

Geometry of Multidimensional Universes

R. Coquereaux* and A. Jadczyk**,***

CERN, CH-1211 Geneva 23, Switzerland

Abstract. Let G be a compact group of transformation (global symmetry group) of a manifold E (multidimensional universe) with all orbits of the same type (one stratum). We study G invariant metrics on E and show that there is one-to-one correspondence between those metrics and triples $(g_{\mu\nu}, A_\mu^a, h_{\alpha\beta})$, where $g_{\mu\nu}$ is a (pseudo-) Riemannian metric on the space of orbits (space-time), A_μ^a is a Yang-Mills field for the gauge group $N|H$, where N is the normalizer of the isotropy group H in G , and $h_{\alpha\beta}$ are certain scalar fields characterizing geometry of the orbits (internal spaces). The scalar curvature of E is expressed in terms of the component fields on M . Examples and model building recipes are also given. The results generalize those of non-abelian Kaluza-Klein theories to the case where internal spaces are not necessarily group manifolds.

1. Introduction

First special, and later general, theories of relativity invoked a picture of the universe as being modelled on a four-dimensional space-time manifold. On the other hand, in order to describe regularities of discrete quantum numbers characterizing elementary particles, a concept of “internal” (as opposed to “external”, i.e., space-time) symmetry, and with it that of internal space, was introduced. The idea behind what we call a “multidimensional universe” (denote it by E) is that external and internal spaces are nothing but two aspects of one geometrical entity E , and all elementary forces in nature should be but a reflection of a unique geometry. By some, not yet fully understood, mechanism, certain configurations of a simple, multidimensional field theory are distinguished, and give rise to a “spontaneous compactification” of extra dimensions (see [1] and references therein). This idea is at the root of the so-called “dimensional

* On leave of absence from Centre de Physique Théorique, F-13288 Luminy, Marseille, France

** On leave of absence from Institute of Theoretical Physics, University of Wrocław, Cybulskiego 36, PL-50-205 Wrocław, Poland

*** Work done in the framework of the Project MRI.7. of the Polish Ministry of Science, Higher Education and Technology

reduction.” As a result, the multidimensional universe splits into a four-dimensional space-time M and a compact internal space S . At the same time, the original simple field(s) on E split(s) into components which are identified with the conventional fields on M , like scalars, tensors, Yang-Mills fields (in supergravity, also spinors), etc. In theories of the Kaluza-Klein type, the “simple theory” is a multidimensional gravitation, and the “distinguished configurations” are those Riemannian metrics on E which admit a given compact group G of isometries. The case where the internal space S is a group manifold itself has been studied in many papers (see, e.g., [2–8]) and the geometrical structure of this kind of dimensional reduction is by now well understood. A G -invariant metric g_{AB} splits into a gravitational field $g_{\mu\nu}$ on M , Yang-Mills field A_μ^i with G as the gauge group, and scalar fields $h_{ij}(x)$ describing the metric in the internal space (these are generalizations of Jordan-Thierry-Brans-Dicke scalars (see [4, 6, 7])). Recently, attempts have been made [9–13] to generalize these results to a case where S is not necessarily a group manifold, but rather a homogeneous space of the type G/H . No straightforward generalization of the original Kaluza-Klein idea has, however, been obtained (in particular, E of [12] is the associated bundle so that it hardly even makes sense to consider G invariance) and the construction given in the present paper may be thought of as an alternative to those of [12, 13]. Homogeneous spaces have also been used (see [14] and references therein) to provide solutions to supergravity theories (where there are matter fields besides the metric); however, the so-called “Kaluza-Klein ansatz” used in these papers is not, in general, G invariant.

In the present paper we solve the following general problem: what are the most general algebraic and geometric properties of an extended universe E under the only requirement that G (a compact group) be a group of internal (global) symmetries? In Sect. 2 we show that E can be written *locally* as the product $M \times S$ with $S = G/H$, and that a local symmetry group $K = N|H$ arises in a natural way, N being the normalizer of H in G . In Sect. 3, we characterize all G invariant metrics on E and show that the local symmetry group K is at the same time the gauge group; we prove that there is one-to-one correspondence between G invariant metrics on E and triples $(g_{\mu\nu}, A_\mu^a, h_{\alpha\beta})$, where $g_{\mu\nu}$ is a metric on M , A_μ^a are Yang-Mills fields corresponding to the gauge group $K = N|H$, and $h_{\alpha\beta}$ are scalar fields. We also express the scalar curvature of E in terms of these fields [formulae (3.5.7) and (3.5.8)]. Examples (see Tables 1 and 2) are given in Sect. 4. We want to stress the fact that a principal bundle structure, so characteristic in mathematical descriptions of gauge fields, arises automatically and naturally in the present approach – the principal bundle emerges as a specific submanifold of the extended universe E (see Fig. 2).

Besides its physical aspects and motivations this paper contains, uses, or refers to quite a number of mathematical techniques and results. Most of them are either standard or are simple exercises in differential geometry. However, we believe that the main results of Sect. 3 are new. The reader who is not interested in a “mathematical balast” may get the idea of the present paper by reading the summaries in Sects. 2.6 and 3.6, and also Sect. 4, where examples and model building recipes are discussed.

2. Bundle Structure of the Extended Space-Time

2.1. Bundle Structure of E

Let G be a compact group of transformations of a manifold E . For each $u \in E$, let $G(u)$ denote the orbit of G through u :

$$G(u) = \{ua : a \in G\}.$$

Then $G(u)$ is a compact submanifold of E , called also a fibre or an “internal space.” When u_1 and u_2 belong to the same orbit, then their isotropy groups (“little groups”), denoted by H_{u_1} and H_{u_2} respectively, are conjugate. But isotropy groups associated to points in different orbits need not be conjugate – then E decomposes into “strata.” By the “principal orbit theorem” [15, 16] the stratum consisting of orbits with maximal dimension is an open dense submanifold of E . In our case it is natural to assume from the very beginning that E , being a model of an extended space-time, consists of one stratum only, i.e., that all isotropy groups H_u ($u \in E$) are mutually conjugate. With the above in mind, we state a theorem that we will comment on and explain later in this section:

Let E be a manifold with a right action of a compact Lie group G , and suppose that all isotropy groups H_u ($u \in E$) are conjugate to a standard one, say $H_{u_0} = H$. Let M be the set of all orbits, $G \setminus H$ the coset space of right classes Ha along H , and let N be the normalizer of H in G . Then M is a manifold and $E(M, G \setminus H)$ is an associated bundle with base M , fibre $G \setminus H$ and group $N \setminus H$.

This statement can be found, without proof, in [17, p. 276, Exercise 4.1] (see also [18, Chap. XII] and [19, p. 93]). Because of its importance for us we shall give some more information about several ingredients of the above result.

2.2. The Normalizer and the Local Symmetry Group

First of all we shall comment on the definition and meaning of N , the normalizer of H in G . It is defined as the largest subgroup of G in which H is normal [or invariant, or distingué (in French)] or, equivalently,

$$N = \{a \in G \mid aH = Ha\}.$$

Since H is normal in N , it is clear that $N \setminus H$ is a group (right and left cosets coincide). What is relevant for us is that $N \setminus H$ can be identified with the automorphism group of the homogeneous space $G \setminus H$. Here by an automorphism of $G \setminus H$ we mean an invertible mapping $\alpha : G \setminus H \rightarrow G \setminus H$, which commutes with the right action of G :

$$\alpha([a]b) = \alpha([a])b,$$

where $[a] = Ha \in G \setminus H$ and $b \in G$.

To see the relation between $N \setminus H$ and the automorphism group of $G \setminus H$, observe that for every $n \in N$ the mapping $\alpha_n : G \setminus H \rightarrow G \setminus H$, defined by

$$\alpha_n([a]) \doteq [na],$$

commutes with the right action of G on $G \setminus H$, and depends on the equivalence class $[n]$ of n in $N \setminus H$ only. Conversely, given an automorphism α of $G \setminus H$, let $n \in G$ be

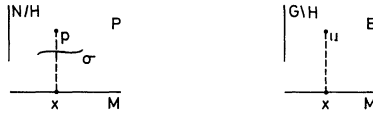


Fig. 1

such that $\alpha(H) = Hn$. Then $\alpha(Ha) = \alpha(H)a = Hna$ for all $a \in G$. In particular for $a = h \in H$ we get $Hnh = Hn$, i.e., $n \in N$. It follows that α is of the form α_n , as above. It is precisely because of this identification of elements of $N|H$ with automorphism of $G \setminus H$ that $N|H$ plays the role of local symmetry group (local gauge group) in our framework.

2.3. Construction of an Associated Bundle $E(G \setminus H, M)$

Let us consider a principle bundle $P(N|H, M)$ with a base M and structure group $N|H$. We shall discuss here the geometry involved in the constructions of the associated bundle $E(G \setminus H, M)$. The procedure described is nothing but a particular case of the standard construction of an associated bundle from a principal one (see, e.g., [20, Chap. XVI.14]).

