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Nahm's Equations and the Classification of Monopoles

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Abstract. Solutions of Nahm's system of ordinary differential equations are produced by variational methods. This leads to an explicit parametrisation of the solutions to the Bogomolny equation over $\mathbb{R}^3$.

Introduction

The Bogomolny equations are a non-linear system of partial differential equations for a connection and Higgs field defined on a bundle over $\mathbb{R}^3$. With suitable boundary conditions at infinity the solutions, mathematical *monopoles*, fall into discrete classes indexed by a "topological charge." If, as we shall always suppose, the group of the bundle is SU(2) this charge is an integer $k \geqq 0$. Within each topological type the solutions are parametrised by continuous variables or moduli spaces M_k.

These monopoles have been studied from a number of different points of view. In the simplest case, taking $k=1$, there is an explicit "fundamental solution" due to Prasad and Sommerfield which exhibits a qualitative soliton or particle-like structure. More generally Taubes has shown ([8] Chap. 4) that, for each value of the charge k, solutions to the equations exist which are approximate superpositions of k copies of the Prasad–Sommerfield monopole centred on widely separated points in $\mathbb{R}^3$.

On the other hand Hitchin and Nahm have developed independent but closely related methods of constructing monopoles stemming from twistor geometry [5]. In the final formulation of Hitchin the construction of a monopole becomes equivalent to finding an algebraic curve with certain properties, and this method was used by Hurtubise [7] to describe all monopoles of charge 2.

However neither from the analytic nor the geometric points of view was it possible to immediately identify the full moduli space M_k for larger values of k. This is the problem that we take up here, and we shall verify a conjecture of Atiyah and Murray [11] relating these moduli spaces to the rational functions on the Riemann sphere, $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.

The manifolds M_k, for $k \geqq 1$, have dimension $4k-1$. It turns out to be more convenient to describe a slightly larger space $\tilde{M}_k$, of dimension $4k$, which is a circle bundle over M_k. The description in terms of rational functions is then:

Theorem. *Given an isomorphism $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$, compatible with the usual metrics, there is a natural one-to-one correspondence between the extended moduli space $\tilde{M}_k$ and the complex manifold R_k of rational maps $f: \mathbb{CP}^1 \to \mathbb{CP}^1$, of degree k and such that $f(\infty) = 0$.*

Of course any such map f has a unique representation $f(z) = p(z)/q(z)$ with p, q coprime polynomials, q monic of degree k and the degree of p at most $k-1$.

The precise definition of the circle bundle $\tilde{M}_k$ over the moduli space is given in the body of the paper, at the end of Sect. 1. Alternatively it follows from that definition and those of Sect. 3 that the moduli space M_k itself corresponds to the quotient $\tilde{M}_k/_\sim$, where we let $f \sim e^{i\theta} f$ for θ in $\mathbb{R}$.

It is easy to give an intuitive interpretation of this parametrisation of the solutions to the Bogomolny equation. The generic function f in R_k can be written as:

$$f(z) = \sum_{i=1}^{k} a_i/(z - z_i).$$

If the points z_i in $\mathbb{C}$ are very far apart it seems likely that this function corresponds to a solution of the type constructed by Taubes based on the k points:

$$(z_i, \log|a_i|) \in \mathbb{C} \times \mathbb{R}.$$

The additional parameters $\arg(a_i) \in S^1$ can be interpreted as certain "group orientations" relative to a fixed trivialisation at infinity. At the other extreme, taking $f(z) = 1/z^k$ we get the essentially unique axi-symmetric solution of Prasad and Rossi.

The method of proof that we use grew out of two previous pieces of work on the (anti) self dual Yang–Mills equations, of which the Bogomolny equations are a special case. On the one hand the moduli of solutions to these equations over compact complex surfaces (that is, four real dimensions) can, in some cases, be related to the moduli of holomorphic bundles—entirely complex analytic objects. This relation is established by the study of a second order partial differential equation for a metric on a holomorphic bundle, but as explained in [3], following the ideas of Atiyah and Bott [2], it can be viewed as an infinite dimensional example of a general principle for choosing distinguished points in orbits of complex Lie groups.

On the other hand, continuing the same line of enquiry, it was found in [4] that, once a complex structure is fixed on $\mathbb{R}^4$, the Yang–Mills instantons there may be similarly related to holomorphic bundles on the (projective) plane. This relation was established by studying the ADHM construction of instantons but again turned on the same general principle. In that case too it was easiest to describe an extended moduli space corresponding to a trivialisation at infinity.

Here we make a start on the application of these methods to monopoles, in strict analogy with the work on instantons and using Nahm's version of the ADHM construction. Of course this means that our results depend crucially on those of Hitchin and Nahm establishing the existence of what we may regard as a transform between the Bogomolny equations and a system on non-linear ordinary differential equations; *Nahm's equations*.

Our strategy of proof is then:

(i) Break the natural symmetry of the problem by dividing Nahm's equations into one "complex equation" and one "real equation," in such a way that the complex equation is invariant under a complex group.

(ii) Show, by variational arguments, that in each orbit of the complex group there is an essentially unique solution of the real equation.

(iii) Identify the equivalence classes of solutions to the complex equation using complex algebraic methods.

Once begun on this programme we find that the analogy with the monads of [4] guides us to a simple classification. However it is, at first, remarkable that we are led in this way to study again a version of the second order differential equation for a metric on a holomorphic bundle mentioned above, and some of the techniques that we need can be borrowed from that theory. This is clearly connected to the wonderful fact, explicitly pointed out by Nahm [13], that his method can be viewed as a transform from one version of the self duality equation to another.

It should not be disguised that the work here represents in many ways a first step towards a full understanding of these moduli spaces of monopoles. There are many interesting aspects that deserve further treatment. For example, one could hope to describe the rational map f in terms of the monopole directly, and one could hope to extend these results to other gauge groups (compare [10]) and to unify them with the closely related description of monopoles on hyperbolic space due to Atiyah [1]. Finally, one could hope to give a proper explanation of the dual nature of Nahm's construction and the relation of the "Nahm complexes" that we define below to the monad construction of bundles.

Here, except for one technicality, we fix our attention on Nahm's equations (which have interest in their own right) and can be thought of, roughly, as the "momentum space" description of a monopole.

Section 1. Re-writing Nahm's equations

Here we shall put Nahm's equations into the form that we need, starting from the Theorem of Hitchin [6] but using slightly different notation.

Proposition (1.1) *[6]. There is a natural equivalence between:*

(a) *Monopoles for the group* SU(2) *with charge k, up to gauge transformation.*

(b) *Conjugacy classes under $O(k,\mathbb{R})$ of matrix valued functions $T_1(s)$, $T_2(s)$, $T_3(s)$ of one real variable $s\in(0,2)$ satisfying:*

(i) $\dfrac{dT_i}{ds}+\dfrac{1}{2}\sum\varepsilon_{ijk}[T_j,T_k]=0.$

(ii) $T_i^*(s)=-T_i(s)$.

(iii) $T_i(2-s)=T_i(s)^T$.

(iv) *T_i extends to a meromorphic function on a neighborhood of $[0,2]$ with simple poles at $s=0,2$ but otherwise analytic.*

(v) *The residues of the matrices T_i at the pole $s=0$ define an irreducible representation of* SU(2).

To motivate our treatment of these equations we observe first:

Proposition (1.2) (cf. [13]). *The system of differential equations: $dT_i/ds+1/2\sum\varepsilon_{ijk}[T_j,T_k]=0$ is equivalent to the anti-self duality equation for a connection 1-form:*

$$A = \sum_{i=1}^{3} T_i \, dp_i$$

on "momentum space" $\mathbb{R}^4 = \{(s, p_1, p_2, p_3)\}$ *that is, that* $dA + A \wedge A$ *be anti-self dual.*

(This, together with other simple calculations in this section, is left to the reader to verify. Note that the space with coordinates p_i is dual to the original $\mathbb{R}^3$ since a point (x_1, x_2, x_3) in physical space goes over in Nahm's scheme to the 1-form $\sum x_i \, dp_i$.)

