

The iterative procedure described in the previous section starts by taking $H^0(\underline{I},\underline{z}) = H(\underline{I},\underline{z})$, with $H(\underline{I},\underline{z})$ the initial Hamiltonian of Sect. 1. Estimates (2.4)–(2.7) follow from (1.9)–(1.13). We assume that the Hamiltonians, H^j, have been constructed for $0 \leqq j \leqq k \leqq k_0 - 1$, satisfying the bounds of Sect. 2. In the present section we construct C^k, $\tilde{C}^k$, and H^{k+1}.

Define $\mathbb{X}_k = \{\underline{v} \in \mathbb{Z}^N \,|\, |\underline{v}| \leqq M_k \text{ and } d(\operatorname{supp}\underline{v}) \leqq L_k\}$. Given a function $h(\underline{I})$, defined on some region $V' \subset \mathbb{R}^N$ the "resonances of h on the scale k" are

$$R(k,h;V') = \left\{\underline{I} \in V' \left| \left\langle \frac{\partial h}{\partial \underline{I}}(\underline{I}), \underline{v} \right\rangle^{-1} \right| \leqq C\exp[(3/2)|\underline{v}| + L_k],\right.$$
$$\left.\text{for some } \underline{v} \in \mathbb{X}_k, \underline{v} \neq 0\right\}. \tag{3.1}$$

Here $\langle .,. \rangle$ is the usual inner product for N-vectors. Define $\tilde{\rho}_k \equiv 8\rho_{k+1}$, $B_{k-1} = \{\underline{I} \,|\, \underline{I} \in S(2\tilde{\rho}_k, \underline{I}') \text{ for some } \underline{I}' \in \partial V_{k-1}\}$. Define $\bar{V}_k = (V_{k-1} \backslash B_{k-1}) \backslash R(k, h^k; V_{k-1})$ and $V_k = \bigcup_{I \in \bar{V}_k} S(\tilde{\rho}_k/2, \underline{I})$. Here $S(r, \underline{I}) = \{\underline{I}' \in \mathbb{R}^N \,|\, |\underline{I} - \underline{I}'| < r\}$.

For any $\underline{I}$ such that $(\underline{I},\underline{z}) \in W(\tilde{\rho}_k, \xi_k; V_k)$, for some $\underline{z}$, there is a path γ connecting $\underline{I}$ to some $\underline{I}' \in \bar{V}_k$ and made up of $2N$ pieces of length at most $\tilde{\rho}_k$ along which only one coordinate of $\underline{I}$ varies. Furthermore, for every $\underline{I}'' \in \gamma$, $(\underline{I}'', \underline{z}) \in W(\tilde{\rho}_k, \xi_k; V_k)$.

Then if $\underline{\nu}\in\mathbb{X}_k$, $\underline{\nu}\neq\underline{0}$

$$|\langle\underline{\omega}^k(\underline{I}),\underline{\nu}\rangle^{-1}|=|\langle\underline{\omega}^k(\underline{I}'),\underline{\nu}\rangle^{-1}|\times\left\{|1+\langle\underline{\omega}^k(I'),\underline{\nu}\rangle^{-1}\int_\gamma d\underline{I}''\left\langle\frac{\partial\underline{\omega}^k}{\partial\underline{I}}(\underline{I}''),\nu\right\rangle|\right\}^{-1}$$

$$\leqq 2C\exp[(3/2)|\underline{\nu}|+L_k]. \tag{3.2}$$

The last inequality combined (3.1) with the fact that (2.5) and (2.6) imply $|\partial\omega_j^k/\partial I_i|\leqq 2$ on $W(\tilde{\rho}_k,\xi_k;V_k)$, for all i,j, and the observation that since $\rho_0<E_0$, $\rho_k<E_k$ for all k. Thus the second term in curly brackets is bounded by $2^2N|\underline{\nu}|\tilde{\rho}_k e^{(3/2)|\underline{\nu}|++L_k}\leqq 1/2$.

Define (on $W(\tilde{\rho}_k,\xi_k;V_k)$) the generating function for our change of variables

$$\Phi(\underline{I}',\underline{z})=\sum_{\substack{\underline{\nu}\in\mathbb{X}_k\\ \underline{\nu}\neq\underline{0}}}\left(\frac{f_{\underline{\nu}}^k(\underline{I}')}{i\langle\underline{\omega}^k(\underline{I}'),\underline{\nu}\rangle}\right)\underline{z}^{\nu}. \tag{3.3}$$

Specify the change of variables $(\underline{I}',\underline{z}')\leftrightarrow(\underline{I},\underline{z})$ by

$$\underline{I}=\underline{I}'+\frac{\partial\Phi^k}{\partial\underline{\phi}}(\underline{I}',\underline{z}),\quad \underline{z}'=\underline{z}\exp\left[i\frac{\partial\Phi^k}{\partial\underline{I}'}(\underline{I}',\underline{z})\right]. \tag{3.4}$$

We show below that the implicit function theorem allows us to invert (3.4) to obtain $(\underline{I}',\underline{z}')$ in terms of $(\underline{I},\underline{z})$ or vice versa.