In the direct product $P \times G \setminus H$, define the following relation

$$(p, [a]) \sim (p', [a']) \Leftrightarrow \exists [n] \in N|H \text{ so that } p' = p[n], \quad [a'] = [n^{-1}][a],$$

where $p[n]$ is obtained from p by using the right action of $[n] \in N|H$ on the principal bundle P , and $[n^{-1}][a] = [n^{-1}a]$. In other words, $N|H$ acts on the typical fibre $G \setminus H$ by automorphisms, those discussed above: $[a] = \alpha_n([a'])$. It is easy to see that the above relation is an equivalence relation. We shall denote an equivalence class by the symbol $p \cdot [a]$. Let us recall what is the intuitive meaning of this: writing $u = p \cdot [a]$ means that the “geometrical object” u has “co-ordinate” $[a]$ in the “frame” p . Of course, $u = p \cdot [a] = (p[n^{-1}]) \cdot ([n][a])$, so that the group $N|H$ plays the role of the group of transformations of “frames”. The space of equivalence classes [quotient of $(P \times G \setminus H)$ by the equivalence relation] is, by definition, the associated fibre bundle $E = E(M, G \setminus H)$ with “geometrical objects” u in the fibre, M in the basis and transition functions valued in $N|H$. The situation is schematically described in Fig. 1.

In a local trivialization determined by a local cross-section σ (gauge), the element p of P can be represented as $p = (x, [n])$ (i.e., $p = \sigma(x)[n]$), where $x \in M$ and $[n] \in N|H$. The element $u \in E$ can be written as (x, y) , where $y = [a] \in G \setminus H$ and $u = \sigma(x) \cdot [a]$.

2.4. Global Action of G on $E(G \setminus H, M)$

Let us ask what are the transformations $\beta: G \setminus H \rightarrow G \setminus H$ of the typical fibre which pass through the equivalence relation defining E to induce transformations of E itself. If $u = (x, y)$ and we want to define $\beta(u)$ as $(x, \beta(y))$, we must check that this definition is gauge independent, i.e., that $p \cdot \beta[a] = p[n^{-1}] \cdot \beta([n][a])$ or $\beta([n][a]) = [n]\beta([a])$. In other words, the left action of the structure group must commute with β . Since the structure group is, in general, non-abelian, it is clear that β cannot be the left multiplication by $N|H$. But it can well be the right multiplication by an element of G so that $ug = (p \cdot [a])g = (p \cdot [ag])$, since we have shown in Sect. 2.2 that $N|H$ is precisely the set of all transformations of $G \setminus H$ which

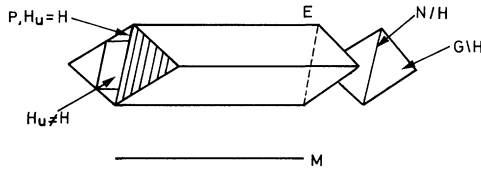


Fig. 2

commute with such $\beta - s$. The situation is now the following: we have a (non-principal) fibre bundle $E(G \setminus H, M)$ with structure group $N|H$, with G acting on E from the right and operating transitively on each fibre, so that the fibres of E coincide with the orbits of G . However, this action is not free; indeed, if $u = p \cdot [a] \in E$, then the isotropy group of u is $H_u = a^{-1}Ha$. In particular, all isotropy subgroups are mutually conjugate. Here and below e denotes the identity of G .

We will now show how P can be identified with a submanifold of E . For each $p \in P$, let $u(p) \in E$ be defined by $u(p) = p \cdot [e]$, i.e., $u(p)$ is the geometrical object uniquely defined by the requirement that it has “co-ordinates” $[e]$ in the “frame” p . It is easy to see that the mapping $p \rightarrow u(p)$ from P into E is an embedding, which also satisfies $u(p[n]) = u(p) \cdot [n]$ for all $p \in P, n \in N$. In particular $u \in E$ is of the form $u = u(p)$ for some $p \in P$ if and only if the isotropy group H_u of u is precisely H .

2.5. Bundle Structure of E (cont.)

Let us summarize the discussion given in Sects. 2.2–2.4: given a compact Lie group, G and a subgroup $H \subset G$, we started with a principal bundle $P(N|H, M)$, N being the normalizer of H in G , and showed that on the associated bundle $E(G \setminus H, M)$ the group G operates from the right in a natural way. All isotropy groups of G in E are mutually conjugate and P can be identified with a subset (immersed submanifold) of E on which all isotropy groups are exactly H . With the above construction in mind, it is now easy to understand why the theorem stated in Sect. 2.1 holds true. Suppose that E is a manifold with right action of G , and in such a way that all the isotropy groups H_u are conjugate to a standard one $H = H_{u_0}$. Let M be the space of all orbits, and let P be defined as the set of all $u \in E$ such that $H_u = H: P \doteq \{p \in E | pH = p\}$. Observe that if $p \in P, a \in G$, then $pa \in P$ if and only if $a \in N$ and, since H acts trivially on the points of P , it is the quotient group $N|H$ which freely acts on the fibres of P . Therefore P has the algebraic structure of the principal bundle with base M and structure group $N|H$. It is also clear that E can be identified with the bundle associated to P via the natural action of $N|H$ on $G \setminus H$. Indeed, we have a natural surjection $P \times G \setminus H \rightarrow E$ given by

$$(p, [a]) \mapsto p \cdot [a] \doteq pa, \quad p \in P \subset E, \quad a \in G.$$

Analytically one needs the so-called “slice theorem” [18, Chap. VIII] (see also [21] for the non-compact case).

The resulting structure is represented in Fig. 2.

2.6. Summary

We shall now summarize the main ideas of this section in plain terms. We started with an extended space-time E with a fixed, *global*, compact, internal symmetry

(classification) group G . The orbits of G are internal spaces of E . They are homogeneous spaces of G and, under very mild assumptions, one can *locally* represent E as a product $M \times G \backslash H$. The group G operates on E just by right translations: if $p = (x, [y])$ is a point in $M \times G \backslash H$, then a group element $a \in G$ transforms p into $(x, [ya])$. Here $[y]$ denotes the right coset $[y] = Hy$, $y \in G$. We have also introduced *local* transformations $(x, [y]) \rightarrow (x, f(x, [y]))$ characterized by the fact that they *commute* with the global ones. We have shown that every local transformation is described by a function $x \rightarrow n(x)$, where $n(x)$ belong to the group $N|H$, N being the normalizer (see the definition in Sect. 2.2) of H in G . In a particular, well understood, case where the internal space is a *group manifold* (i.e., $H = \{e\}$ and $G \backslash H = G$) the global and local symmetry groups happen to be isomorphic, the first acting on the internal space from the right and the second from the left. If the internal space is a homogeneous space $G \backslash H$ with a non-trivial isotropy group H , then the situation becomes more complicated and the resulting local symmetry group is no longer isomorphic to the global one. Examples are given in Sect. 4. It will be seen in the next section that the local symmetry group, identified here as $N|H$ for geometrical reasons, will play the role of a gauge group, both kinematically and dynamically.

3. Metric and Curvature

3.1. Decomposition of \mathcal{G}

We first introduce convenient decompositions of the Lie algebra $\mathcal{G} = \text{Lie}(G)$. To make the discussion simple, both H and G are assumed to be compact, connected Lie groups, with $H \subset G$. Since G is compact, we can choose a bi-invariant Riemannian metric on G or, equivalently, an $\text{Ad}G$ invariant p -definite scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ in \mathcal{G} (if G is simple then $\langle\langle \cdot, \cdot \rangle\rangle$ is unique up to a multiplication by a positive constant). Let $\mathcal{H} = \text{Lie}(H)$ and denote by \mathcal{S} the orthogonal complement of \mathcal{H} in \mathcal{G} . Since $\langle\langle \cdot, \cdot \rangle\rangle$ is, in particular, $\text{Ad}H$ invariant, we get what is called a reductive pair [22, Vol. II, Chap. X.2]

$$\mathcal{G} = \mathcal{H} + \mathcal{S}, \quad (\text{Ad}H)(\mathcal{S}) = \mathcal{S}. \quad (3.1.1)$$

It is then possible to identify \mathcal{S} , endowed with the above linear action of H , with the vector space tangent to the homogeneous space $S \doteq G \backslash H$ at the origin. For a given homogeneous space, the above reductive decomposition need not be unique, however, this non-uniqueness will not cause any trouble in the following.

With $\mathcal{N} \doteq \text{Lie}(N)$ being the Lie algebra of the normalizer N of H in G , we decompose \mathcal{N} into

$$\mathcal{N} = \mathcal{H} + \mathcal{K}, \quad (3.1.2)$$

where

$$\mathcal{K} \doteq \mathcal{N} \cap \mathcal{S} \quad (3.1.3)$$

is the orthogonal complement of \mathcal{H} in \mathcal{N} . Again we have a reductive pair $(\text{Ad}H)(\mathcal{K}) = \mathcal{K}$, but now, since H is normal in N , we also have $[\mathcal{H}, \mathcal{N}] \subset \mathcal{H}$, what implies $[\mathcal{H}, \mathcal{K}] \subset \mathcal{H}$. Since $\mathcal{K} \cap \mathcal{H} = 0$, we get

$$[\mathcal{H}, \mathcal{K}] = 0. \quad (3.1.4)$$

It is useful at this point to introduce the centralizer (or commutant) of \mathcal{H} in \mathcal{G} :

$$\mathcal{Z}_H \doteq \{x \in \mathcal{G} \mid [x, \mathcal{H}] = 0\}. \quad (3.1.5)$$