Because of this, it is more natural to introduce a fourth matrix $T_0(x)$, satisfying the same conditions (ii), (iii) of Proposition (1.1) (b) and extend Nahm's equations to:

$$\frac{dT_i}{ds} + [T_0, T_i] + \frac{1}{2} \sum \varepsilon_{ijk} [T_j, T_k] = 0 \quad (i = 1, 2, 3), \tag{1.3}$$

which is the anti-self duality equation for $T_0 \, ds + \sum T_i \, dp_i$.

Of course, in the usual way, this is an equation with redundancy due to a gauge group. If

$$u: (0, 2) \mapsto U(k) \tag{1.4}$$

satisfies

$$u(2 - s) = u^T(s)^{-1}, \tag{1.5}$$

then the action:

$$\begin{cases} u(T_i) = u T_i u^{-1} \\ u(T_0) = u T_0 u^{-1} - \left(\dfrac{du}{ds}\right) u^{-1} \end{cases} \tag{1.6}$$

sends one solution of (1.3) to another and preserves the conditions (ii), (iii) of (1.1)(b). In particular, we can always transform to a gauge in which $T_0 = 0$. (We postpone discussing the poles for the moment.)

Now the choice of isomorphism $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ made in the hypothesis of the theorem allows us to naturally introduce complex coordinates:

$$z = s + ip_1, \quad w = p_2 + ip_3.$$

On $\mathbb{C}^2$ it is familiar that it is best to split a connection into its $(1, 0)$ and $(0, 1)$ components, and similarly that the curvature splits into bi-type. Further, it is well known that the $(0, 2)$ *forms are self dual* so a unitary anti-self dual connection has curvature of type $(1, 1)$ and defines an integrable holomorphic structure (i.e., $\bar{\partial}$-operator). Moreover, the holomorphic structure transforms naturally under the group of *complex* automorphisms of the bundle. We next re-write Nahm's equations from precisely this point of view, in a form suited to our case when only one variable is effective.

For skew adjoint matrices $T_0(s)$, $T_i(s)$ put:

$$\alpha = \tfrac{1}{2}(T_0 + iT_1), \quad \beta = \tfrac{1}{2}(T_2 + iT_3), \tag{1.7}$$

so α and β are complex $k \times k$ matrix valued functions. Conversely, from α, β we can recover the T_i and we shall usually suppose that the "symmetry condition":

$$\alpha(2-s)=\alpha^T(s), \quad \beta(2-s)=\beta^T(s), \tag{1.8}$$

corresponding to Proposition (1.1)(b)(iii) holds. Then the set of Eqs. (1.3) is equivalent to the pair:

$$\frac{d\beta}{ds}+2[\alpha,\beta]=0 \quad \text{(The "complex equation")}, \tag{1.9}$$

$$\hat{F}(\alpha,\beta)\equiv\frac{d}{ds}(\alpha,\alpha^*)+2([\alpha,\alpha^*]+[\beta,\beta^*])=0 \quad \text{(The "real equation")}, \tag{1.10}$$

and the complex equation is invariant under the "complex gauge group." That is, if

$$g:(0,2)\mapsto \mathrm{GL}(k,\mathbb{C}) \tag{1.11}$$

satisfies

$$g(2-s)=g^T(s)^{-1}, \tag{1.12}$$

and if we set:

$$\begin{cases} \alpha' = g(\alpha) = g\alpha g^{-1} - \dfrac{1}{2}\dfrac{dg}{ds}g^{-1} \\ \beta' = g(\beta) = g\beta g^{-1} \end{cases} \tag{1.13}$$

(which we often write $(\alpha',\beta')=g(\alpha,\beta)$), then

$$\frac{d\beta'}{ds}+2[\alpha',\beta']=g\left(\frac{d\beta}{ds}+2[\alpha,\beta]\right)g^{-1}.$$

This becomes clearer if we define operators (in effect the $\bar{\partial}$, ∂ operators associated to a connection) on $\mathbb{C}^k$-valued functions f by:

$$\begin{cases} \bar{d}_\alpha f = \dfrac{1}{2}\dfrac{df}{ds}+\alpha f; & d_\alpha f = \dfrac{1}{2}\dfrac{df}{ds}-\alpha^* f \\ \bar{d}_\beta f = \beta f & d_\beta f = -\beta^* f \end{cases} \tag{1.14}$$

with similar operators, having the same names, on matrix valued functions γ:

$$\bar{d}_\alpha\gamma=\frac{1}{2}\frac{d\gamma}{ds}+[\alpha,\gamma] \quad \text{etc.}$$

Then the complex equation is equivalent to the operator equation:

$$[\bar{d}_\alpha,\bar{d}_\beta]=0, \tag{1.15}$$

which is plainly preserved under the action of the complex automorphisms:

$$\begin{cases} \bar{d}_{g(\alpha)} = g\circ\bar{d}_\alpha\circ g^{-1} \\ \bar{d}_{g(\beta)} = g\circ\bar{d}_\beta\circ g^{-1}. \end{cases} \tag{1.16}$$

Next examine the poles at $s=0,2$. If we suppose that α,β satisfy the complex equation

$$\frac{d\beta}{ds}+2[\alpha,\beta]=0$$

and have simple poles with residues a,b at $s=0$, then these residues satisfy:

$$b=2[a,b]. \tag{1.17}$$

So the action of b shifts one eigenvector of a to another in the usual way:

$$av=\lambda v \Rightarrow a(bv)=(\lambda+\tfrac{1}{2})bv. \tag{1.18}$$

In the formal definition that we give below, we shall suppose that $\mathrm{Tr}(a)=0$ and that the action of b on an eigenvector v with $av=(k-1/4)v$ generates $\mathbb{C}^k$, so in some base:

$$a=\mathrm{diag}\left(-\left(\frac{k-1}{4}\right),\ldots,\left(\frac{k-1}{4}\right)\right), \tag{1.19}$$

$$b=\begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & & & & & \cdot \\ \cdot & 1 & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix} \tag{1.20}$$

The whole point of introducing these ideas is that the solutions of the complex equation, modulo the complex gauge group, are "locally trivial;" just as for holomorphic bundles. Thus, if we solve locally for g:

$$\alpha'=g(\alpha)=g\alpha g^{-1}-\frac{1}{2}\frac{dg}{ds}g^{-1}=0,$$

i.e.,

$$\frac{dg}{ds}=2g\alpha,$$

then the complex equation becomes $d\beta'/ds=0$. Hence the general local solution of the complex equation is:

$$\alpha=\tfrac{1}{2}g^{-1}\frac{dg}{ds},\quad \beta=g^{-1}\beta' g$$

for a *constant* matrix β'. Thus the *global* classification that we seek in Sect. 3 comes down to understanding the poles at $s=0,2$.

With these brief preliminaries completed, we can phrase our main definitions:

Definition (1.21). *A Nahm complex is a pair of $(k\times k)$ matrix valued functions $\alpha(s)$, $\beta(s)$, for $s\in(0,2)$ and a vector $v\in\mathbb{C}^k$ satisfying:*

(i) $\dfrac{d\beta}{ds}+2[\alpha,\beta]=0.$

(ii) $\alpha(2-s)=\alpha^T(s), \quad \beta(2-s)=\beta^T(s).$

(iii) *α and β are smooth in $(0,2)$ and meromorphic in some neighborhoods of $s=0,2$ with simple poles at $s=0,2$, and residues a,b at $s=0$.*

(iv) $\operatorname{Tr}(a)=0$ *and v is a vector in the $-(k-1/4)$ eigenspace of a such that the vectors $\{b^j v\}_{j=0}^{k-1}$ span $\mathbb{C}^k$.*

Definition (1.22). *Two Nahm complexes $(\alpha,\beta,v),(\alpha',\beta',v')$ are equivalent if there is a continuous map:*

$$g\colon[0,2]\to \mathrm{GL}(k,\mathbb{C}),$$

smooth in the interior, such that:

(i) $g(\alpha,\beta)=(\alpha',\beta')$ in $(0,2)$ (cf. (1.13)).