The choice of $\Phi^k(\underline{I}',\underline{z})$ is motivated by classical perturbation theory [11]. Unlike previous studies we introduce two cutoffs, one to take advantage of the decay arising from analyticity ($|\underline{\nu}|\leqq M_k$, an "ultraviolet cutoff") and one to take advantage of the decay with distance ($d(\operatorname{supp}\underline{\nu})\leqq L_k$, an "infrared cutoff").

Derivatives of $\Phi^k(\underline{I}',\underline{z})$ are estimated on $W(\tilde{\rho}_k,\xi_k-\delta;V_k)$,

$$\left|\frac{\partial\Phi^k}{\partial\underline{\phi}}\right|=\left|\sum_{\substack{\underline{\nu}\in\mathbb{X}_k\\ \underline{\nu}\neq\underline{0}}}f_{\underline{\nu}}^k(\underline{I}')(\underline{\nu})\underline{z}^{\underline{\nu}}/(i\langle\underline{\omega}^k(\underline{I}'),\underline{\nu}\rangle)\right|. \tag{3.5}$$

We bound the denominator by (3.2), and $f_{\underline{\nu}}^k(\underline{I}')$ by $\varepsilon_k\rho_k e^{-\xi_k|\underline{\nu}|}$, combining (2.4) with Cauchy's theorem. Also, $|\underline{z}^\nu|\leqq e^{-(\xi_k-\delta)|\nu|}$ on $W(\tilde{\rho}_k,\xi_k-\delta;V_k)$. The number of terms in the sum is bounded by noting that the number of vectors $\underline{\nu}\in\mathbb{Z}^N$ with $|\underline{\nu}|=M$ and $d(\operatorname{supp}\underline{\nu})=L$ is bounded by $N\cdot 2^{2M}\cdot 2^L$. Fix the leftmost point in $\operatorname{supp}\underline{\nu}$ to be i. If there are j sites in $\operatorname{supp}\underline{\nu}$, there are $\binom{L}{j-1}$ ways of choosing them. Furthermore, if $\operatorname{supp}\underline{\nu}$ is fixed there are at most 2^{2M} vectors $\underline{\nu}$, with $|\underline{\nu}|=M$, and specified support. Summing from $j=0$ to L, and estimating the number of choices of i by N yields the stated result. Thus, (3.5) is bounded by

$$\sum_{L=0}^{L_k}\sum_{M=1}^{M_k}2C\varepsilon_k\rho_k e^{L_k}\cdot N2^L\cdot M2^{2M}e^{-(\delta-3/2)M}\leqq 2^2C\varepsilon_k\rho_k e^{(1+\ln 2)L_k}N. \tag{3.6}$$

Next note that

$$\left|\frac{\partial\Phi^k}{\partial\underline{I}'}(\underline{I}',\underline{z})\right|=\left|\sum_{\substack{\underline{\nu}\in\mathbb{X}_k\\ \underline{\nu}\neq\underline{0}}}\frac{\partial f_{\underline{\nu}}^k}{\partial\underline{I}'}(\underline{I}')\underline{z}^\nu/i\langle\underline{\omega}^k(\underline{I}'),\underline{\nu}\rangle\right.$$

$$\left.-if_{\underline{\nu}}^k(\underline{I}')<\frac{\partial\underline{\omega}^k}{\partial\underline{I}'}(\underline{I}'),\underline{\nu}>\underline{z}^{\underline{\nu}}/(i\langle\underline{\omega}^k(\underline{I}'),\underline{\nu}\rangle)^2\right|.$$

Bound the denominators of these terms by (3.2), bound $f_{\underline{\nu}}^k(\underline{I})$ by $\varepsilon_k \rho_k e^{-\xi_k|\underline{\nu}|}$, and $|\partial f_{\underline{\nu}}^k/\partial \underline{I}'|$ by $N\varepsilon_k e^{-\xi_k|\underline{\nu}|}$ by combining (2.4) with Cauchy's theorem. Note that $|\underline{z}^{\nu}| \leqq e^{(\xi_k - \delta/2)|\underline{\nu}|}$ on $W(\tilde{\rho}_k, \xi_k - \delta/2; V_k)$, and finally note that (2.5) and (2.6) imply $|\langle \partial \underline{\omega}^k/\partial \underline{I}'(\underline{I}'), \underline{\nu}\rangle| \leqq 2N|\underline{\nu}|$. Bounding the number of terms in the sum just as we did in (3.6) we find (on $W(\tilde{\rho}_k, \xi_k - \delta/2; V_k)$),

$$\sup \left| \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z}) \right| \leqq 2^6 \varepsilon_k E_k C^2 N^2 \exp[(2 + \ln 2)L_k]. \tag{3.7}$$