The centralizer contains, in particular, the centre \mathcal{C}_H of \mathcal{H} :

$$\mathcal{C}_H \doteq \mathcal{Z}_H \cap \mathcal{H}. \quad (3.1.6)$$

It follows from (3.1.3) and (3.1.4) that $\mathcal{H} \subset \mathcal{S} \cap \mathcal{Z}_H$, and it is easy to see (compare also [23, Sect. 3.2.44, Theorem 2]) that in fact

$$\mathcal{H} = \mathcal{S} \cap \mathcal{Z}_H. \quad (3.1.7)$$

In other words \mathcal{H} is composed of those vectors which are tangent to $G \setminus H$ at the origin and are invariant under H . (This remark will be important in Sect. 4, when we shall count the scalar fields.) Actually \mathcal{H} , being an orthogonal complement (with respect to a bi-invariant metric) of the ideal \mathcal{K} , is itself an ideal and, *a fortiori*, Lie subalgebra of \mathcal{N} . We can therefore identify, \mathcal{H} with the Lie algebra $\text{Lie}(N|H)$, and for obvious notational reasons we call

$$K \doteq N|H. \quad (3.1.8)$$

Finally, we observe that

$$\mathcal{Z}_H = \mathcal{H} + \mathcal{C}_H. \quad (3.1.9)$$

Indeed, it follows from (3.1.1)–(3.1.7) that $\mathcal{H} + \mathcal{C}_H \subset \mathcal{Z}_H$, so that it is enough to show that $\mathcal{Z}_H \subset \mathcal{H} + \mathcal{C}_H$. Let $x \in \mathcal{Z}_H$ and decompose $x = x_H + x_S$ with $x_H \in \mathcal{H}$ and $x_S \in \mathcal{S}$. Then, for every $y \in \mathcal{H}$, we have $[x_H, y] = [x, y] - [x_S, y]$, where the first term vanishes (since $x \in \mathcal{Z}_H$), and, by (3.1.1), the second term is in \mathcal{S} . On the other hand, since \mathcal{H} is an algebra, $[x_H, y]$ is in \mathcal{H} , and so $[x_H, y] \in \mathcal{H} \cap \mathcal{S} = 0$. It follows that $x_H \in \mathcal{C}_H$ and therefore $x_S = x - x_H \in \mathcal{S} \cap \mathcal{Z}_H = \mathcal{H}$. In a particular case, if H is semisimple [i.e., if H contains no direct $U(1)$ factors] then the Lie algebras of $K = N|H$ and of the centralizer of H are isomorphic.

Finally, we introduce \mathcal{L} , the orthogonal complement of \mathcal{N} in \mathcal{G} :

$$\mathcal{G} = \mathcal{N} + \mathcal{L}, \quad (\text{Ad } N)\mathcal{L} = \mathcal{L}. \quad (3.1.10)$$

In particular,

$$\mathcal{S} = \mathcal{H} + \mathcal{L}, \quad (\text{Ad } H)\mathcal{L} = \mathcal{L}, \quad (3.1.11)$$

and it should be noticed [compare also the discussion in 3.3 Ad*i*] that, in fact, \mathcal{L} is orthogonal to \mathcal{H} with respect to *any* $\text{Ad } H$ invariant scalar product on \mathcal{S} . This follows from the fact, that according to (3.1.7), the representation of H on \mathcal{L} does not contain the trivial representation.

In practice, in particular in model-building as discussed in Sect. 4, one often starts with a direct product $H \cdot K$ of two subgroups H and K of G and the question arises whether $H \cdot K$ is the normalizer N of H in G or not. From Eqs. (3.1.7) and (3.1.2) one can deduce the following criterion: $\mathcal{N} \doteq \mathcal{H} + \mathcal{K}$, $[\mathcal{H}, \mathcal{K}] = 0$, is the normalizer of \mathcal{H} in \mathcal{G} if and only if

$$\mathcal{L} \cap \mathcal{Z}_H = 0, \quad (3.1.12)$$

where \mathcal{L} is defined as the orthogonal complement of \mathcal{N} in \mathcal{G} . In particular, if the present representation of $H \cdot K$ on \mathcal{L} is faithful and irreducible, then $\mathcal{N} = \text{Lie}(H \cdot K)$ is the normalizer of \mathcal{H} in \mathcal{G} ; the normalizer N of H in G is in that case equal to $H \cdot K$ modulo a discrete group (indeed N is very often not connected).

The whole story can be visualized and summarized as follows :

$$\mathcal{G} = \overbrace{\mathcal{H} \oplus \mathcal{H}}^{\mathcal{N}} \oplus \underbrace{\mathcal{L}}_{\mathcal{S}}$$

$$\begin{aligned} [\mathcal{H}, \mathcal{H}] &\subset \mathcal{H} \\ [\mathcal{H}, \mathcal{H}] &\subset \mathcal{H} \\ [\mathcal{H}, \mathcal{H}] &= 0 \\ [\mathcal{H}, \mathcal{L}] &\subset \mathcal{L} \\ [\mathcal{H}, \mathcal{L}] &\subset \mathcal{L} \end{aligned}$$

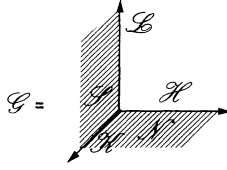


Fig. 3

$$\begin{aligned} \mathcal{G} &= \text{Lie}(G) \\ \mathcal{H} &= \text{Lie}(H) \\ \mathcal{N} &= \text{Lie}(N) \\ \mathcal{H} &\simeq \text{Lie}(N|H) \\ \mathcal{S} &= T_0(S = G \setminus H) \\ \mathcal{L}_H &= \text{centralizer of } \mathcal{H} \text{ in } \mathcal{G} \\ \mathcal{C}_{\mathcal{H}} &= \text{centre of } \mathcal{H} \end{aligned}$$

N is locally a direct product of two normal subgroups: $N \simeq HK$ (it may be disconnected)

3.2. Adapted Basis

We fix once and for all a basis T_i in \mathcal{G} , with $[T_i, T_j] = C_{ij}^k T_k$, adapted to the decompositions

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{S} \quad \text{and} \quad \mathcal{S} = \mathcal{H} \oplus \mathcal{L}, \tag{3.2.1}$$

and introduce notation

$$T_i = \{T_{\hat{\alpha}} \in \mathcal{H}, T_{\alpha} \in \mathcal{S}\} \quad \text{and} \quad T_{\alpha} = \{T_{\hat{\alpha}} \in \mathcal{H}, T_{\alpha} \in \mathcal{L}\}, \tag{3.2.2}$$

distinguishing between basis vectors in the different components of \mathcal{G} . For each $a \in G$, let $A(a)_i^j$ denote the matrix of the adjoint representation of G :

$$(\text{Ad } a)T_i = aT_i a^{-1} = A(a)_i^j T_j. \tag{3.2.3}$$

It should be noticed that for $n \in N$ the matrix $A(n)$ has the following structure

$$A(n) = \left[\begin{array}{c|c} A_{\hat{\beta}}^{\hat{\alpha}}(n) & 0 \\ \hline 0 & A_{\beta}^{\alpha}(n) \end{array} \right]. \tag{3.2.4}$$

Indeed, if $n \in N$, then $nHn^{-1} = H$ and, infinitesimally $nT_{\hat{\alpha}}n^{-1} = A(n)_{\hat{\alpha}}^{\hat{\beta}} T_{\hat{\beta}}$, in particular $A(n)_{\hat{\alpha}}^{\hat{\beta}} = 0$.

This implies that the submatrices

$$(A(n)_{\beta}^{\alpha}) \doteq \mathbb{R}(n) \tag{3.2.5}$$

form a representation of N in $\mathcal{S} = \mathcal{H} + \mathcal{L}$.

3.3. Vertical Moving Frame

Let ε_i be the fundamental vector fields on E corresponding to infinitesimal transformations of E generated by T_i :

$$\varepsilon_i(u) = \frac{d}{dt}(ue^{tT_i})|_{t=0}, \quad u \in E. \tag{3.3.1}$$

(These vectors are Killing vector fields of every G invariant metric.) Since G acts on E from the right, we have

$$[\varepsilon_i, \varepsilon_j](u) = C_{ij}^k \varepsilon_k(u), \quad u \in E. \tag{3.3.2}$$

It should be noticed that, at any given $u \in E$, the family $\varepsilon_i(u)$ is overcomplete in the vertical tangent space at u . Something special happens, however, on the submanifold $P \subset E$: the vector fields $\varepsilon_{\hat{\alpha}}$, included in the family $\{\varepsilon_i\}$, vanish on P , whereas the ε_α -s are linearly independent at every point of P . (We remind the reader that the principal bundle P was defined in Sect. 2.5 as the set of all points invariant under H .)

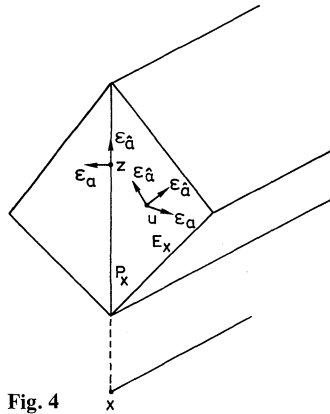


Fig. 4

$$T_i \left\{ \begin{array}{l} T_{\hat{\alpha}} \in \mathcal{H} \longrightarrow \varepsilon_{\hat{\alpha}} \\ T_{\hat{\alpha}} \in \mathcal{K} \longrightarrow \varepsilon_{\hat{\alpha}} \\ T_a \in \mathcal{L} \longrightarrow \varepsilon_a \end{array} \right\} \varepsilon_i$$

Now, since the vector fields ε_α are linearly independent on P , they are also independent in some neighbourhood U of P in E . In particular, the commutator of two vertical fields being again vertical, we have

$$[\varepsilon_\alpha, \varepsilon_\beta](u) = f_{\alpha\beta}^\gamma(u) \varepsilon_\gamma(u), \quad u \in U, \tag{3.3.3}$$

where the $f_{\alpha\beta}^\gamma$ are the structure functions of the vertical moving frame ε_α ; notice that they depend generally on the point u at which they are calculated. However, they are constant on P :

$$f_{\alpha\beta}^\gamma(p) \equiv C_{\alpha\beta}^\gamma \quad \text{for } p \in P. \tag{3.3.4}$$

Notice also that the property for ε_α to constitute a moving frame in the vertical space fails if one goes too far from P : the vectors ε_α may become linearly dependent there.