(ii) $g(2-s)=g^T(s)^{-1}$.

(iii) $g(0)v=v'$.

Notes. (a) We have sacrificed elegance in these definitions in order to ease our proofs later. For example, we have not in fact defined a group action, and we allow α,β to be only smooth in the interior so that we may use bump functions there.

(b) In Definition (1.22) (iii), it suffices to suppose that $g(1)\in 0(k,\mathbb{C})$.

(c) The use of the word complex in this definition is only indirectly apt. As we have seen, they correspond to degenerate examples of $\bar\partial$-operators, and we have in mind the corresponding "Dolbeault complex,"

$$V \xrightarrow[\begin{pmatrix} d_\alpha \\ \bar d_\beta\end{pmatrix}]{\bar d} V\oplus V \xrightarrow[(-\bar d_\beta \;\; \bar d_\alpha)]{\bar d} V, \quad \bar d^2=0,$$

where V is the space of $\mathbb{C}^k$-valued functions on $(0,2)$. These are like the *monads* of Horrocks and Barth (mentioned in [4]) in that the fibre of the bundle at, say, $0\in\mathbb{R}^3$ on which Nahm defines his solution to the Bogomolny equations is: $\operatorname{Ker}\bar d/\operatorname{Im}\bar d$ (interpreted relative to suitable boundary conditions).

A solution $\{T_i\}$ of Nahm's equations, as in Proposition 1.1(b), gives, by the discussion above, matrix valued functions:

$$\alpha=\frac{i}{2}T_1; \quad \beta=\tfrac{1}{2}(T_2+iT_3),$$

which satisfy all the conditions for a Nahm complex (the residues a,b having the right properties by the usual representation theory of SU(2)). We can moreover choose a *unit* vector v in the $-(k-1/4)$ eigenspace of a (that is to say, a *weight vector* with respect to the torus in SU(2) determined by the choice of complex structure.) This choice of weight vector defines the circle bundle $\tilde M_k$ over the moduli space of monopoles, and we then have a natural map of $\tilde M_k$ to the set of equivalence classes of Nahm complexes. In the next section we shall show that this map is in fact an isomorphism.

Section 2. The Real Equation

We have to investigate now the solutions to the real equation (1.10) within an equivalence class of Nahm complexes or, more generally, the properties of the assignment:

$$(\alpha, \beta) \mapsto \hat{F}(\alpha, \beta) = \frac{d}{ds}(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]).$$

First note that everything that we do is in effect invariant under the *unitary* gauge group (as in (1.6)), and it will often be convenient to offset this by working in the homogeneous space: $\mathscr{H} = \mathrm{GL}(k, \mathbb{C})/U(k)$, which we identify with the positive self-adjoint matrices. Thus for a transformation:

$$g: (0,2) \to \mathrm{GL}(k, \mathbb{C}),$$

we set:

$$h = h(g) = g^* g: (0,2) \to \mathscr{H}. \tag{2.1}$$

Next we ask how $\hat{F}$ transforms under these complex gauge transformations:

Lemma (2.2). *If* $g: (0,2) \to \mathrm{GL}(k, \mathbb{C})$, *then for any* $(\alpha', \beta') = g(\alpha, \beta)$:

$$g^{-1} \hat{F}(\alpha', \beta') g = \hat{F}(\alpha, \beta) - 2[\bar{d}_\alpha(h^{-1} d_\alpha h) + \bar{d}_\beta(h^{-1} d_\beta h)].$$

Proof. This follows easily from the simpler identities:

$$\begin{cases} g^{-1} \cdot \bar{d}_{\alpha'} \cdot g = \bar{d}_\alpha \\ g^{-1} \cdot \bar{d}_{\beta'} \cdot g = \bar{d}_\beta \end{cases}, \quad \begin{cases} g^{-1} \cdot d_{\alpha'} \cdot g = d_\alpha + (h^{-1} d_\alpha h) \\ g^{-1} \cdot d_{\beta'} \cdot g = d_\beta + (h^{-1} d_\beta h) \end{cases},$$

and

$$\hat{F}(\alpha, \beta) = 2([d_\alpha, \bar{d}_\alpha] + [d_\beta, \bar{d}_\beta]).$$

Our treatment of the real equation will be based on a variational description and a differential inequality, which we derive in turn.

A Variational Description

Lemma (2.3). *Fix* (α, β) *and write* $(\alpha', \beta') = g(\alpha, \beta)$. *Then the equation* $\hat{F}(\alpha', \beta') = 0$ *is the Euler-Lagrange equation for the functional of* g *given by:*

$$\mathscr{L}_\varepsilon(g) = \frac{1}{2} \int_\varepsilon^{2-\varepsilon} |\alpha' + \alpha'^*|^2 + 2|\beta'|^2 \, ds.$$

That is, $\hat{F}(\alpha', \beta') = 0$ *if and only if for each* $\varepsilon > 0$ *this integral is stationary with respect to variations* δg *supported in* $(\varepsilon, 2-\varepsilon)$.

Proof. By the group invariance, we may suppose $g = 1$ so $\alpha' = \alpha$, $\beta' = \beta$, and also we can suppose the variation δg is self-adjoint. Then

$$\delta \mathscr{L} = \mathrm{Re} \int \mathrm{Tr}((\alpha + \alpha^*) \delta(\alpha + \alpha^*) + 2\beta(\delta\beta)^*) \, ds.$$

But $\delta\alpha = [\delta g, \alpha] - \frac{1}{2}\frac{d}{ds}(\delta g)$, so $\delta(\alpha + \alpha^*) = [\delta g, \alpha - \alpha^*] - \frac{d}{ds}(\delta g)$, and $\delta\beta = [\delta g, \beta]$.

Thus

$$\delta\mathscr{L} = \mathrm{Re}\int -\mathrm{Tr}((\alpha+\alpha^*)\frac{d}{ds}(\delta g)) + \mathrm{Tr}((\alpha+\alpha^*)[\delta g, \alpha-\alpha^*] + 2\beta[\beta^*, \delta g])$$

$$= \int \mathrm{Tr}\left(\delta g\left\{\frac{d}{ds}(\alpha+\alpha^*) + 2[\alpha,\alpha^*] + 2[\beta,\beta^*]\right\}\right)ds$$

$$= \int \mathrm{Tr}(\delta g \hat{F}(\alpha,\beta))\,ds$$

as required.

We see that, in a formal way, to find a point in an equivalence class of Nahm complexes satisfying $\hat{F}(\alpha,\beta)=0$ we have to minimize the Lagrangian $\mathscr{L}$ of Lemma (2.3), just as for the monads in [4] (compare the discussion of [3]). Of course, the integral over the whole range diverges, so we work first over the ranges $(\varepsilon, 2-\varepsilon)$, as above. This has the advantage that we can then work in a locally trivial complex gauge, as explained in Sect. 1, and in that gauge we get a better grip on our Lagrangian.