The implicit function theorem allows us to invert (3.4) as

$$\underline{z} = \underline{z}' \exp(i\underline{\Delta}(\underline{I}', \underline{z}')), \quad \underline{I}' = \underline{I} + \underline{\Xi}'(\underline{I}, \underline{z}), \tag{3.8}$$

with $\underline{\Xi}'(\underline{I}, \underline{z})$ analytic in $W(\tilde{\rho}_k/2, \xi_k - \delta; V_k)$ and $\underline{\Delta}(\underline{I}', \underline{z}')$ analytic in $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$ provided

$$\sup_{j,k} \left| \frac{\partial^2 \Phi^k}{\partial I_j \partial \phi_k}(\underline{I}', \underline{z}) \right| \leqq (8N)^{-1}, \quad \sup \left| \frac{\partial \Phi^k}{\partial \underline{\phi}}(\underline{I}', \underline{z}) \right| \leqq \tilde{\rho}_k/2^3 N. \tag{3.9}$$

In both cases the supremum is over $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$. Inequalities (3.9) follow from the implicit function theorem of Appendix 3 of [11] as modified in Appendix 1 of the present work. Combining (3.6) and (3.7) with a dimensional estimate, and the definitions of (2.8) we see that (3.9) are implied by (2.1).

From the definitions of $\underline{\Delta}(\underline{I}', \underline{z}')$ and $\underline{\Xi}'(\underline{I}, \underline{z})$ we see that

$$\underline{\Delta}(\underline{I}', \underline{z}') = -\frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z}),$$

and

$$\underline{\Xi}'(\underline{I}', \underline{z}') = -\frac{\partial \Phi^k}{\partial \underline{\phi}}(\underline{I}', \underline{z}), \tag{3.10}$$

on $W(\tilde{\rho}_k/2, \xi_k - 2\delta; V_k)$. We also define

$$\underline{\Delta}'(\underline{I}, \underline{z}) = \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I} + \underline{\Xi}'(\underline{I}, \underline{z}), \underline{z}),$$

and

$$\underline{\Xi}(\underline{I}', \underline{z}') = \frac{\partial \Phi^k}{\partial \underline{\phi}}(\underline{I}', \underline{z}' \exp(i\underline{\Delta}(\underline{I}', \underline{z}')), \tag{3.11}$$

the first being defined on $W(\tilde{\rho}_k/2, \xi_k - \delta; V_k)$ and the second on $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$. We obtain transformations

$$C^k : (\underline{I}', \underline{z}') \to (\underline{I}, \underline{z}) \text{ with } \begin{cases} \underline{I} = \underline{I}' + \underline{\Xi}(\underline{I}', \underline{z}') \\ \underline{z} = z' \exp(i\underline{\Delta}(\underline{I}', \underline{z}')) \end{cases},$$

$$\tilde{C}^k : (\underline{I}, \underline{z}) \to (\underline{I}', \underline{z}') \text{ with } \begin{cases} \underline{I}' = \underline{I} + \underline{\Xi}'(\underline{I}, \underline{z}) \\ \underline{z}' = \underline{z} \exp(i\underline{\Delta}'(\underline{I}, \underline{z})) \end{cases}, \tag{3.12}$$

mapping $W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$ into $W(\tilde{\rho}_k/2, \xi_k - 2\delta; V_k)$ with C^k and $\tilde{C}^k$ real and canonical on $V_k \times T^N$ and satisfying $C^k \circ \tilde{C}^k = \tilde{C}^k \circ C^k =$ identity on their common domain.

Define

$$\begin{aligned} H^{k+1}(\underline{I}', \underline{z}') &\equiv H^k \circ C^k(\underline{I}', \underline{z}') \\ &= h^k(\underline{I}' + \underline{\Xi}(\underline{I}', \underline{z}')) + f^k(\underline{I}' + \underline{\Xi}(\underline{I}', \underline{z}'), \underline{z}' \exp(i\underline{\Delta}(\underline{I}', \underline{z}'))) \\ &\equiv h^{k+1}(\underline{I}') + f^{k+1}(\underline{I}', \underline{z}'), \end{aligned} \tag{3.13}$$

with

$$h^{k+1}(\underline{I}') = h^k(\underline{I}') + f_{\underline{0}}^k(\underline{I}'), \tag{3.14}$$

and

$$f^{k+1}(\underline{I}', \underline{z}') = H^{k+1}(\underline{I}', \underline{z}') - h^{k+1}(\underline{I}'). \tag{3.15}$$

Let

$$f^{k[\leqq]}(\underline{I}, \underline{z}) = \sum_{\substack{\underline{v} \in \mathbb{X}_k \\ \underline{v} \neq \underline{0}}} f_{\underline{v}}^k(\underline{I}) \underline{z}^{\underline{v}},$$

and

$$f^{k[\geqq]}(\underline{I}, \underline{z}) = \sum_{\underline{v} \notin \mathbb{X}_k} f_{\underline{v}}^k(\underline{I}) \underline{z}^{\underline{v}}. \tag{3.16}$$