From the Definition (3.3.1), and taking into account that $\varepsilon_{\hat{\alpha}}(z) = 0$ for $z \in P$, we get the relation

$$\varepsilon_\alpha(pn) = R(n)_{\alpha}^{\beta} \varepsilon_\beta(p), \quad p \in P, \quad n \in N, \tag{3.3.5}$$

which will be used in (3.4.4).

For calculation purposes, one also needs the expression of $\varepsilon_\alpha(f_{\beta\gamma}^\delta)$ in terms of the structure constants C_{ij}^k of the group G . Indeed, when one wants to compute geometrical quantities [such as, e.g., scalar curvature of a homogeneous space (3.5.8)], one performs the calculation at a point $z \in P$, but what is needed are the values of the structure functions *and* their derivatives in the directions transversal to P . To obtain such a formula we first observe that, from (3.3.2) and (3.3.3), we have

$$f_{\beta\gamma}^\delta(u)\varepsilon_\delta(u) = C_{\beta\gamma}^i \varepsilon_i(u). \quad (3.3.6)$$

Then, taking the commutator of both sides with ε_α and using (3.3.4), we get

$$\varepsilon_\alpha(f_{\beta\gamma}^\delta)(p) = C_{\beta\gamma}^\delta C_{\alpha\delta}^\delta, \quad p \in P. \quad (3.3.7)$$

In particular, owing to (3.1.4), we get $\varepsilon_\alpha(f_{\beta\gamma}^\delta)(p) = 0$ in agreement with (3.3.4).

3.4. Characterization of G Invariant Metrics on E

Let g now be a G invariant metric on E . We are going to show that it determines and is determined by

- i) a G invariant metric h_x on every fibre E_x of E ;
- ii) a G invariant horizontal distribution $(Z_u)_{u \in E}$ on $E(G \setminus H, M)$ or, equivalently, a principal connection in $P(N|H, M)$;
- iii) a metric γ on M .

A given manifold may not always admit a non-degenerate pseudo-Riemannian metric of given signature. In the following we shall always assume that problems of this type do not arise.

In subparagraphs Ad i)–Ad iii) we prove the direct proposition, while in Ad iv) we will show the converse.

Ad i) g being a metric on E , we know, *a fortiori*, how to compute the scalar product of two vertical vectors. Since g is G invariant, we obtain a G invariant metric h_x on every fibre E_x of E .

Notice that a co-ordinate representation of h_x is obtained as follows: choose a local cross-section (gauge) of P , $\sigma: M \rightarrow P$ – such a section “marks the origin” on each fibre – then define $h_{\alpha\beta}(x) \doteq g_{\alpha\beta}(\sigma(x))$, where $g_{\alpha\beta}(p) \doteq g_p(\varepsilon_\alpha(p), \varepsilon_\beta(p))$, $p \in P$, are numerical functions on P . The matrix $h = (h_{\alpha\beta})$, being associated with a G invariant metric on a space isomorphic to $G \setminus H$ is automatically $\text{Ad}H$ invariant [22, Vol. II, Chap. X, Proposition 3.1], and therefore satisfies the constraints.

$$\mathbb{R}(a)^T \text{lh} \mathbb{R}(a) = \text{lh}, \quad a \in H, \quad (3.4.1)$$

where $\mathbb{R}(a)$, $a \in H \subset N$, is given by (3.2.5).

According to (3.4.1) the matrix $\text{lh}(x)$ defines an $\text{Ad}H$ invariant scalar product in \mathcal{S} . In (3.1.11) we have defined a splitting of \mathcal{S} into the direct sum of two subspaces \mathcal{K} and \mathcal{L} , using an auxiliary bi-invariant metric on G . We now realize that \mathcal{K} and \mathcal{L} are also orthogonal with respect to the scalar product induced by $h_{\alpha\beta}(x)$. Indeed, \mathcal{K} and \mathcal{L} carry disjoint representations of H and therefore they are orthogonal with respect to every $\text{Ad}H$ invariant scalar product (see, e.g., [24, Chap. VIII, Sect. 3])

$$\text{lh} = \begin{pmatrix} h_{ab} & 0 \\ 0 & h_{ab} \end{pmatrix}. \quad (3.4.2)$$

Owing to the formulae (3.2.5) and (3.1.4), we also have

$$\mathbb{R}(a) = \begin{pmatrix} \delta_b^a & 0 \\ 0 & R_b^a \end{pmatrix}, \tag{3.4.3}$$

so that the constraints (3.4.1) are effective on h_{ab} only.

The field $x \rightarrow h_x$ of G invariant metrics on fibres of E can be identified with a cross-section of a bundle associated to P . Actually, by (3.3.5) we find

$$g_{\alpha\beta}(pn) = R(n)_\alpha^{\alpha'} R(n)_{\beta'}^{\beta} g_{\alpha'\beta'}(p), \quad n \in N, \tag{3.4.4}$$

and

$$g_{\alpha\beta}(p) = R(h)_\alpha^{\alpha'} R(h)_{\beta'}^{\beta} g_{\alpha'\beta'}(p), \quad h \in H, \tag{3.4.5}$$

so that the numerical functions $g_{\alpha\beta}(z)$ on P can indeed be identified with sections of an appropriate associated bundle [22, Vol. I, Chap. II, Example 5.2]. The relations (3.4.4) and (3.4.5) define the type of transformations of these scalar fields under $N|H$ gauge transformations.

Notice that it is enough to know the functions $h_{\alpha\beta}(x) = g_{\alpha\beta}(\sigma(x))$, where $\sigma: M \rightarrow P$ is a local gauge. Using G invariance and the constraints (3.4.1), $h_{\alpha\beta}$ can be then unambiguously propagated all over the bundle E .

Ad ii) For every $u \in E$, let V_u denote the internal, or vertical, tangent space, i.e., subspace of $T_u(E)$ consisting of all vectors tangent to the fibre at u . Let Z_u be the orthogonal complement of V_u in $T_u(E)$ with respect to our given G invariant metric g . Since g is G invariant, we have $(Z_u)a = Z_{ua}$, $a \in G$, i.e., we have a G invariant horizontal distribution on E . In order to show that the distribution determines a principal connection, we must prove that, at $p \in P$, Z_p is tangent to P . This follows by observing that Z_p is orthogonal to the vectors $\varepsilon_a(p)$ which span the orthogonal complement to P at p . Now, $(Z_p)_{p \in P}$ is an N , and therefore also an $N|H$ invariant horizontal distribution on P , i.e., a principal connection [22, Vol. I, Chap. II.1].

Ad iii) The scalar product $\gamma_x(v, w)$ of two vectors, tangent to M at a point x , is obtained as follows: choose an arbitrary point u in the orbit labelled by x , and let v^* and w^* be the vectors in Z_u which project onto v and w respectively. Then define $\gamma_x(v, w) \doteq g_u(v^*, w^*)$. The result is independent of the choice u on the orbit because of G invariance of the metric g .

Ad iv) Conversely, it is easy to see that given a metric γ on M , principal connection $(Z_p)_{p \in P}$, and G invariant metrics h_x on the fibres of E , one constructs a G invariant metric on E . Indeed, given $\xi, \eta \in T_u E$, let $p \in P$ be such that $u = pa$ for some $a \in G$, and let $\pi(\xi)$ and $\pi(\eta)$ denote the projections of ξ and η on M . Denote by ξ^* and η^* the horizontal lifts of $\pi(\xi)$ and $\pi(\eta)$ to Z_p . Then the vectors $\xi - \xi^*a$ and $\eta - \eta^*a$ are vertical, and the scalar product of ξ and η in E can be defined as

$$g_u(\xi, \eta) = \gamma_x(\pi(\xi), \pi(\eta)) + h_x(\xi - \xi^*a, \eta - \eta^*a). \tag{3.4.6}$$

3.5. Curvature

Let x^μ be a co-ordinate system on M , and let e_μ be the horizontal lift of the vector fields ∂_μ from M to E . The vectors $e_A = (e_\mu, \varepsilon_\alpha)$ then form a basis at every point

$u \in U$, where U is an appropriate neighbourhood of P . By their very definition the vector fields e_μ are orthogonal to ε_α , and the metric takes the form

$$g_{AB} = \begin{pmatrix} g_{\mu\nu}(x), & 0 \\ 0, & g_{\alpha\beta}(x, y) \end{pmatrix}, \quad (3.5.1)$$

where $(x, y) \in M \times G \setminus H$ is a local product representation of E in a given gauge $\sigma: M \rightarrow P$. For $p = (x, \sigma(x) = g) \in P$, the connection form can be written

$$\omega(p) = g^{-1} A_\mu(x) dx^\mu g + g^{-1} dg \quad \text{with} \quad A_\mu(x) = A_\mu^{\hat{a}}(x) T_{\hat{a}},$$

and the horizontal lift $e_\mu(p)$ can be written

$$e_\mu(p) = (\partial_\mu)_p - A_\mu^{\hat{a}}(x, g) \varepsilon_{\hat{a}}(g), \quad (3.5.2)$$

where

$$A_\mu(x, g) = g^{-1} A_\mu(x) g \quad \text{and} \quad A_\mu(x, g) = A_\mu^{\hat{a}}(x, g) T_{\hat{a}}.$$