Suppose that over some range $(\varepsilon, 2-\varepsilon)$, $\alpha=0$ and β is a constant matrix. Then the Lagrangian integrand is

$$\frac{1}{4}\left|\frac{dg}{ds}g^{-1} + g^{*-1}\frac{dg^*}{ds}\right|^2 + 2|g\beta g^{-1}|^2, \tag{2.4}$$

which is easily rewritten in terms of $h=g^*g$ as

$$\tfrac{1}{4}\mathrm{Tr}\left(h^{-1}\frac{dh}{ds}\right)^2 + 2\,\mathrm{Tr}(\beta h^{-1}\beta^* h). \tag{2.5}$$

Now $\frac{1}{4}\mathrm{Tr}(h^{-1}\,dh)^2$ is the standard $\mathrm{GL}(k)$-invariant Riemannian metric $|\ |^2_{\mathscr{H}}$ on the homogeneous space: $\mathscr{H} = \mathrm{GL}(k)/\mathrm{U}(k)$. So in this gauge the Lagrangian (regarded as a functional of $h(s)$) is

$$\mathscr{L}_\varepsilon = \int_\varepsilon^{2-\varepsilon} \frac{1}{2}\left|\frac{dh}{ds}\right|^2_{\mathscr{H}} + V_\beta(h)ds, \tag{2.6}$$

where $V_\beta(h) = \mathrm{Tr}(\beta h^{-1}\beta^* h) \geqq 0$.

(So the solution sought is the path in the complete Riemannian manifold $\mathscr{H}$ of a particle moving under the influence of a potential-V_β.) Since $V_\beta(h)$ is smooth and non-negative, it follows easily from the direct method of the calculus of variations that for any $h_+, h_- \in \mathscr{H}$ there is a continuous path:

$$h: [\varepsilon, 2-\varepsilon] \to \mathscr{H}\begin{cases} h(\varepsilon) = h_+ \\ h(2-\varepsilon) = h_- \end{cases}, \tag{2.7}$$

smooth in $(\varepsilon, 2-\varepsilon)$ and minimizing the Lagrangian $\mathscr{L}$ amongst all such paths. In particular, for any choice of g with $g^*g = h$; $g(0,\beta) = (\alpha',\beta')$ satisfies the real equation:

$$\hat{F}(\alpha',\beta') = \frac{d}{ds}\left(h^{-1}\frac{dh}{ds}\right) + 2[\beta,[\beta^*,h]] = 0.$$

So we have our main *existence* property.

Proposition (2.8). *If $\varepsilon > 0$ and if α, β satisfy the complex equation over $[\varepsilon, 2-\varepsilon]$, then for any h_+, h_- in $\mathscr{H}$ there is a continuous $g: [\varepsilon, 2-\varepsilon] \to \mathrm{GL}(k, \mathbb{C})$, with $h = h(g) = h_+, h_-$ respectively at $\varepsilon, 2-\varepsilon$ and such that $(\alpha', \beta') = g(\alpha, \beta)$ satisfies the real equation $\hat{F}(\alpha', \beta') = 0$ in $(\varepsilon, 2-\varepsilon)$.*

This follows directly from the discussion above and the existence of a locally trivial complex gauge.

A Differential Inequality

Recall that if $f(s)$ is a continuous function, we can define

$$\frac{d^2 f}{ds^2} \geqq 0 \quad \text{weakly}$$

if for all smooth, compactly supported, positive, test functions γ,

$$\int \frac{d^2\gamma}{ds^2} \cdot f \geqq 0.$$

For $h \in \mathscr{H}$ with eigenvalues λ_i, define

$$\Phi(h) = \log \max(\lambda_i)_{i=1}^{k} \in \mathbb{R} \tag{2.9}$$

so that if $h(s)$ is continuous, $\Phi(h(s))$ is also.

Lemma (2.10). *If as usual $(\alpha', \beta') = g(\alpha, \beta)$ over some interval in $(0, 2)$ then, with $h = g^* g$:*

$$\frac{d^2}{ds^2} \Phi(h) \geqq -2(|\hat{F}(\alpha, \beta)| + |\tilde{F}(\alpha', \beta')|),$$

$$\frac{d^2}{ds^2} \Phi(h^{-1}) \geqq -2(|\tilde{F}(\alpha, \beta)| + |\tilde{F}(\alpha', \beta')|)$$

in the weak sense.

Proof. Given fixed (α, β) this depends only upon h. The set of self-adjoint matrices with two or more eigenvalues equal has codimension at least three in $\mathscr{H}$. We can approximate any h in C^2 by a map whose eigenvalues are distinct for each s and, since the inequality is uniform, deduce the general case by taking the limit. So, without loss of generality, suppose that for all s the eigenvalues of $h(s)$ are distinct.

Then, using the unitary gauge group, we can suppose

$$g = \mathrm{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_k}) \quad \text{for } t_1(s) > t_2(s) \ldots > t_k(s).$$

Thus $\Phi(h) = 2t_1$. We compare the $(1,1)$ entry in the two matrices $\hat{F}(\alpha, \beta)$, $\hat{F}(\alpha', \beta')$, using Lemma (2.2),

$$g^{-1}\hat{F}(\alpha', \beta')g = \hat{F}(\alpha, \beta) - 2[\bar{d}_\alpha(h^{-1}d_\alpha h) + \bar{d}_\beta(h^{-1}d_\beta h)].$$

Now

$$d_\alpha h = \left(\frac{1}{2}\frac{d}{ds} - [\alpha^*,]\right)(\mathrm{diag}(e^{2t_i})).$$

So

$$h^{-1}d_\alpha h = \operatorname{diag}\left(\frac{dt_i}{ds}\right) + \alpha^* - h^{-1}\alpha^* h,$$

and

$$\bar{d}_\alpha(h^{-1}d_\alpha h) = \tfrac{1}{2}\operatorname{diag}\left(\frac{d^2t_i}{ds^2}\right) + \frac{1}{2}\frac{d}{ds}(\alpha^* - h^{-1}\alpha^* h)$$
$$+\left[\alpha, \operatorname{diag}\left(\frac{dt_i}{ds}\right)\right] + [\alpha, \alpha^* - h^{-1}\alpha^* h].$$

Likewise

$$\bar{d}_\beta(h^{-1}d_\beta h) = [\beta, \beta^* - h^{-1}\beta^* h].$$

For any diagonal matrix Λ the diagonal entries of $[\Lambda, *]$ are zero, and similarly the diagonal entries of $g^{-1}\hat{F}g$ are the same as those of $\hat{F}$. Hence

$$(\hat{F}(\alpha,\beta) - \hat{F}(\alpha',\beta'))_{(1,1)}$$
$$= -2\left(\frac{1}{2}\frac{d^2t_1}{ds^2} + ([\alpha, \alpha^* - h^{-1}\alpha^* h] + [\beta, \beta^* - h^{-1}\beta^* h])_{(1,1)}\right).$$

But the $(1,1)$ entry of $[\alpha, \alpha^* - h^{-1}\alpha^* h]$ is

$$\sum_{j>1} \{|\alpha_{1j}|^2(1 - e^{2(t_1 - t_h)}) - |\alpha_{j1}|^2(1 - e^{2(t_j - t_1)})\},$$

which is *negative* since $t_1 > t_j$ for $j > 1$. Similarly for the β term. Thus

$$\frac{d^2t_1}{ds^2} \geqq -|(\hat{F}(\alpha',\beta') - \hat{F}(\alpha,\beta))_{(1,1)}| \geqq -(|\hat{F}(\alpha',\beta') + |\hat{F}(\alpha,\beta)|),$$

and one gets the bound on $\Phi(h^{-1}) = -2t_k$ by considering the (k,k) entry in just the same way.

This inequality gives easily the essential uniqueness of our solutions.