Since (3.3) insures that

$$\underline{\omega}^k(\underline{I}') \cdot \frac{\partial \Phi^k}{\partial \underline{\phi}}(\underline{I}', \underline{z}) + f^{k[\leqq]}(\underline{I}', \underline{z}) - f_0^k(\underline{I}') = 0, \tag{3.17}$$

we may rewrite (3.15) using the fundamental theorem of calculus, as

$$f^{k+1}(\underline{I}', \underline{z}') = f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}'), \tag{3.18}$$

where

$$f^I(\underline{I}', \underline{z}') = \int_0^1 dt \int_0^t ds \sum_{i,j=1}^N \frac{\partial^2 h^k}{\partial I_i \partial I_j}(\underline{I}' + s\underline{\Xi}) \Xi_i \Xi_j, \tag{3.19}$$

$$f^{II}(\underline{I}', \underline{z}') = \int_0^1 dt \sum_{j=1}^N \frac{\partial f^{k[\leqq]}}{\partial I_j}(\underline{I}' + t\underline{\Xi}, \underline{z}' \exp(i\underline{\Delta})) \Xi_j, \tag{3.20}$$

and

$$f^{III}(\underline{I}', \underline{z}') = f^{k[\geqq]}(\underline{I}' + \underline{\Xi}, \underline{z}' \exp(i\underline{\Delta})). \tag{3.21}$$

We have omitted the arguments on the functions $\underline{\Xi}(\underline{I}', \underline{z}')$ and $\underline{\Delta}(\underline{I}', \underline{z}')$ to save space. Using (2.4) we see that

$$\sup \left| \frac{\partial h^{k+1}}{\partial \underline{I}}(\underline{I}') \right| \leqq E_k + \varepsilon_k = E_{k+1} \tag{3.22}$$

on $W(\tilde{\rho}_k/2, \xi_k - 3\delta; V_k)$.

Bound $\underline{\Xi}(\underline{I}', \underline{z}')$ on $W(\tilde{\rho}_k/2, \xi_k - 3\delta; V_k)$ by (3.6), bound $\partial^2 h^k / \partial I_i \partial I_j$ by (2.5) and (2.6) on $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$, and bound the number of terms in the sum in (3.19) by N^2

to obtain (on $W(\tilde{\rho}_k/2, \xi_k - 3\delta; V_k)$),

$$\sup|f^I(\underline{I}', \underline{z}')| \leqq 2^3 \varepsilon_k^2 \rho_k^2 C^2 e^{4L_k} N^4. \tag{3.23}$$

To bound (3.20) note that $|\partial f_{\underline{v}}^k/\partial I_j(\underline{I}' + t\underline{\Xi})| < \varepsilon_k e^{-\xi_k|\underline{v}|}$ on $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$. Bound $\underline{\Xi}(\underline{I}', \underline{z}')$ by (3.6) and control the sum over $\underline{v}$ just as we did in (3.6) obtaining (on $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$)

$$\sup|f^{II}(\underline{I}', \underline{z}')| \leqq 2^3 \varepsilon_k^2 \rho_k C e^{3L_k} N^2. \tag{3.24}$$

To bound (3.21) note that if $\underline{v} \notin \mathbb{X}_k$, then either $d(\operatorname{supp} \underline{v}) > L_k$ or $d(\operatorname{supp} \underline{v}) \leqq L_k$ and $|\underline{v}| > M_k$. Since (2.4) and (2.7) imply that on $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$ $|f_{\underline{v}}^k(\underline{I}' + \underline{\Xi})| \leqq \min(\varepsilon_0\rho_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)d(\operatorname{supp} v)} e^{-\xi_k|\underline{v}|}, \varepsilon_k\rho_k e^{-\xi_k|\underline{v}|})$, we sum over all $\underline{v} \notin \mathbb{X}_k$, using the first of these estimates to control those terms with $d(\operatorname{supp} \underline{v}) > L_k$, and the second estimate on the remainder. (Once again bound the number of terms with $|\underline{v}| = M$ and $d(\operatorname{supp} \underline{v}) = L$ by $N 2^L 2^{2M}$.) This gives

$$\begin{aligned}\sup|f^{III}(\underline{I}', \underline{z}')| \leqq \{&(\varepsilon_0\rho_0 N)(2\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(L_k+1)} \\ &+ 2^{L_k+2} N \varepsilon_k \rho_k e^{(2\ln 2 - 2\delta)(M_k+1)}\},\end{aligned} \tag{3.25}$$

on $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$.