Therefore the inverse metric in P can be written as

$$g^{-1} = \gamma^{\mu\nu}(x) (\partial_\mu - A_\mu^{\hat{a}}(x, g) \varepsilon_{\hat{a}}(g)) \otimes (\partial_\nu - A_\nu^{\hat{b}}(x, g) \varepsilon_{\hat{b}}(g)) + h^{\hat{a}\hat{b}}(x, g) (\varepsilon_{\hat{a}}(g) \otimes \varepsilon_{\hat{b}}(g)).$$

It is enlightening – although unnecessary – to express g^{-1} in P in terms of the vector fields $e_{\hat{a}}(g) = g^{-1} \varepsilon_{\hat{a}}(g) g$ (which are only defined through the choice of a gauge, and satisfy $[\varepsilon_{\hat{a}}, e_{\hat{b}}](g) = 0$). These vector fields $e_{\hat{a}}$ can be thought of as right invariant vector fields in the copy of $K = N \setminus H$ above x . Then one obtains on P

$$g^{-1} = \gamma^{\mu\nu}(x) (\partial_\mu - A_\mu^{\hat{a}}(x) e_{\hat{a}}(g)) \otimes (\partial_\nu - A_\nu^{\hat{b}}(x) e_{\hat{b}}(g)) + h^{\hat{a}\hat{b}}(x) (e_{\hat{a}}(g) \otimes e_{\hat{b}}(g)).$$

This writing clearly exhibits the G invariance (but destroys the explicit gauge invariance).

The commutation relations of the basis e_A are

$$[e_\mu, e_\nu] = f_{\mu\nu}^\alpha \varepsilon_\alpha, \quad (3.5.3)$$

$$[e_\mu, \varepsilon_\alpha] = 0, \quad (3.5.4)$$

$$[\varepsilon_\alpha, \varepsilon_\beta] = f_{\alpha\beta}^\gamma \varepsilon_\gamma, \quad (3.5.5)$$

with

$$f_{\mu\nu}^{\hat{a}}(p) = -F_{\mu\nu}^{\hat{a}}(p), \quad p \in P, \quad (3.5.6)$$

where $F_{\mu\nu}^{\hat{a}}$ are the components of the curvature two-form of the connection D_μ . Information about the structure functions $f_{\alpha\beta}^\gamma$ on P is given by (3.3.4) and (3.3.7). These are the necessary ingredients for calculation of the scalar curvature R of the Levi-Civita connection of g_{AB} . Taking also into account the fact that ε_α are Killing vector fields for $g_{\alpha\beta}$, the result is

$$\begin{aligned} R(E) = R(M) + R(G \setminus H) - \frac{1}{4} F_{\mu\nu}^{\hat{a}} F_{\hat{a}}^{\mu\nu} - \frac{1}{2} h^{\alpha\beta} h^{\gamma\delta} (D_\mu h_{\alpha\gamma} D^\mu h_{\beta\delta} \\ + D_\mu h_{\alpha\beta} D^\mu h_{\gamma\delta}) - \nabla_\mu (h^{\alpha\beta} D_\mu h_{\alpha\beta}). \end{aligned} \quad (3.5.7)$$

Here $R(M)$ is the scalar curvature of the Levi-Civita connection ∇_μ of $g_{\mu\nu}$, D_μ and $F_{\mu\nu}^{\hat{a}}$ are, respectively, the covariant derivative and the curvature of the principal

connection A_μ^a and $R(G\backslash H)$ is given by

$$R(G\backslash H) = h^{\beta\beta'} \left(\frac{1}{2} C_{\alpha\beta}^\gamma C_{\beta'\gamma}^\alpha - \frac{1}{4} h^{\alpha\alpha'} h_{\gamma\gamma'} C_{\alpha\beta}^\gamma C_{\alpha'\beta'}^{\gamma'} + C_{\alpha\beta}^{\hat{\alpha}} C_{\beta'\hat{\alpha}}^\alpha - C_{\alpha\beta}^\alpha C_{\gamma\beta'}^\gamma \right). \quad (3.5.8)$$

The indices μ, ν are lowered with the metric $g_{\mu\nu}$, while the indices α, β (respectively a, b) are lowered with the help of the matrix $h_{\alpha\beta}$ (respectively h_{ab}) given by (3.4.2). When G is unimodular (in particular when G is compact), the last term in (3.5.8) vanishes. If H is trivial ($H = \{e\}$), the third term of (3.5.8) vanishes and if $G\backslash H$ is a symmetric space ($[\mathcal{L}, \mathcal{L}] \subset \mathcal{H}$), the first two terms of (3.5.8) vanish. The last term in (3.5.7) gives rise to a total derivative in the Lagrangian $R\sqrt{g(M)}$ and can be neglected. While varying this Lagrangian the constraints (3.4.1) are to be taken into account.

Expression (3.5.4) agrees with the scalar curvature given in [4], where the group case is treated (H trivial) although it is not written in the same way; however, we believe that there is an erroneous sign inside the kinetic term of the scalar fields in [13].

3.6. Summary

We summarize the results given in this section. We have considered an extended space-time E having a local product structure $E \simeq M \times G\backslash H$, with global symmetry group G acting on E from the right. We have shown that there is one-to-one correspondence between G invariant Riemannian metrics g_{AB} on E and triples $(g_{\mu\nu}, A_\mu^{\hat{a}}, h_{\alpha\beta})$, where $g_{\mu\nu}$ is a Riemannian metric on M , $A_\mu^{\hat{a}}$ are gauge potentials corresponding to the local symmetry group $N|H$, and $h_{\alpha\beta}$ are certain scalar fields determining an $\text{Ad}H$ invariant metric on $\mathcal{G}\backslash\mathcal{H}$. The curvature $R(E)$ of the metric g_{AB} splits into

- a) $R(M)$ – the curvature of space-time metric $g_{\mu\nu}$,
- b) $R(G\backslash H)$ – the curvature of the G invariant metric on $G\backslash H$ which gives the potential term for scalars $h_{\alpha\beta}$,
- c) Yang-Mills Lagrangian of $A_\mu^{\hat{a}}$,
- d) kinetic term for the scalars $h_{\alpha\beta}$,

[see the formulae (3.5.7) and (3.5.8)]. When varying the scalars $h_{\alpha\beta}$, the constraints (3.4.1) have to be taken into account. To make the discussion given in this paper as simple as possible, we have considered Riemannian metrics instead of vielbeins. In many cases, especially when spinors are taken into account, the vielbein formalism is a necessity. In such a case we would have $h(x) = \delta_{\alpha\beta} \psi^\alpha(x) \otimes \psi^\beta(x)$, with $\psi^\alpha(x)$ being the soldering forms (vielbein) on the internal space at x . Taking into account the fact that ε_α can be thought of as an orthogonal basis on $G\backslash H$ with respect to a fixed metric $\langle\langle, \rangle\rangle$, induced by a bi-invariant metric on G , it is natural to write $\psi^\alpha(x) = \theta^\alpha(x) + \phi^\alpha(x)$, where θ^α are the duals of ε_α . The $\{\psi^\alpha\}$ constitute a set of $s = \dim(G\backslash H)$ one-forms, orthogonal with respect to h , and their deviations ϕ^α play the role of Higgs fields.

4. Comments and Examples

4.1. Counting the Number of Scalar Fields

The set of all G invariant metrics which can be defined on a given homogeneous space $S = G\backslash H$ is itself a (connected) manifold, which we shall call $R(G; S)$. The

reader should be aware of the fact that a given manifold S may admit several homogeneous structures [for example, the homogeneous spaces $SO(8)/SO(7)$, $SU(4)/SU(3)$, $Spin\ 7/G_2$, $USp(4)/USp(2)$ are all diffeomorphic to the standard seven-sphere S^7]; therefore we stress the fact that, in the following, we will consider a manifold S with a given homogeneous structure $G\backslash H$, and we shall consider those metrics on S which are invariant with respect to the action of G on S . Both G and H are assumed to be compact and connected.

It is well known [22] that G invariant metrics on $G\backslash H$ are in one-to-one correspondence with AdH invariant bilinear symmetric forms on the tangent space \mathcal{S} at the origin of $S=G\backslash H$. Indeed, owing to the transitivity of G action, one can transport such a scalar product from the origin to any point of S , and the transport is unambiguous because of the assumed AdH invariance. In order to find the dimension d of the manifold $R(G, S)$, one has therefore to decompose the representation AdH on the vector space \mathcal{S} into irreducible ones:

$$\mathcal{S} = \bigoplus_i (V_i \otimes \mathbb{R}^{r_i}), \quad (4.1.1)$$

where the index i runs over inequivalent irreducible representations of H on V_i , and r_i is the multiplicity with which V_i occurs in S . Since the dimension d of $R(G; S)$ is equal to the dimension of the space of symmetric operators on \mathcal{S} commuting with the representation AdH , it follows that

$$d = \sum_i \frac{r_i(r_i + 1)}{2}. \quad (4.1.2)$$

From the formula (3.1.7) it follows that the dimension k of the gauge group K coincides with r_0 , where $i=0$ denotes the trivial representation of H

$$k = \dim \mathcal{K} = r_0. \quad (4.1.3)$$

It is sometimes natural to restrict the attention to G invariant metrics on S with a fixed volume element. We shall denote by d_0 the dimension of the manifold of conformal equivalence classes of G invariant metrics on $G\backslash H$ ($d_0 = d - 1$).