Lemma (2.12). *Let (α,β) be a solution to the complex equation (1.9) in some interval $(\varepsilon, 2-\varepsilon)$ $\varepsilon \geqq 0$. Suppose that:*

$$\left.\begin{array}{l}(\alpha',\beta') = g_1(\alpha,\beta) \\ (a'',\beta'') = g_2(\alpha,\beta)\end{array}\right\} \quad \textit{for } g_i\colon [\varepsilon, 2-\varepsilon] \to \mathrm{GL}(k,\mathbb{C}) \textit{ continuous},$$

each satisfy the real equation: $\hat{F}(\alpha',\beta') = \hat{F}(\alpha'',\beta'') = 0$. Then if $h_1 = g_1^* g_1$, $h_2 = g_2^* g_2$ take on equal values at the end points ε, $2-\varepsilon$, then in fact $h_1(s) = h_2(s)$ throughout $[\varepsilon, 2-\varepsilon]$.

Proof. We may suppose that $g_2 = 1$ and $(\alpha'',\beta'') = (\alpha,\beta)$. Then by Lemma (2.10) the logarithm of the maximum eigenvalue of h, $2t_1 = \Phi(h)$ satisfies:

$$\frac{d^2t_1}{ds^2} \geqq 0.$$

But $t_1 = 0$ at $s = \varepsilon$, $2 - \varepsilon$, so by the obvious convexity argument (still valid if the differential inequality holds in the weak sense): $t_1 \leqq 0$ in $[\varepsilon, 2-\varepsilon]$. For the same reason the logarithm of the minimum eigenvalue of h, $2t_k = -\Phi(h^{-1})$, satisfies $t_k \geqq 0$ in $[\varepsilon, 2-\varepsilon]$, and since $t_1 \geqq t_k$, by definition we get $t_1 = t_2 = \cdots = t_k$ in $[\varepsilon, 2-\varepsilon]$ and $h_1(s) = 1$.

Corollary (2.13). *Suppose in addition that* (α, β) *satisfies the symmetry condition* (1.8):

$$\begin{cases} \alpha(2-s) = \alpha^T(s) \\ \beta(2-s) = \beta^T(s)\text{'} \end{cases}$$

and that $h(2-\varepsilon) = h^T(\varepsilon)^{-1}$. *Then if* $(\alpha', \beta') = g(\alpha, \beta)$ *satisfies the real equation it follows that*

$$h(2-s) = h^T(s)^{-1}, \qquad s \in (\varepsilon, 2-\varepsilon),$$

and α', β' *also satisfy the symmetry conditions.*

Proof. The obvious symmetry argument, taking $h_2 = (h^T)^{-1}$ in (2.12).

Note. This uniqueness is a reflection of the harmony between the kinetic and potential terms in the Lagrangian,

$$\int \frac{1}{2} \left| \frac{dh}{ds} \right|^2_{\mathscr{H}} + V_\beta(h)\, ds.$$

The manifold $\mathscr{H} = \mathrm{GL}(k)/\mathrm{U}(k)$ has non-positive *curvature* and the function V_β on $\mathscr{H}$ is geodesically *convex* [9]. More generally, for any simply connected manifold with a complete Riemannian metric of negative curvature and positive, geodesically convex, potential function there is a unique stationary path for the corresponding Lagrangian between any two points.

Armed with these principles, we will soon let ε tend to 0 to get solutions of the real equation over the full range. First, it is convenient to examine the state of affairs near the poles using power series.

Lemma (2.14). *Suppose* (α, β, v) *is a Nahm complex. Then there is an equivalent Nahm complex* (α', β', v') *such that:*

(i) $\hat{F}(\alpha', \beta')$ *is bounded in* $(0,2)$.
(ii) $|\alpha - \alpha'^*|$ *is bounded in* $(0,2)$.
(iii) $|v'| = 1$.
(iv) *The residues of the matrices* T_i *corresponding to* α', β' *(at the pole* $s = 0$*) are conjugate by* $U(k)$ *to the standard irreducible representation* τ_1, τ_2, τ_3 *of* SU(2) *on* $\mathbb{C}^k$ *(so,* $\tau_1 = [\tau_2, \tau_3]$ etc.).

Proof. We work in a neighborhood of $s = 0$. Then since α, β have simple poles a, b, we have

$$\hat{F}(\alpha, \beta) = \frac{1}{s^2}(-(a^* + a) + 2([a, a^*] + [b, b^*])) + \frac{1}{s}\Psi + (\text{holomorphic}),$$

say. Then it is easy to see that, given our assumptions on the poles (a, b), we may arrange that (ii), (iii), (iv) hold, in a neighborhood of 0, by a *constant* complex gauge transformation, and that this implies that the $1/s^2$ term of $\hat{F}$ vanishes.

So suppose, without loss of generality, that a, b have the standard form corresponding to the representation S^{k-1} of SU(2), hence also for the residues τ_i:

$$\begin{cases} a = \frac{i}{2}\tau_1 \\ b = \frac{1}{2}(\tau_2 + i\tau_3). \end{cases} \tag{2.15}$$

Now consider a complex gauge transformation

$$g = 1 + \frac{\chi}{2}s + \cdots, \quad \text{with } \chi = \chi^*.$$

Using the formula:

$$g^{-1}\hat{F}(\alpha', \beta')g = \hat{F}(\alpha, \beta) - 2\{\bar{d}_\alpha(h^{-1}(d_\alpha h)) + \bar{d}_\beta(h^{-1}(d_\beta h))\},$$

one finds that the residue Ψ of $\hat{F}$ at 0 transforms by

$$\Psi' = \Psi + 2([b, [b^*, \chi]] + [a, [a, \chi]]) - [a, \chi],$$

so we can choose χ to make $\Psi' = 0$ if and only if Ψ lies in the image of

$$\chi \xrightarrow{D} 4[b, [b^*, \chi]] + 4[a, [a, \chi]] - 2[a, \chi].$$

But
$$\begin{aligned} -D(\chi) &= [\tau_2 + i\tau_3, [\tau_2 - i\tau_3, \chi]] + [\tau_1, [\tau_1, \chi]] + i[\tau_1, \chi] \\ &= \sum_{i=1}^{3} [\tau_i, [\tau_i, \chi]] \end{aligned}$$

Now $\chi \to \sum_{i=1}^{3} [\tau_i, [\tau_i, \chi]]$ is the *Casimir operator* on the representation

$$S^{k-1} \otimes S^{k-1} \cong S^{2k-2} \oplus \cdots \oplus S^0,$$

and this map is equal to the homethety by factor $l(l+1)/2$ on S^l for each l. Thus we can solve $\Psi' = 0$ for χ if and only if Ψ has no component in S^0, that is if

$$\operatorname{Tr} \Psi = 0.$$

But this is always the case since the pole of $\hat{F} = d/ds(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*])$ comes entirely from commutator terms.

In this way we find a complex gauge transformation g, analytic in a neighborhood of $s = 0$, such that $\hat{F}(\alpha', \beta')$ is bounded there. We extend g smoothly to be 1 in a neighborhood of $s = 1$ and define it for $s > 1$ by $g(2 - s) = g^T(s)^{-1}$.

Now we can state our final results in this section. The uniqueness has essentially been done already.

Proposition (2.16). *Suppose* (T_1, T_2, T_3), (T'_1, T'_2, T'_3) *are two solutions of Nahm's* *equations (1.1)* (b) *and that v, v' are corresponding unit weight vectors. Then if the Nahm complexes (α, β, v), (α', β', v') defined by these are equivalent $(T_1, T_2, T_3; v)$ is in fact conjugate by $O(k, \mathbb{R})$ to $(T'_1, T'_2, T'_3; v')$.*

Proof. Suppose $g: [0,2] \to \mathrm{GL}(k, \mathbb{C})$ maps $(\alpha, \beta; v)$ to $(\alpha', \beta'; v')$, with $g(2-s) = g^T(s)^{-1}$. Then considering the representations at $s=0$ and the fact that $|v| = |v'| = 1$, we must have $g(0)$ *unitary*, so $h = h(g) = 1$ at $s = 0, 2$. So by Lemma (2.12), with $\varepsilon = 0$, we get $h = 1$ throughout $(0,2)$, and since α, α' are each self-adjoint (i.e., $T_0 = T'_0 = 0$) it follows that $g = g(1)$ is a constant unitary matrix. But $g(1)$ is orthogonal by hypothesis, so $(T_i; v)$, $(T'_i; v')$ are conjugate by $O(k)$ and correspond to the same point in the circle bundle $\tilde{M}_k$ over the moduli space.