Combining (3.23)–(3.25) with a pair of dimensional estimates on $W(\tilde{\rho}_k/4, \xi_k - 4\delta; V_k) \supset W(\rho_{k+1}, \xi_{k+1}; V_k)$ and some algebra involving the definitions (2.8), we see

$$\sup\left\{\left|\frac{\partial f^{k+1}}{\partial \underline{I}'}(\underline{I}', \underline{z}')\right| + \rho_{k+1}^{-1}\left|\frac{\partial f^{k+1}}{\partial \underline{\phi}}(\underline{I}', \underline{z}')\right|\right\} \leqq \varepsilon_{k+1}. \tag{3.26}$$

This section closes by iterating (2.5) and (2.6). Note that

$$\frac{\partial \omega_i^{k+1}}{\partial \underline{I}_j'}(\underline{I}') = \frac{\partial \omega_i^{k+1}}{\partial I_j'} + \frac{\partial^2 f_0^k(I')}{\partial I_i' \partial I_j'}. \tag{3.27}$$

If $i = j$, bound the second derivative of f_0^k on $W(\rho_{k+1}, \xi_{k+1}; V_k)$ by $2N\varepsilon_k\rho_k^{-1}$, using (2.4) and a dimensional estimate, to obtain the $k+1^{st}$ case of (2.5). Similarly if $i \neq j$, bound the derivative of f_0^k using (2.7) to obtain (2.6).

4. Decay Estimates

In this section we prove the decay estimates (2.7). The proof is based on the following proposition that is proved in [19].

Proposition 4.1. *On* $W(\rho_{k+1}, \xi_{k+1}; V_k)$,

$$\sup\left|\frac{\partial^2 f^{k+1}}{\partial \phi_i \partial \phi_j}(\underline{I}, \underline{z})\right| \leqq \varepsilon_0\rho_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|},$$

$$\sup\left|\frac{\partial^2 f^{k+1}}{\partial I_i \partial \phi_j}(\underline{I}, \underline{z})\right| \leqq \varepsilon_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|},$$

$$\sup\left|\frac{\partial^2 f^{k+1}}{\partial I_i \partial I_j}(\underline{I}, \underline{z})\right| \leqq \varepsilon_0\rho_0^{-1}(\varepsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|}. \tag{4.1}$$

By definition, one has

$$f_{\underline{\nu}}^{k+1}(\underline{I})=\left(\frac{1}{2\pi i}\right)^N \oint \frac{d\underline{z}}{\left(\prod_k z_k\right)} f^{k+1}(\underline{I},\underline{z})\underline{z}^{\underline{\nu}}, \tag{4.2}$$

where each integral on the right-hand side of (4.2) runs over the contour $|z_i|=1$. Take i and j respectively the leftmost and rightmost points in supp $\underline{\nu}$. Rewrite (4.2) as

$$f_{\underline{\nu}}^{k+1}(\underline{I})=\frac{(-1)}{\nu_i \nu_j}\left(\frac{1}{2\pi i}\right)^N \oint \frac{dz}{\left(\prod_k z_k\right)} \frac{\partial^2 f^{k+1}}{\partial\phi_i \partial\phi_j}(\underline{I},\underline{z})\underline{z}^{\underline{\nu}}. \tag{4.3}$$

(This is just the multivariable version of the observation that if $\tilde{g}_n$ are the coefficients in the Laurent expansion of the function $g(z)$, then $\widetilde{(zg'(z))}_n=(1/n)\tilde{g}_n$.) Combining the first estimate in (4.1) with (4.3) we see that we have written $f_{\underline{\nu}}^{k+1}(\underline{I})$ as the $\underline{\nu}^{\text{th}}$ Laurent coefficient of a function analytic on $W(\rho_{k+1},\xi_{k+1};V_k)$ and bounded by $\varepsilon_0\rho_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|}$. Cauchy's theorem then guarantees that

$$\sup|f_{\underline{\nu}}^{k+1}(\underline{I})| \leqq \varepsilon_0\rho_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|}e^{-\xi_{k+1}|\underline{\nu}|}, \tag{4.4}$$

verifying the first inequality in (2.7).

Suppose l is the point in supp $\underline{\nu}$ farthest from j. Then just as above we find

$$\frac{\partial f_{\underline{\nu}}^{k+1}}{\partial I_j}(\underline{I})=\left(\frac{1}{2\pi i}\right)^N \oint \frac{d\underline{z}}{\left(\prod_k z_k\right)}\left(\frac{1}{i\nu_l}\right)\frac{\partial^2 f^{k+1}}{\partial\phi_l \partial I_j}(\underline{I},\underline{z})\underline{z}^{\underline{\nu}}. \tag{4.5}$$

Proposition 4.1 guarantees that $|(1/\nu_l)\cdot(\partial^2 f^{k+1}/\partial\phi_l\partial I_j)|$ is bounded by $\varepsilon_0(\varepsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|l-j|}$, and then Cauchy's theorem implies the second estimate of (2.7) holds. Finally,

$$\sup\left|\frac{\partial^2 f_{\underline{\nu}}^{k+1}}{\partial I_i \partial I_j}(\underline{I})\right| = \sup\left|\left(\frac{1}{2\pi i}\right)^N \oint \frac{d\underline{z}}{(\prod z_k)}\frac{\partial^2 f^{k+1}}{\partial I_i \partial I_j}(\underline{I},\underline{z})\underline{z}^{\underline{\nu}}\right|$$
$$=\varepsilon_0\rho_0^{-1}(\varepsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|}e^{-\xi_{k+1}|\nu|}, \tag{4.6}$$

which completes the proof of (2.7).