In order to find out the decomposition (4.1.1), one can use tables [25] – in practice one looks at the branching rule of AdG into $N=HK$. However, one has to remember that what we need are decompositions into real-irreducible representations, while the tables (and most papers on the subject) give the branching rules in terms of *complex*-irreducible ones. Special care has to be taken if H or K is one of the groups $SU(n)$, $Spin(4n+2)$ or E_6 . Indeed, these groups admit some representations (ϱ) which are not self-conjugate. In such cases ϱ and $\bar{\varrho}$ will appear simultaneously in the reduction of the adjoint representation of G , and one has to collect together such pairs to build R -irreducible representations.

4.2. A Class of Almost Trivial Examples

a) $S = G/\{e\}$, i.e., $H = \{e\}$. Now S itself is a group, and the number of scalars is the number of right invariant metrics on G . The isotropy group $H = \{e\}$ is trivial and its irreducible representations are one-dimensional. We have $\mathcal{S} = \mathcal{G} = \mathcal{N} = \mathcal{K}$ and $\mathcal{H} = \mathcal{L} = 0$. The number of gauge potentials (i.e., the dimension of the gauge

group) is $k=g=\dim(G)$. The number of scalars is $d=g(g+1)/2$. In that case $d_0 = \dim \text{SL}(g)/\text{SO}(g)$.

b) $G = G_1 \times G_1$, $H = \text{diag } G = \{(a, a) | a \in G_1\}$. The homogeneous space $S = G \backslash H$ can be naturally identified with G_1 , the action of G on S being given by $x \rightarrow a^{-1}xb$, $(a, b) \in G_1 \times G_1$. In particular H acts on S by $x \rightarrow a^{-1}xa$, so that the number of scalars is equal to the number of bi-invariant metrics on G_1 . This can be determined by decomposing the adjoint representation of G into irreducible representations and applying the formula (4.1.2). The gauge group $K = N|H$ is easily seen to be isomorphic to the centre of G_1 . In particular if G_1 is simple, then $d=1$ and $k=0$.

c) $S = G \backslash H$ is an irreducible symmetric space. In that case we have a reductive decomposition $\mathcal{G} = \mathcal{H} + \mathcal{L}$ with $[\mathcal{L}, \mathcal{L}] \subset \mathcal{H}$, and the adjoint representation of $\text{Ad}H$ on \mathcal{L} is irreducible. It follows that $\mathcal{K} = 0$ and $\mathcal{S} = \mathcal{L}$, so that the gauge group is at most discrete ($k=0$) and only one scalar field is present ($d=1$). All irreducible symmetric spaces have been thoroughly studied and classified [26]. Example: $S^7 = \text{SO}(8) \backslash \text{SO}(7)$ is a symmetric space admitting, up to a scale, only one $\text{SO}(8)$ invariant metric.

d) $S = G \backslash H$ is an isotropy-irreducible homogeneous space. The cases discussed in c) fall into this category but there are many isotropy-irreducible homogeneous spaces which are not symmetric. They are classified in [27]. Here again $d=1$ and $N|H$ are discrete. Example: $S^7 = \text{Spin}7 \backslash G_2$ is a simply connected isotropy-irreducible, but non-symmetric space. It admits, up to a scale, only one $\text{Spin}7$ invariant metric (n.b. the same as the one in b) [28]).

Another example of this type: $G = \text{Spin}8$, $H = \text{SU}(3)/\mathbb{Z}_3$, $S = G \backslash H$ is isotropy irreducible (not symmetric) with $N \simeq H$ and a discrete gauge group K . The decomposition of $\text{Ad}G$ into real irreducible representations of H reads: $28 = 8 \oplus [10 + \overline{10}]$. Notice (see the end remark of Sect. 4.1) that $10 + \overline{10}$ is to be understood as \mathbb{R} -irreducible and therefore the number of scalars (i.e., $\text{Spin}8$ invariant metrics on S) is $d=1$ (and not $1+1$). The 8 in the decomposition is of course \mathcal{H} itself.

e) $S = G \backslash H$ is normal space. Any metric on S which is induced by a bi-invariant metric on G is G -invariant. Such metrics on S are called normal. In a sense normal metrics are the “most natural” G -invariant metrics on S . A homogeneous space $S = G \backslash H$ is called normal if every G invariant metric on S is normal. Notice that isotropy irreducible spaces are normal, but they are not the only spaces of normal type [29].

4.2. Model Building

To build a model one has to choose a global group G together with two subgroups H and K so that $N = H \times K$ is the normalizer of H in G . Then $S = G \backslash H$ is the internal space and $K = N|H$ is the gauge group. We can notice that S itself admits a principal bundle structure with basis $L = G \backslash N$ and structure group $K = N|H$ [18, Chap. XII] schematically.

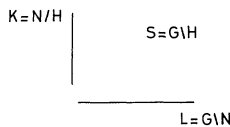


Fig. 5

Table 1. Simply connected irreducible symmetric spaces $L=G\backslash N$, where N is not a simple Lie group. In column \mathcal{L} , the real irreducible representation $\text{Ad}(N)$ on \mathcal{L} is expressed in terms of reducible complex representations. Decompositions not appearing in this column should be computed for given values of p, q (or n)

G	N	Remarks	\mathcal{L}
SU(4)	SO(4)		$3 \otimes 3$
SU($p+q$)	$S(U_p \times U_q)$	$p \geq 2, q \geq 1$	
SO($p+q$)	$\text{SO}(p) \times \text{SO}(q)$	$p \geq 2, q \geq 2$	
SO($4+q$)	$\text{SO}(4) \times \text{SO}(q)$	$p=4, q \geq 2$	
SO($2n$)	$U(n)$	$n \geq 3$	
USp($2n$)	$U(n)$	$n \geq 1$	
USp($2(p+q)$)	$\text{USp}(2p) \times \text{USp}(2q)$	$p \geq 1, q \geq 1$	
E_6	$\text{SU}(6) \times \text{SU}(2)$		$20 \otimes 2$
E_6	$\text{SO}(10) \times U(1)$		$45 \otimes 1 + (16 + \sqrt{6}) \otimes 1 + 1 \otimes 1$
E_7	$\text{SO}(12) \times \text{SU}(2)$		$32 \otimes 2 + \sqrt{32} \otimes 2$
E_7	$E_6 \times U(1)$		$27 \otimes 1 + \sqrt{27} \otimes 1$
E_8	$E_7 \times \text{SU}(2)$		$56 \otimes 2$
F_4	$\text{USp}(6) \times \text{SU}(2)$		$14 \otimes 2$
G_2	SO(4)	(2 cases)	$4 \otimes 2$

In this table, $\text{USp}(2n)$ denotes $\text{USp}(2n, \mathbb{C}) = U(n, \mathbb{H})$ and $S(U(p) \times U(q)) \doteq (U(p) \times U(q)) \cap \text{SU}(p+q)$. Notice the following local isomorphism $\text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2)$ $S(U_p \times U_q) \sim \text{SU}(p) \times \text{SU}(q) \times U(1)$

Table 2. Same as in 1 but $L=G\backslash N$ are simply connected irreducible and isotropy-irreducible (but not symmetric) spaces where N is not a simple Lie group

G	N	Remarks	\mathcal{L}
$\text{SU}(pq)/\mathbb{Z}_m$	$\{\text{SU}(p)/\mathbb{Z}_p\} \times \{\text{SU}(q)/\mathbb{Z}_q\}$	$p \geq q \geq 2, pq > 4$ $m = \text{l.c.m.}(p, q)$	
F_4	$\text{SO}(3) \times G_2$		$5 \otimes 7$
F_4	$\{\text{SU}(3) \times \text{SU}(3)\}/\mathbb{Z}_3$		$3 \otimes 6 \times \sqrt{3} \otimes \bar{6}$
E_6/\mathbb{Z}_3	$\{\text{SU}(3)/\mathbb{Z}_3\} \times G_2$		$8 \otimes 14$
E_6/\mathbb{Z}_3	$\frac{\text{SU}(3) \times \text{SU}(3) \times \text{SU}(3)}{\mathbb{Z}_3 \times \mathbb{Z}_3}$		$3 \otimes 3 \otimes 3 + \sqrt{3} \otimes \sqrt{3} \otimes \bar{3}$
E_7/\mathbb{Z}_2	$\{\text{USp}(6)/\mathbb{Z}_2\} \times G_2$		$14 \otimes 14$
E_7/\mathbb{Z}_2	$\text{SO}(3) \times F_4$		$3 \otimes 26$
E_7/\mathbb{Z}_2	$\{\text{SU}(3) \times \text{SU}(6)\}/\mathbb{Z}_6$		$\sqrt{3} \otimes 15 + 3 \otimes \sqrt{15}$
E_8	$G_2 \times F_4$		$14 \otimes 52$
E_8	$\{\text{SU}(3) \times E_6\}/\mathbb{Z}_3$		$3 \otimes 27 + \sqrt{3} \otimes \bar{27}$

Given H and K it is natural to restrict oneself to those cases where the representation of $N=H \cdot K$ on \mathcal{G}/\mathcal{N} is faithful and irreducible. As it was mentioned in Sect. 3.1, N is then, modulo a discrete group, the normalizer of H in G . Symmetric irreducible spaces are listed in [26, 30] and the non-symmetric ones can be found in [27]. When one goes through these lists one realizes that N is often a simple Lie group. In such a case either $H=N$ (in which case the gauge group K is trivial) or $H=\{e\}$ (modulo discrete groups), so that S is a group manifold itself, $S=N=K=G$. The $G\backslash N$ simply connected with non-simple N are very rare and are all listed in Tables 1 and 2. Table 1 is extracted from [30] and

Table 3. In order to simplify the reading of Tables 1 and 2. Table 3 recalls the usual Cartan classification of Lie groups

G	Dimension	Centre	Complex extension
$A_m = \text{SU}(m+1, \mathbb{C})$	$m(m+2)$	Z_{m+1}	$Sl(m+1, \mathbb{C})$
$B_m = \text{Spin}(2m+1, \mathbb{R})$	$m(2m+1)$	Z_2	$\text{Spin}(2m+1, \mathbb{C})$
$C_m = \text{U}(m, \mathbb{H})$	$m(2m+1)$	Z_2	$\text{Sp}(2m, \mathbb{C})$
$D_m = \text{spin}(2m, \mathbb{R})$	$m(2m-1)$	Z_4 if $m=2l+1$ $Z_2 \times Z_2$ if $m=2l$	$\text{Spin}(2m, \mathbb{C})$
G_2	14	1	
F_4	52	1	
E_6	78	Z_3	
E_7	133	Z_2	
E_8	248	1	

G : simple, compact, simply connected real Lie group.