Proposition (2.17). *If (α, β, v) is a Nahm complex there is a solution $\{T'_i\}$ of Nahm's equations and a weight vector v' such that the Nahm complex defined by (T'_i, v') is equivalent to (α, β, v).*

Proof (Extended). We may suppose that (α, β, v) satisfies the conditions (i)–(iv) of Lemma (2.14). They by Proposition (2.8) there is, for each $\varepsilon > 0$ a g_ε, and corresponding h_ε, with $h_\varepsilon = 1$ at $s = \varepsilon, 2-\varepsilon$ and $g_\varepsilon(\alpha, \beta)$ satisfies the real equation in $(\varepsilon, 2-\varepsilon)$. By the symmetry principle, Corollary (2.9), $h_\varepsilon(2-s) = h_\varepsilon^T(s)^{-1}$.

Next we use the differential inequality again.

Lemma (2.18). *For h_ε as above and some constant $C = C(\alpha, \beta)$,*

$$\Phi(h_\varepsilon) + \Phi(h_\varepsilon^{-1}) \leqq Cs(2-s) + 2.$$

Proof. Put $\Phi = \Phi(h_\varepsilon) + \Phi(h_\varepsilon^{-1})$ so $\Phi = 2$ at $s = \varepsilon$, $2-\varepsilon$, and

$$\frac{d^2\Phi}{ds^2} \geqq -2|\hat{F}(\alpha, \beta)| \geqq -2C,$$

say by Lemma (2.10) and the boundedness of $\hat{F}(\alpha, \beta)$.

Consider the function $f_\varepsilon(s) = C(s-\varepsilon)(2-\varepsilon-s) + 2$. We have

$$\frac{d^2}{ds^2}(f_\varepsilon - \Phi) \leqq 0$$

and $f_\varepsilon - \Phi = 0$ at the end points $\varepsilon, 2-\varepsilon$. Hence by convexity, $f_\varepsilon - \Phi \geqq 0$ in $(\varepsilon, 2-\varepsilon)$ so

$$\Phi(s) \leqq C(s-\varepsilon)(2-\varepsilon-s) + 2 \leqq Cs(2-s) + 2.$$

Corollary (2.19). (a) *The matrices $h_\varepsilon(s)$, $h_\varepsilon^{-1}(s)$ are uniformly bounded, independent of ε and s. Moreover, for s near* 0:

$$|h_\varepsilon(s) - 1| \leqq \text{const.}\, s.$$

(b) *As $\varepsilon \to 0$ the h_ε converge in $C^\infty(0,2)$ to a continuous $h: [0,2] \to \mathscr{H}$ (smooth in $(0,2)$) such that:*

(i) $(\alpha', \beta') = h^{1/2}(\alpha, \beta)$ *satisfies the real equation* $\hat{F}(\alpha', \beta') = 0$.
(ii) $h(2-s) = h^T(s)^{-1}$ *and* $h(0) = h(2) = 1$.
(iii) $|h(s) - 1| \leqq \text{const}\, s$.

All of this follows readily from the lemma above and the fact that the h_ε satisfy the equation, $\hat{F}(h_\varepsilon^{1/2}(\alpha,\beta))=0$ (cf. (2.2)). For (2.19) (a) the bounds on $\Phi(h_\varepsilon)$, $\Phi(h_\varepsilon^{-1})$ of Lemma (2.18) control the eigenvalues, and so the norm, of h_ε. Likewise these eigenvalues are pinched toward 1 as s tends to ε. Given this uniform bound and the fact that h_ε satisfy an elliptic equation, one easily gets convergence in the interior (compare (2.20) below).

It remains only to check the behavior near the poles. We set $g=h^{1/2}$; $(\alpha',\beta')=g(\alpha,\beta)$.

Lemma (2.20). $|\alpha-\alpha'|$, $|\beta-\beta'|$ *are bounded in* $(0,2)$.

Proof. All our information derives from the bound $|h-1|\leqq \text{const.}\, s$ which we shall use repeatedly.

By definition,

$$\alpha'=g\alpha g^{-1}-\frac{1}{2}\left(\frac{dg}{ds}\right)g^{-1},\quad \beta'=g\beta g^{-1}.$$

So the proof of the lemma is equivalent to showing that dg/ds or alternatively dh/ds are bounded.

But (α',β') satisfies the real equation so (cf. (1.14) and Lemma (2.2)):

$$\left(\frac{1}{2}\frac{d}{ds}+[\alpha,\ \ ]\right)h^{-1}\left(\frac{1}{2}\frac{d}{ds}-[\alpha^*,\ \ ]\right)h-[\beta,h^{-1}[\beta^*,h]]=2\hat{F}(\alpha,\beta)$$

is regular. Expand this to get that

$$\frac{1}{4}\frac{d}{ds}\left(h^{-1}\frac{dh}{ds}\right)+\frac{1}{2}\left(\left[\alpha,h^{-1}\frac{dh}{ds}\right]+h^{-1}\frac{dh}{ds}h^{-1}[\alpha^*,h]-\cdots\right.$$

$$\left.\cdots-h^{-1}\left[\frac{d\alpha^*}{ds},h\right]-h^{-1}\left[\alpha^*,\frac{dh}{ds}\right]\right)-[\alpha,h^{-1}[\alpha^*,h]]-[\beta,h^{-1}[\beta^*,h]]$$

is uniformly bounded in $(0,2)$.

Next use the fact that the terms $[\alpha^*,h]$, $[\beta^*,h]$ are bounded, and similarly $[d\alpha^*/ds,h]\leqq \text{const.}\, s^{-1}$, to obtain

$$\frac{d}{ds}\left(h^{-1}\frac{dh}{ds}\right)+2\left(\left[\alpha,h^{-1}\frac{dh}{ds}\right]+h^{-1}\frac{dh}{ds}h^{-1}[\alpha^*,h]-h^{-1}\left[\alpha^*,\frac{dh}{ds}\right]\right)\leqq Cs^{-1}. \tag{2.21}$$

(M, C will be positive constants, varying from line to line in the usual way.)

Expanding, in a similar fashion, the middle term of (2.21) and using the fact that $\alpha-\alpha^*$ is bounded, we get:

$$\left|\frac{d}{ds}\left(h^{-1}\frac{dh}{ds}\right)\right|\leqq M\left|\frac{dh}{ds}\right|+Cs^{-1}. \tag{2.22}$$

Given $s_0>0$ and $\sigma\in[\frac{1}{2}s_0,\frac{3}{2}s_0]$, we integrate this to get

$$\left|\left(h^{-1}\frac{dh}{ds}\right)_{s=\sigma}-\left(h^{-1}\frac{dh}{ds}\right)_{s=s_0}\right|\leqq (Ms_0)G+C, \tag{2.23}$$

where $G = \sup_{[\frac{1}{2}s_0, \frac{3}{2}s_0],} \left|\frac{dh}{ds}\right|$. Taking in particular σ to be a point

where G is attained and rearranging gives

$$\left|\left(\frac{dh}{ds}\right)_{s=\sigma} - \left(\frac{dh}{ds}\right)_{s=s_0}\right| \leqq M s_0 G + C,$$

so once $s_0 \leqq \frac{1}{2}M^{-1}$ (say),

$$G \leqq 2\left|\left(\frac{dh}{ds}\right)_{s=s_0}\right| + C_2.$$

Then, returning to (2.23):

$$\left|\left(\frac{dh}{ds}\right)_{s=\sigma} - \left(\frac{dh}{ds}\right)_{s=s_0}\right| \leqq \left|\left(h^{-1}\frac{dh}{ds}\right)_{s=\sigma} - \left(h^{-1}\frac{dh}{ds}\right)_{s=s_0}\right|$$
$$+ M_1 s_0 \left|\frac{dh}{ds}\right|_{s=s_0} \leqq M s_0 \left|\left(\frac{dh}{ds}\right)_{s=s_0}\right| + C$$

for all $\sigma \in [\frac{1}{2}s_0, \frac{3}{2}s_0]$. Hence integrating again and dividing by s_0 gives

$$\left|\frac{h(\frac{3}{2}s_0) - h(\frac{1}{2}s_0)}{s_0} - \left(\frac{dh}{ds}\right)_{s=s_0}\right| \leqq M s_0 \left|\left(\frac{dh}{ds}\right)_{s=s_0}\right| + C.$$