5. An Estimate on the Volume of Phase Space Lost at the k[th] Iterative Step

We complete the induction argument of Sect. 2 by estimating $\mathrm{vol}(V_{k-1}\backslash V_k)$. (Set $V_{-1}=V$.) Note that $\bar{V}_k \subset V_k$ and $\bar{V}_k$ is obtained from V_{k-1} by first omitting the set of points, B_{k-1}, whose distance from the boundary of V_{k-1} is less than $2\tilde{\rho}_k$, and then omitting all resonant points, $R(k,h^k;V_{k-1})$. Thus

$$\mathrm{vol}(V_{k-1}\backslash V_k) \leqq \mathrm{vol}\,B_{k-1} + \mathrm{vol}\,R(k,h^k;V_{k-1}). \tag{5.1}$$

It was shown in [5, 11] that

$$\mathrm{vol}\,B_{k-1} \leqq [1-(1-2\tilde{\rho}_k/\rho_k)^N]\,\mathrm{vol}\,V. \tag{5.2}$$

Remark. The argument of [5, 11] does not apply directly to the case $k=0$, directly to the case $k=0$, since $V_{-1}=V$ is not constructed by the same procedure as subsequent sets V_k. Since V is assumed to be a sphere, though, it is easy to estimate the volume of the set of points within a distance $\tilde{\rho}_0$ of its boundary directly, and one obtains

$$\operatorname{vol} B_{-1} \leqq (1-(1-2\tilde{\rho}_0/r)^N)\operatorname{vol} V. \tag{5.3}$$

If $\underline{\omega}^k$ is single-valued,

$$\operatorname{vol}(R(k,h^k,V_{k-1})) = \int_{\omega^k(R(k,h^k,V_{k-1}))} \left|\det\left[\frac{\partial \underline{\omega}^k}{\partial \underline{I}}\right]^{-1}\right| d\underline{\omega}. \tag{5.4}$$

The single valuedness of $\underline{\omega}^k$ follows from (2.5), (2.6) and the fundamental theorem of calculus. Pick a path, γ, contained in $W(\rho_0,\xi_0;V)$ from $\underline{I}$ to $\underline{I}'$, of length at most $3|\underline{I}'-\underline{I}|$, and made up of line segments along which only one coordinate of $\underline{I}$ is allowed to vary. Then

$$\underline{\omega}^0(\underline{I}) - \underline{\omega}^0(\underline{I}') = \int_\gamma d\underline{I}''\cdot\frac{\partial \underline{\omega}^0}{\partial \underline{I}}(\underline{I}''). \tag{5.5}$$

By (2.5) and (2.6) the magnitude of the integral on the right-hand side of (5.5) may be bounded from below by $3/4\;|\underline{I}-\underline{I}'|$.

The argument of Appendix H of [11] shows that (on $W(\rho_k,\xi_k;V_{k-1})$)

$$\left|\sum_{j=0}^{k}\left(\frac{\partial f_0^j}{\partial \underline{I}}(\underline{I}') - \frac{\partial f_0^j}{\partial \underline{I}}(\underline{I})\right)\right| \leqq 3|\underline{I}-\underline{I}'|\left(\sum_{j=0}^{k}(\varepsilon_j\rho_j^{-1})\right). \tag{5.6}$$

Thus

$$\begin{aligned}|\underline{\omega}^k(\underline{I}') - \underline{\omega}^k(\underline{I})| &= \left|\underline{\omega}^0(\underline{I}) - \underline{\omega}^0(\underline{I}') + \sum_{j=0}^{k-1}\left(\frac{\partial f_0^j}{\partial \underline{I}}(\underline{I}) - \frac{\partial f_0^j}{\partial \underline{I}}(\underline{I}')\right)\right| \\ &\leqq 1/2|\underline{I}-\underline{I}'|,\end{aligned} \tag{5.7}$$

so ω^k is single-valued.

We now estimate $\sup|\det(\partial\underline{\omega}^k/\partial\underline{I})^{-1}|$ on $W(\rho_k,\xi_k;V_{k-1})$,

$$\det\left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}}\right)^{-1} = \exp\left\{-\operatorname{tr}\ln\left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}}\right)\right\} \equiv \exp\{-\operatorname{tr}\ln(\mathbb{I}-\mathbb{D})\}, \tag{5.8}$$

where by (2.5) and (2.6) we see that $\mathbb{D}$ is a matrix whose diagonal entries are bounded in magnitude by c_1N^{-1} (with $c_1<1/2$), and whose off diagonal entries are bounded (in magnitude) by $\theta(k;i,j)\leqq \varepsilon_0\rho_0^{-1}$. An easy induction argument shows

$$|[\mathbb{D}^k]_{ij}| \leqq \begin{cases} c_1N^{-k} & \text{if } i=j \\ (\varepsilon_0\rho_0^{-1})N^{-(k-1)} & \text{if } i\neq j.\end{cases} \tag{5.9}$$