The usual isomorphisms are:

$$A_1 = B_1 = C_1, B_2 = C_2, A_3 = D_3, D_2 = A_1 \times A_1.$$

Usual notations are:

$$\text{U}(m, \mathbb{H}) = \text{USp}(2m, \mathbb{C}) = \text{Sp}(2m, \mathbb{C}) \cap \text{SU}(2m, \mathbb{C}).$$

Notice that $\text{Spin}(n)$ is the two-fold covering of $\text{SO}(n)$

Table 2 is extracted from [27]. We also give the reduction χ (over the reals) of $\text{Ad}G$ with respect to $N = H \cdot K$. Examples with reducible representations of N on \mathcal{L} can be obtained by taking products of irreducible ones. Table 3, recalling the usual Cartan classification, is also given in order to ease the reading of Tables 1 and 2.

Let us now analyze in some detail several cases from the tables.

a) $G = E_8, N = (E_6 \times \text{SU}(3))/Z_3, H = E_6.$

$S = G/H$ is of dimension 170, $L = G \setminus N$ is isotropy irreducible but not symmetric. The connected component of the identity of the gauge group is $K = N \setminus H = \text{SU}(3)/Z_3$. The reduction of $\text{Ad}G$ with respect to real irreducible representations of $N = H \cdot K$ is

$$[248] = \underbrace{[78 \otimes 1]}_{\mathcal{H}} + \underbrace{[1 \otimes 8]}_{\mathcal{H}} + \underbrace{[27 \otimes 3 + \overline{27} \otimes \overline{3}]}_{\mathcal{L}}.$$

The reduction of the subspace $\mathcal{S} = K + \mathcal{L}$ with respect to H is

$$\mathcal{S} = 8[1] + 3[27 + \overline{27}].$$

The dimension of the space of G invariant metrics on S is therefore

$$d = \frac{8 \times 9}{2} + \frac{3 \times 4}{2} = 42.$$

b) $G = \text{SO}(10), N = \text{SU}(5) \times \text{U}(1), H = \text{U}(1).$

$S = G/H$ is of dimension 20, $L = G \setminus N$ is a Hermitian symmetric space. The gauge group is $\text{SU}(5)$, and the reduction of $\text{Ad}G$ with respect to $N = H \cdot K$ reads

$$[45] = [1 \otimes 1] + [1 \otimes 24] + [1 \otimes 10 + \overline{1} \otimes \overline{10}].$$

The reduction of \mathcal{S} with respect to H is

$$\mathcal{S} = 24[1] + 10[1 + \overline{1}].$$

(Observe that in the real domain an Abelian group may have two-dimensional irreducible representations.) Therefore

$$d = \frac{24 \times 25}{2} + \frac{10 \times 11}{2} = 355.$$

c) $G = U(q+1, \mathbb{H})$, $N = U(q, \mathbb{H}) \times SU(2)$, $H = U(q, \mathbb{H}) (= USp(2q, \mathbb{C}))$.

$S = G \setminus H = S^{4q+3}$ (spheres), $L = G \setminus N = \mathbb{H}P^q$ is a quaternionic projective space ($G \setminus N$ is symmetric). The gauge group is $SU(2)$. The reduction of $\text{Ad } G$ with respect to $H \cdot K$ reads

$$[(q+1)(2q+3)] = [q(2q+1) \otimes 1] + [1 \otimes 3] + [2q \otimes 2],$$

and the reduction of \mathcal{S} with respect to H is

$$\mathcal{S} = 3[1] + 1[4q].$$

Therefore

$$d = \frac{3 \times 4}{2} + \frac{1 \times 2}{2} = 7.$$

Notice that for $q=1$ ($U(2, \mathbb{H}) \simeq SO(5)$) we obtain the $SU(2)$ foliation of S^7 over $\mathbb{H}P^1 \simeq S^4$, i.e., the usual $k=1$ instanton bundle [31, 32].

d) $G = SU(4)$, $N = SU(3) \times U(1)$, $H = SU(3)$.

$S = G \setminus H = S^7$, $L = SU(4) \setminus SU(3) \times U(1)$ is an irreducible Hermitian symmetric space. The gauge group is $U(1)$ and the reduction of $\text{Ad } G$ with respect to $H \cdot K$ is

$$[15] = [1 \otimes 8] + [1 \otimes 1] + [3 \otimes 1 + \bar{3} \otimes \bar{1}].$$

The H -reduction of \mathcal{S} reads,

$$\mathcal{S} = 1 + 1[3 + \bar{3}],$$

and therefore

$$d = 2.$$

4.3. Comments

Here we collect various comments, remarks and information which are not used in the present paper but are closely related to the subject:

On the Sign of Scalar Curvature r in the Internal Space S . If S is a compact, non-Abelian group G then, according to [33]:

there exist right invariant metrics on S so that $r < 0$, $r = 0$, $r > 0$;

for a bi-invariant metric on G , r is positive.

If S is a homogeneous space $G \setminus H$ with G compact non-Abelian then, according to [34]:

if S is of normal type, then every G invariant metric on S has $r > 0$;

if S is not normal, then there exist G invariant metrics on S with $r < 0$, $r = 0$, and $r > 0$.

On the (Un)boundedness of the Scalar Curvature of the Internal Space. If the scalar curvature of E is considered as a possible Lagrangian then, writing formally "Lagrangian = kinetic term - potential," it is clear that the scalar curvature $r(x)$ of

the internal space at x is to be interpreted as “minus the potential.” It is natural to freeze the volume of the internal space and to allow only for squashing deformations of S . The space of all G invariant metrics on S with a fixed volume is denoted by $R_0(G; S)$, and has the dimension $d_0 = d - 1$. Except in very special cases [for example $G = SO(3)$ or $G = SL(2, R)$ non-compact] the scalar curvature r considered as a function on R_0 is still unbounded from above [35]. For homogeneous spaces little is known, see however [36]. An interesting possibility is to allow for solvable (non-compact) groups, in which case r as a function on R_0 is always non-positive [35] (Sect. 6) (see also Ref. [6] for a discussion of “flat” groups).

On the Critical Points of the Scalar Curvature in S . Consider the space of Riemannian metrics of a given fixed volume form dv on a compact manifold S . It is known [37] that the critical points of the functional

$$A[g] = \int_S r[g] dv$$

are precisely the Einstein metrics on S . If $S = G \backslash H$ is a homogeneous space, then $r[g]$ is constant on S and $A[g] = r[g] \cdot V_0$. It is shown in [35] that every compact simple Lie group G , except for $SO(3)$ and possibly G_2 and $USp(4n + 2)$ admits at least two conformally inequivalent right invariant Einstein metrics. Notice that these critical points are usually saddle points of the functional A on R_0 , and not always local extrema. The results of [35] have been extended to a class of homogeneous spaces in [38]. A classification of homogeneous spaces admitting Einstein metrics and the structure of moduli of such metrics is a difficult problem, still unsolved. For a recent account see [39, 40]. (Notice that an isotropy irreducible homogeneous space is always an Einstein space with respect to its unique invariant metric [27].) It is clear that these problems are relevant for a semi-classical approximation of a quantum theory containing the scalars $h_{\alpha\beta}$. Notice that making an expansion of $h_{\alpha\beta}$ around some non-trivial critical point (Einstein metric on S) amounts to give a “non-zero expectation value” to the Higgs fields $\Phi_\alpha(x)$ so that, because of the minimal coupling of the gauge fields of $N \backslash H$ to the scalars, we have a spontaneous symmetry breaking and some of the gauge fields may acquire masses.