So, again taking s_0 sufficiently small compared with M^{-1}, one sees

$$\left|\left(\frac{dh}{ds}\right)_{s=s_0}\right| \leqq C,$$

since

$$|h(\tfrac{3}{2}s_0) - h(\tfrac{1}{2}s_0)| \leqq \text{const}\, s_0 \quad \text{by} \quad (2.19)\,(a).$$

Now, continuing our proof of Proposition (2.17), we transfer back to the "$T_0 = 0$ gauge." That is, solve the equation:

$$\frac{du}{ds} = u(\alpha' - \alpha'^*) \text{ in } (0, 2), \quad u(s) \in U(k), \tag{2.28}$$

starting from $u(1) = 1$ (compare (1.6), (1.7), (1.13)). Since $|\alpha' - \alpha'^*|$ is bounded $u(s)$ extends continuously to $[0, 2]$. It is easy to see that the "residues" transform by $u(0)$ and that we retain our bound on the finite part. Thus, if we let

$$T_1 = u(\alpha' + \alpha'^*)u^{-1}, \quad T_2 = u(\beta' - \beta'^*)u^{-1}, \quad T_3 = u(\beta' + \beta'^*)u^{-1},$$

we get solutions to Nahm's equations with a continuous complex gauge transformation $u \circ g$ from (α, β) to $\{T_i\}$ such that in $(0, 1)$, $|T_i(s) - (1/s)u(0)\tau_i u(0)^{-1}|$ is bounded. Finally we need:

Proposition (2.25). *Suppose that matrices $T_i(s)$ satisfy conditions* (i), (ii), (iii) *of Prop.* (1.1(b), *but in place of* (iv), (v) *we are given only that for some $u \in U(k)$:*

$|T_i(s) - u\tau_i u^{-1}/s|$ *is bounded in* $(0, 1)$. *Then in fact the* T_i *satisfy all the conditions of Prop.* (1.1)(b).

This additional regularity can be proved directly, but to avoid a further digression we will sidestep the issue by observing that, working through Hitchin's description ([6], pp. 147–155), Nahm's construction accepts this data to produce a monopole. Then Hitchin's main theorem shows that the T_i are meromorphic with simple poles.

This completes the proof of Proposition (2.17) which, together with Proposition (2.16), shows that the circle bundle $\tilde{M}_k$ over the moduli space of monopoles is naturally identified with the set of equivalence classes of Nahm complexes.

Section 3. The Classification of Nahm Complexes

We begin with this general remark. As we have seen, a solution (α, β) of the complex equation (1.9) is locally trivial in that it can be thrown by a complex gauge transformation into the form $(0, \beta')$ with β' constant. Otherwise said, $\beta(s) \in M_{k \times k}(\mathbb{C})$ is a one-parameter family of matrices in the same conjugacy class (compare [6] p. 170). As discussed briefly in [4], in good cases an orbit in a linear representation space V is closed, but the same is not in general true of the associated orbit in $\mathbb{P}(V)$. In our case the prescribed behavior at the end points $s = 0, 2$ corresponds to demanding that the path $[\beta(s)]$ in $\mathbb{P}(M_{k \times k}(\mathbb{C}))$ converges to a given point [b] outside the orbit, which requires $|\beta(s)| \to \infty$. The classification is, in effect, of a point β' in $M_{k \times k}(\mathbb{C})$ and a one-parameter subgroup p in $\mathrm{GL}(k, \mathbb{C})$ such that $[p(s)\beta'] \to b$ as $s \to 0$.

Now we give formal statements.

Proposition (3.1). *There is a one-to-one correspondence between*

(a) *equivalence classes of Nahm complexes.*

(b) *equivalence classes under* $O(k, \mathbb{C})$ *of pairs* (B, w), *where*

 (i) B *is a symmetric* $k \times k$ *matrix and* $w \in \mathbb{C}^k$ *a column vector.*

 (ii) w *generates* $\mathbb{C}^k$ *as a* $\mathbb{C}[B]$ *module* (*or* w *is a "cyclic vector" for* B).

Proposition (3.2). *The assignment of the rational map*

$$(B, w) \mapsto f \in R_k,\ f(z) = w^T(z1 - B)^{-1} w \in \mathbb{C} \cup \{\infty\}$$

induces a one-to-one correspondence between equivalence classes of pairs (B, w) *as in Prop. (3.1)(b) and the based rational maps of degree k;* R_k.

Proof of Proposition 3.1 (Extended). First we should define the pair (B, w) corresponding to a Nahm complex (α, β, v). We put $B = \beta(1)$, which is symmetric by (1.21)(ii). Then consider the differential operator, $\bar{d}_\alpha = 1/2(d/ds) + \alpha$, acting on $\mathbb{C}^k$-valued functions of s. Recall that near 0, $\alpha = a/s +$ (holomorphic), and that the vector v lies in the $-(k-1)/4$ eigenspace of a. In a suitable basis $\{b^j v\}$:

$$a = \mathrm{diag}\left(-\frac{(k-1)}{4}, \ldots, \frac{k-1}{4} \right).$$

Thus there is a unique solution $u(s)$ to $\bar{d}_\alpha u = 0$, such that $s^{-(k-1)/2} u(s) \to v$ as $s \to 0$ (Compare [5] pp. 590–593, [6] p. 183). Now define $w = u(1) \in \mathbb{C}^k$. The next four lemmas complete the proof of Proposition (3.1).

Lemma (3.4). *Up to the action of* $O(k, \mathbb{C})$, *the pair* (B, w) *depends only upon the equivalence class of the Nahm complex* (α, β, v).

Proof. Suppose $g(\alpha, \beta, v) = (\alpha', \beta', v')$ with g as in Definition (1.22). Then by definition $\beta'(1) = B'$ is conjugate to $\beta(1) = B$ by $g(1) \in O(k, \mathbb{C})$. Moreover $\bar{d}_{\alpha'} = g \circ \bar{d}_\alpha \circ g^{-1}$, so if $u(s)$ is as above, $\bar{d}_{\alpha'}(g(s)u(s)) = 0$. Now $g(0)v = v'$ and, g being continuous at the end points,

$$s^{-(k-1)/2}(g(s)u(s)) \to g(0)v = v'$$

as $s \to 0$. Hence $u'(s) = g(s)u(s)$, and in particular $w' = g(1)w$, and (B', w') is equivalent to (B, w).

Lemma (3.5). *The pair* (B, w) *assigned to a Nahm complex satisfies the conditions of Proposition* (3.1) (b); *that is,* w *is a cyclic vector for* B.

Proof. By assumption v is a cyclic vector for the residue b of β at 0. But this is an open condition and

$$(s\beta(s), s^{-(k-1)/2}u(s)) \to (b, v) \quad \text{as } s \to 0,$$

so for small s, $u(s)$ is a cyclic vector for $\beta(s)$. But this condition is independent of s (i.e., preserved by our "parallel transport" $\bar{d}_\alpha$) so the same is true of w and B.