Thus, $|\operatorname{tr}\mathbb{D}^k| \leqq c_1N^{-(k-1)}$, and

$$\sup\left|\det\left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}}\right)^{-1}\right| \leqq \exp\sum_{j=1}^{\infty}\left(\frac{1}{j}\right)\sup|\operatorname{tr}\mathbb{D}^j| \leqq \exp 2c_1. \tag{5.10}$$

Inserting this estimate in (6.4) and using the definition of $R(k, h^k; V_{k-1})$ we find

$$\mathrm{vol}(R(k, h^k; V_{k-1}) \leqq 2^2 \sum_{\substack{\underline{v}\in\mathbb{X}_k \\ \underline{v}\neq \underline{0}}} \int_{\substack{\omega: \\ |\langle \underline{\omega}, \underline{v}\rangle| \leqq (2C)^{-1} e^{-(3/2)|\underline{v}| - L_k} \\ \underline{\omega}\in\underline{\omega}^k(R(k,h^k, V_{k-1}))}} \tag{5.11}$$

Note that (5.5) and (5.6) when combined with (2.5), (2.6) and the fact that $V_{k-1} \subset V =$ sphere of radius r imply that

$$\|\underline{\omega}^k(\underline{I}) - \underline{\omega}^k(\underline{0})\| \leqq (1 + 2^2/N)r, \tag{5.12}$$

where $\|\cdot\|$ is the usual Euclidean norm, and $\underline{I} \in V_{k-1}$. (We have used the fact that $|\underline{x}| \leqq N\|x\|$.)

This implies that any term with fixed $\underline{v}$ on the right-hand side of (5.11) may be bounded by the volume of a slice of thickness $(2C)^{-1}e^{-(3/2)|\underline{v}| - L_k}$ out of an N-dimensional sphere of radius $(1 + 2^2/N)r$. Hence

$$\begin{aligned}\mathrm{vol}(R(k, h^k; V_{k-1})) \leqq 2^2 \sum_{\substack{\underline{v}\in\mathbb{X}_k \\ \underline{v}\neq \underline{0}}} (2C)^{-1} e^{-(3/2)|\underline{v}|} e^{-L_k} \pi^{(N-1)/2} \\ \times [r(1 + 2^2/N)]^{N-1} / \Gamma[1 + (N-1)/2].\end{aligned} \tag{5.13}$$

Since $\mathrm{vol}\, V = (\pi^{N/2} r^N)/\Gamma(1 + N/2)$, we obtain

$$\begin{aligned}\mathrm{vol}(R(k, h^k; V_{k-1})) &\leqq 2\pi^{-(1/2)}(2Cr)^{-1}\{\Gamma(1 + N/2)/\Gamma[1 + (N-1)/2] \\ &\times (1 + 2^2/N)^{N-1} \Big\{ \sum_{\substack{\underline{v}\in\mathbb{X}_k \\ \underline{v}\neq \underline{0}}} e^{-(3/2)|\underline{v}|} e^{-L_k} \Big\} \mathrm{vol}\, V.\end{aligned} \tag{5.14}$$

The sum over $\underline{v}$ is bounded in the now standard fashion by $2(2/e)^{L_k}$. A little algebra, some standard recursion relations for the Γ function (see, e.g. [12]) and the fact that $\rho_0 < r$ by assumption give

$$\mathrm{vol}(R(k, h^k; V_{k-1})) \leqq 2^9 (C\rho_0)^{-1} N^{3/2} (2/e)^{L_k}. \tag{5.15}$$

If we combine this result with (5.2) and (5.3), a little algebra shows

$$\mathrm{vol}(V_{k-1} \setminus V_k) \leqq \mathrm{vol}\, B_{k-1} + \mathrm{vol}\, R(k, h^k; V_{k-1}) \leqq (\tfrac{1}{4})\lambda \exp[-(\tfrac{1}{2})(\tfrac{3}{2})^k]\{\mathrm{vol}\, V\},$$

as claimed in Sect. 2.

Appendix A. The Implicit Function Theorem

We wish to show that the equation

$$\underline{z}' = \underline{z} \exp\left(i \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z}) \right) \tag{A.1}$$

is 1–1 for $(\underline{I}', \underline{z})$ in $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$. Assume not. Then there is some $\underline{\sigma}(\underline{z}) = (\sigma_1 z_1, \sigma_2 z_2, \ldots, \sigma_N z_N)$ such that

$$\underline{\sigma}(\underline{z}) \exp\left(i \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{\sigma}(\underline{z})) \right) = z \exp\left(i \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z}) \right). \tag{A.2}$$