The Case when L is an Irreducible Symmetric Space. The scalar curvature $R(G \backslash H)$ can be computed using (3.5.8), but it can be also obtained via the standard Kaluza-Klein construction applied to S considered as a principal bundle over L with structure group K (Fig. 5). If L is isotropy irreducible, the $\left(1 + \frac{k(k+1)}{2}\right)$ parameter family of G invariant metrics on S is given by

$$h_{ab} = \mu^2 \eta_{ab}, \quad h_{\hat{a}\hat{b}} = \psi_{\hat{a}}^{\hat{a}'} \psi_{\hat{b}}^{\hat{b}'} \eta_{\hat{a}'\hat{b}'},$$

where $-\eta_{ij}$ is the Killing form on G . In particular, when $L = G \backslash N$ is symmetric, $\mu^2 = 1$ and $\psi_{\hat{a}}^{\hat{a}'} = t \delta_{\hat{a}}^{\hat{a}'}$, then (3.5.7) gives

$$R(G \backslash H) = R(L) + R(K) - \frac{1}{4} C_{ab}^c C_c^{ab},$$

where $R(L)=l/2$, $R(K)=ck/4t^2$ and $C_{ab}^{\hat{c}}$ can be considered as the curvature of the canonical connection on S . In the above formula $k=\dim K$, $l=\dim L$ and c is defined by $-\eta_{\hat{a}\hat{b}}=c^{-1} \times$ (the Killing form of K) or, equivalently $c=\text{index}(\text{Ad } K)/\text{index}(\text{Ad } G)$. Notice that by replacing $h_{\alpha\beta}$ by $(t^2)^{-k/s}h_{\alpha\beta}$, $s=\dim S$, we obtain a one-parameter family of metrics with a fixed volume, the scalar curvature being given by

$$R(G \setminus H; t) = t^{2k/s} \left[\frac{l}{2} + \frac{ck}{4t^2} + \frac{k}{4}(c-1)t^2 \right].$$

In particular for $S^7 = \text{USp}(4) \setminus \text{SU}(2)$, we get, using $c=2/3$

$$R(S^7; t) = t^{6/7} \left[2 + \frac{1}{2t^2} - \frac{t^2}{4} \right],$$

$$\dot{R}(S^7; t) = -\frac{5}{7} t^{-15/7} (t^2 - 2)(t^2 - 2/5).$$

The condition $\dot{R}=0$ is a necessary condition for obtaining an Einstein metric. The two Einstein metrics on S^7 corresponding to $t^2=2$ and $t^2=2/5$ have been found in [38] by computing the Ricci tensor. Notice that $t^2=2$ is the $\text{SO}(8)$ invariant metric and $t^2=2/5$ has symmetry $\text{USp}(4) \times \text{SU}(2)$. Notice finally that $t^2=1$ – the normal metric on $\text{USp}(4) \setminus \text{SU}(2)$ – is not Einstein [which should not be surprising since $\text{USp}(4) \setminus \text{SU}(2)$ is not isotropy irreducible]. The scalar curvature $R(S^7; t)$ as a function of t is given in Fig. 6. The $t^2=2/5$ Einstein metric on S^7 have recently been used for providing a new vacuum for $d=11$ supergravity [41].

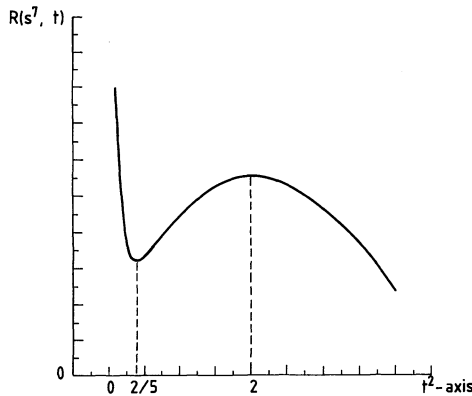


Fig. 6

Spinors. Introducing spinor fields in the extended space-time E is an obvious necessity (and can be achieved, for example, by replacing multidimensional gravitation by supergravity). By studying the Dirac equation on E , one would be led naturally to a system of spinor fields coupled to gauge fields and to the scalar fields. To make a link with the previous comment, notice that zero modes of the Dirac operator on the typical compact internal space S (endowed with a G invariant metric) exist only if the scalar curvature is negative.

Acknowledgements. We wish to thank M. Duff, B. Morel, C. Orzalesi, L. Palla, and P. Sorba for comments and clarifying remarks. Special thanks are due to Professor R. Stora for his continuous interest, encouragement and discussions. We are also indebted to Prof. G. Mack for his comments and suggestions. One of us (A.J.) is also grateful to Prof. R. Haag for many illuminating discussions concerning the structure of multidimensional unified field theories. Thanks are also due to the Theory Division of CERN for kind hospitality.

References

1. Witten, E.: Search for a realistic Kaluza-Klein theory. Nucl. Phys. B **186**, 412 (1981)
2. Kerner, R.: Generalization of the Kaluza-Klein theory for an arbitrary non-abelian gauge group. Ann. Inst. H. Poincaré **9**, 143 (1968)
3. Cho, Y.M.: Higher-dimensional unifications of gravitation and gauge theories. J. Math. Phys. **16**, 2029 (1975)
4. Cho, Y.M., Freund, P.G.O.: Non-abelian gauge fields as Nambu-Goldstone fields. Phys. Rev. D **12**, 1711 (1975)
5. Cho, Y.M., Jang, Pong Soo: Unified geometry of internal space with space-time. Phys. Rev. D **12**, 3789 (1975)
6. Scherk, J., Schwarz, John H.: How to get masses from extra dimensions. Nucl. Phys. B **153**, 61 (1979)
7. Orzalesi, C.A.: Multidimensional unified theories. Fortschr. Phys. **29**, 413 (1981)
8. Coquereaux, R., Duff, M.J.: Unpublished
9. Souriau, J.M.: Relativité multidimensionnelle non stationnaire. Colloque de Royaumont (1959) In: Les Théories relativistes de la gravitation. Paris: CNRS 1962
10. Domokos, G., Kövesi-Domokos, S.: Gauge fields on coset spaces. Nuovo Cimento **44 A**, 318 (1978)
11. Salam, A., Strathdee, J.: On Kaluza-Klein theory. Ann. Phys. **141**, 316 (1982)
12. Percacci, R., Randjbar-Daemi, S.: Kaluza-Klein theories on bundles with homogeneous fibres. ICTP preprint ICTP/82/18 (1982)
13. Cho, Y.M.: Gauge theories on homogeneous fibre bundle. CERN preprint TH. 3414 (1982)
14. Duff, M.J.: Invited lecture at the third Marcel Grossman meeting, Shanghai (1982). CERN preprint TH. 3451 (1982)
15. Montgomery, D., Yang, C.T.: The existence of a slice. Ann. Math. **65**, 108 (1956)
16. Montgomery, D., Yang, C.T.: Orbits of highest dimension. Trans. Am. Math. Soc. **87**, 284 (1958)
17. Abraham, R., Marsden, J.E.: Foundations of mechanics (2nd edn.). London: Benjamin 1978
18. Borel, A., Bredon, G., Floyd, E.E., Montgomery, D., Palais, R.: Seminar on transformation groups. Ann. Math. Stud. Vol. 46. Princeton, New Jersey: Princeton University Press 1960
19. Hsiang, W.Y.: In: Proceedings of the Conference on Transformation Groups, New Orleans 1967, Mostert, P.S. (ed.). Berlin, Heidelberg, New York: Springer 1968
20. Dieudonné, J.: Treatise on analysis, Vol. III. New York, London: Academic Press 1972
21. Palais, S.R.: On the existence of slices for action of non compact Lie groups. Ann. Math. **73**, 295 (1961)
22. Kobayashi, S., Nomizu, K.: Foundations of differential geometry, Vol. I/II. New York, London: Interscience Publ. 1963
23. Lichnerowicz, A.: Geometry of groups of transformations. Leyden: Noordhoff Intern. Publ. 1977
24. Barut, A.O., Raczka, R.: Theory of group representations and applications. Warszawa: PWN-Polish Sci. Publ. 1980
25. Patera, J., Sankoff, D.: Tables of branching rules and representations of simple Lie algebras. Montreal: Les Presses de l'Université de Montréal 1974
26. Helgason, S.: Differential geometry and symmetric spaces. New York: Academic Press 1962
27. Wolf, J.A.: The geometry and structure of isotropy irreducible homogeneous spaces. Acta Math. **120**, 59 (1968)
28. Poncet, J.: Groupes de Lie compacts de transformations. Commun. Math. Helv. **33**, 109 (1959)
29. Berger, H.: Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive. Ann. Scu. Norm. Sup. Pisa **15**, 179 (1961)
30. Wolf, J.A.: Spaces of constant curvature. New York: McGraw-Hill 1967
31. Stora, R.: Yang Mills instantons, geometrical aspects. Erice Lectures (27 June – 9 July 1977)

32. Atiyah, M.F., Hitchin, N.Y., Singer, I.M.: Self-duality in four-dimensional Riemannian geometry. Proc. R. Soc. London A **362**, 425–461 (1978)
33. Milnor, J.: Curvatures of left invariant metrics on Lie groups. Adv. Math. **21**, 21 (1976)
34. Bérard Bergery, L.: Sur la courbure des métriques riemanniennes invariantes des groupes de Lie et des espaces homogènes. Ann. Sci. Ecole Norm. Sup. **11**, 543 (1978)
35. Jensen, G.R.: The scalar curvature of left-invariant Riemannian metrics. Ind. Univ. Math. J. **20**, 1125 (1971)
36. Koiso, N.: Non-deformability of Einstein metrics. Osaka J. Math. **15**, 419 (1979)
37. Nagano, T.: A problem on the existence of an Einstein metric. J. Math. Soc. Jpn. **19**, 30 (1967)
38. Jensen, G.R.: Einstein metrics on principal fiber bundles. J. Differential Geometry **8**, 599 (1973)
39. Berger, M.: Rapport sur les variétés d'Einstein. In: Analyse sur les variétés. Metz 1979. Soc. Math. de France, Astérisque 80 (1980)
40. Bourguignon, J.P.: Deformations des métriques d'Einstein. In: Analyse sur les variétés. Metz 1979. Soc. Math. de France, Astérisque 80 (1980)
41. Awada, M.A., Duff, M.J., Pope, C.N.: $N=8$ supergravity breaks down to $N=1$. Preprint ICTP/82-82/4 (1982)

Communicated by G. Mack

Received January 10, 1983; in revised form February 16, 1983