Lemma (3.6). *For any pair* (B, w) *satisfying the conditions of Proposition* (3.1)(b) *there is at least one corresponding Nahm complex.*

Proof. By hypothesis the vectors $\{B^j w\}_{j=0}^{k-1}$ span $\mathbb{C}^k$, so using this basis we can write:

$$B = \Lambda \tilde{B} \Lambda^{-1}; \quad \tilde{B} = \begin{bmatrix} 0 & . & . & . & . & . & -q_0 \\ 1 & 0 & & & & & -q_1 \\ . & 1 & & & & & . \\ . & & . & & & & . \\ . & & & . & & & . \\ . & & & & . & & . \\ . & . & . & . & . & 1 & -q_{k-1} \end{bmatrix}.$$

Let $\sigma(s)$ be a smooth function for $s \in (0, 1)$ such that $\sigma(s) = s$ in a neighborhood of $s = 0$ and $\sigma(s) = 1$ in a neighborhood of $s = 1$. Put

$$p(s) = \operatorname{diag}(\sigma^{(k-1)/2}, \sigma^{(k-3)/2}, \ldots, \sigma^{-(k-1)/2}).$$

Then as $s \to 0$,

$$p(s)\tilde{B}p(s)^{-1} = \frac{1}{s}\begin{pmatrix} 0 & . & . & . & . & . & . \\ 1 & 0 & & & & & . \\ . & 1 & & & & & . \\ & & . & & & & . \\ . & & & . & & & . \\ & & & & . & & \\ . & . & . & . & . & 1 & 0 \end{pmatrix} + O(1)$$

and

$$-\tfrac{1}{2}p^{-1}\frac{dp}{ds}=\frac{1}{s}\begin{pmatrix} -\frac{(k-1)}{2} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \frac{k-1}{2} \end{pmatrix}$$

Thus

$$\left.\begin{aligned} \beta(s) &= \Lambda p\tilde{B}p^{-1}\Lambda^{-1} \\ \alpha(s) &= \frac{\Lambda}{2}\left(p^{-1}\frac{dp}{ds}\right)\Lambda^{-1} \end{aligned}\right\} s\in(0,1]$$

satisfy the complex equation, have the correct poles and lead to the correct definition of (B, w) if we take $v = \Lambda w$. Finally define (α, β) in $[1, 2)$ by the symmetry condition to get the required Nahm complex.

Lemma (3.7). *Suppose that two Nahm complexes (α, β, v), (α', β', v') have the property that the corresponding pairs (B, w), (B', w') are $0(k, \mathbb{C})$ equivalent. Then (α, β, v) is equivalent to (α', β', v').*

Proof. We defined $u(s)$ so that

$$|s^{-(k-1)/2}u(s) - v| = \varepsilon(s)$$

say, tends to zero with s. Also

$$|s\beta(s) - b| \leqq C.s.$$

Then this means

$$|s^{1-(k-1)/2}u(s) - bv| \leqq |s\beta - b|\,|s^{-(k-1)/2}u| + |b|\,|v - s^{-(k-1)/2}u| \leqq C_2\varepsilon(s),$$

and likewise,

$$|s^{i-(k-1)/2}u(s) - b^i v| \leqq C_i\varepsilon(s) \tag{3.8}$$

for $i = 0, \ldots, k-1$. Similarly for the primed case.

Now if $g_1 \in 0(k, \mathbb{C})$ maps (B, w) to (B', w'), so $B'(1) = \beta'(1) = g_1\beta(1)g_1^{-1}$, $w' = g_1 w$. Then g_1 "propagates" uniquely over $(0, 2)$ to a map g with:

$$g\circ\bar{d}_\alpha\circ g^{-1} = \bar{d}_{\alpha'}; \quad g(1) = g_1,$$

and since $[\bar{d}_\alpha, \bar{d}_\beta] = 0$, $g(\beta^i u'(s)) = \beta'^i u'(s)$ for each i and all s. So

$$g(s^{i-(k-1)/2}\beta^i u) = s^{i-(k-1)/2}\beta'^i u',$$

and (3.8) implies that as s tends to 0 $g(s)$ converges to some Γ, say, uniquely defined by $\Gamma(b^i v) = b'^i v'$. Likewise g tends to a limit at $s = 2$ and gives the equivalence of the two Nahm complexes.

This completes the proof of Proposition (3.1). Finally we have:

Proof of Proposition (3.2). First it is clear that this assignment

$$(B, w) \mapsto f; \quad f(z) = w^T (z - B)^{-1} w$$

depends only on the $0(k, \mathbb{C})$ equivalence class of (B, w). To see that f has degree k consider the expansion for large z:

$$f(z) = w^T \left(\frac{1}{z} + \frac{B}{z^2} + \frac{B^2}{z^3} \dots \right) w.$$

If f had degree less than k, so that $f(z)q(z)$ is a polynomial for some

$$q(z) = \sum_0^{k-2} q_k z^k,$$

then $f(z)q(z) = \sum_{i,j} q_i (w^T B^{i+j} w) z^{-j}$ would imply that the matrix whose (i,j) entry is $(w^T B^{i+j} w)$, $(0 \leqq i, j \leqq k - 1)$, would have rank less than k. But this is the matrix of the non-degenerate inner product in the base $\{B^i w\}_{i=0}^{k-1}$. Hence it is not singular.

To complete the proof of Proposition (3.2) we give the inverse construction. Given

$$f(z) = p(z)/q(z); \quad \deg q = k, \quad \deg p \leqq k - 1,$$

with p and q coprime, let V be the k-dimensional vector space: $\mathbb{C}[t]/\langle q(t) \rangle$. Define an inner product on V by:

$$(\pi, \sigma) = \sum_{\lambda \in Z(q)} \operatorname{res}_\lambda (f \pi \sigma \, dt), \quad (Z(q) = \text{set of zeros of } q).$$

This is well defined since qf is regular. Moreover, it is a non-degenerate form since in the primary decomposition:

$$V \cong \bigoplus_{\lambda \in Z(q)} V_\lambda.$$

The factors V_λ are orthogonal and, if $\lambda \in Z(q)$ is a zero of order l

$$V_\lambda = \{\pi_0 + \pi_1 (t - \lambda) + \dots + \pi_{l-1} (t - \lambda)^{l-1}\}.$$

the restriction of the form to V_λ is the non-degenerate

$$((\pi_i), (\sigma_j)) = p(\lambda) \sum_{i=0}^{l-1} \pi_i \sigma_{l-i-1}.$$

Now define $B: V \to V$ by $B(\pi) = t\pi$ and let the element w in $V = \mathbb{C}[t]/\langle q \rangle$ be that corresponding to $1 \in \mathbb{C}[t]$. So w is tautologically a cyclic vector for B, and the map $g(z)$ corresponding to $(V, (\,,), B, w)$ is given by

$$g(z) = (v, (z - B)^{-1} v) = \sum_{\lambda \in Z(q)} \operatorname{res}_\lambda \left(\frac{f(t)}{z - t} \, dt \right).$$

But

$$\sum_{\lambda \in Z(q) \cup \{z\}} \mathrm{res}_\lambda\left(\frac{f(t)}{z-t}dt\right) = 0$$

by the Cauchy integral formula in the complex t-plane, since:

$$\left|\frac{f(t)}{z-t}\right| = 0(t^{-2}) \quad \text{for large } t.$$

So

$$g(z) = -\mathrm{res}_z\left(\frac{f(t)}{z-t}dt\right) = f(z).$$

Thus (up to $0(k, \mathbb{C})$) this construction gives a right inverse for the other assignment, and likewise one checks it gives a left inverse.

This proof hints that the map f is the primary geometric object, rather than the "dual" apparatus (B, w) used to obtain it.

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