Then

$$|\sigma_j - 1| \leqq \sup\left|\exp\left[i\frac{\partial \Phi^k}{\partial I'_j}(\underline{I}', \underline{\sigma}(\underline{z})) - i\frac{\partial \Phi^k}{\partial I'_j}(\underline{I}', \underline{z})\right] - 1\right|$$
$$\leqq 2^7 \varepsilon_k E_k C^2 N e^{(2+\ln 2)L_k} < (\tfrac{1}{2}), \tag{A.3}$$

by (3.7). Thus we can consider

$$\sum_{j=1}^{N} |\ln \sigma_j| \leqq \sup \sum_{j=1}^{N} \left|\frac{\partial \Phi^k}{\partial I'_j}(\underline{I}', \underline{\sigma}(\underline{z})) - \frac{\partial \Phi^k}{\partial I'_j}(\underline{I}', \underline{z})\right|$$
$$= \sup \sum_{j=1}^{N} \left|\sum_{k=1}^{N} \int_0^1 d\alpha \frac{\partial^2 \Phi^k}{\partial I'_j \partial \phi_k}(\underline{I}', \underline{\sigma}^\alpha(\underline{z}))\cdot \sigma_k^\alpha(\ln \sigma_k)\right|, \tag{A.4}$$

where we applied the fundamental theorem of calculus and $\underline{\sigma}^\alpha(\underline{z}) = (\sigma_1^\alpha z_1, \sigma_2^\alpha z_2, \ldots, \sigma_N^\alpha z_N)$. But (3.9) implies the right-hand side of (A.4) is bounded by $(\frac{1}{8}) \sum_{k=1}^{N} |\ln \sigma_k|$, so we have $\sigma_k \equiv 1$ for $k = 1, \ldots, N$, and (A.1) is 1–1. Standard inverse function theorems (e.g., [13, Theorem 1.7.6.]) guarantee that (A.1) has an analytic inverse on the image B, of $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$. Denote the inverse map by

$$\underline{z}(\underline{I}', \underline{z}') = \underline{z}' e^{i\Delta(I', z')}. \tag{A.5}$$

From (A.1) we see that $\underline{\Delta}(\underline{I}', \underline{z}') = -(\partial \Phi^k / \partial \underline{I}')(\underline{I}', \underline{z})$, so

$$\sup_{\mathrm{B}} |\underline{\Delta}(\underline{I}', \underline{z}')| \leqq \sup \left|\frac{\partial \Phi^k}{\partial \underline{I}'}\right| \leqq 2^6 \varepsilon_k E_k C^2 N e^{(2+\ln 2)L_k} \leqq \delta/2.$$

Hence as $(\underline{I}', \underline{z})$ varies over $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$, $(\underline{I}', \underline{z}')$ must cover at least $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$, so $\underline{\Delta}(\underline{I}', \underline{z}')$ is analytic on that domain.

The argument for inverting $\underline{I} = \underline{I}' + (\partial \Phi^k / \partial \underline{\phi})(\underline{I}', \underline{z})$ is identical to that in [11] so I do not reproduce it.

Appendix B. Dimensional Estimates

Lemma B.1. *If* $e^{-(\xi_0 - \delta)} < |z_0| < e^{(\xi_0 - \delta)}$, *and* $f(z)$ *is analytic for* $e^{-\xi} < |z_0| < e^{\xi}$, *then*

$$\left|z_0^n \frac{d^n f}{dz^n}(z_0)\right| \leqq \frac{n!}{(1 - e^{-\delta})^n} \sup |f(z)|, \tag{B.1}$$

where the supremum runs over all z *such that* $e^{-\xi} < |z| < e^{\xi}$.

Proof.

$$\left|z_0^n \frac{d^n f}{dz^n}(z_0)\right| = \left|\frac{d^n}{d\alpha^n} f((1+\alpha)z_0\right|_{\alpha=0}\Bigg|$$
$$= \left|\frac{n!}{(2\pi i)} \oint_{|\alpha|=r} \frac{f((1+\alpha)z_0)}{\alpha^{n+1}} d\alpha\right|. \tag{B.2}$$

By the analyticity of f, we may choose the circle over which we integrate to have radius r as close to $(1-e^{-\delta})$ as we wish, so the right-hand side of B.2 is bounded by $n!(1-e^{-\delta})^{-n}\sup|f(z)|$, which proves (B.1).

Corollary B.2. *If f is analytic on $e^{-\xi}<|z|<e^{\xi}$, then*

$$\sup\left|\frac{\partial f}{\partial \phi}\right| \leqq 2\sup|f| \quad \textit{and} \quad \sup\left|\frac{\partial^2 f}{\partial \phi^2}\right| \leqq 2^2\sup|f|, \tag{B.3}$$

where the suprema on the left-hand side of (B.3) run over $e^{-(\xi-\delta)}<|z|<e^{(\xi-\delta)}$, while those on the right run over $e^{-\xi}<|z|<e^{\xi}$.

This follows immediately from (B.1) and the choice of δ in (2.8). The extension to $z\in\mathbb{C}^N$ is immediate and is left to the reader.

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