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The Rotation Number for Finite Difference Operators and its Properties

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Abstract. We discuss a rotation number $\alpha(\lambda)$ for second order finite difference operators. If $k(\lambda)$ denotes the integrated density of states, then $k(\lambda)=2\alpha(\lambda)$. For almost periodic operators, $k(\lambda)$ is proved to lie in the frequency-module whenever λ is outside the spectrum; this yields a labelling of the gaps of the spectrum.

Introduction

We study in this paper the Jacobi matrices, acting on $\ell^2(\mathbb{Z})$:

$$(Hu)(n) = -u(n+1) - u(n-1) + V(n)u(n), \tag{1}$$

and more general second order finite difference operators defined later below. These operators have been of interest for a long time in mathematics, and also in physics where they appear in the tight binding approximation of one-dimensional condensed matter systems. They can be viewed as a finite difference analogue of the Schrödinger equation. On the other hand they have recently received renewed interest both in mathematics and physics particularly in the cases where the diagonal elements $V(n)$ are realizations of a sequence of independent random variables or when they constitute an almost periodic sequence. For reviews describing motivations and results in these two situations we refer, respectively, to references [10] and [15].

In the study of the continuous Schrödinger equation, one of the basic tools is the rotation[1] number $\alpha(\lambda)$; it follows from Sturm Liouville theory that it is equal to half the integrated density of states $k(\lambda)$. Recently Johnson and Moser [9] found the following remarkable result: if the potential $V(\cdot)$ of the continuous Schrödinger equation is an almost periodic function, $k(\lambda)=2\alpha(\lambda)$ lies in the

1 Our definition may differ by a factor 2π from the one used by other authors; same remark for the frequency module

frequency module of $V(\cdot)$ whenever λ is outside the spectrum. This result yields, in particular a labelling of the gaps of the spectrum, although this spectrum has a "tendency" to be a Cantor set as can be seen from perturbation theory (we come back to this last point in the concluding remarks of this note).

In the finite difference case, the situation is much poorer and there is only one result related to the above ones: if there is only one frequency in the frequency-module[2] of the sequence $V(n)$, then the integrated density of states, in a gap, lies in the frequency module. It follows from the remark [2] that:

i) the integrated density of states is equal [14] to the trace on some C^* algebra naturally associated to the problem, and

ii) that this trace is [13] in the frequency module. (This provides also an alternative proof for the continuous case if one replaces ii) by the analogue result of [6].)

The limitation of the result to the case of one frequency in the finite difference situation relies on two facts: on one hand there was apparently no good analogue to the rotation number in the finite difference case, forbidding extension of the Johnson-Moser type of argument. On the other hand the known results of [13] in the C^* algebra approach are limited to the one frequency case.

In Sect. I of this note, we discuss a rotation number $\alpha(\lambda)$ for operators of type (1); its relation with the integrated density of states $k(\lambda)$, namely $k(\lambda)=2\alpha(\lambda)$, is established for random or almost periodic potentials. This relation shows that our $\alpha(\lambda)$ is really the good finite difference analogue of the rotation number. In the preprint of this note, we wrote that we believed that such an analogue should have been known from some mathematicians for a long time. In fact we found that, as noted in his original paper [16], Sturm discovered his famous results by considering first the finite difference case! His results for the discrete case have been published later [17]; several approaches appeared in textbooks on the subject. We have not found our proof by orthogonalization in these references, although it was certainly known by someone somewhere; we keep it here for selfcompleteness. The first application to the physics of disordered systems of oscillation properties seems to appear in [18], although this paper never mentions the Sturm-Liouville theory! These ideas also allow us to derive an equation for the density of states of the operator (1), for example when $V(n)$ is a random potential; this remark is the last proposition of Sect. I and already appeared in Schmidt's paper [18], where there was also an almost complete proof of the existence of invariant measures and of their uniqueness for energies within the spectrum.

In Sect. II, we prove the main result of this note: if $V(n)$ is an almost periodic sequence, the integrated density of states $k(\lambda)$ is in the frequency module of V for all real λ in the resolvent set. This gap labelling theorem is obtained through an adaptation of the Johnson-Moser approach; the main burden is to construct a "nice" almost periodic function associated to a given almost periodic sequence.

In Sect. III, we indicate how to extend such results to second order finite difference operators

$$(Hu)(n)=-J_{n,n+1}u(n+1)-J_{n,n-1}u(n-1)+V(n)u(n). \tag{2}$$

2 Note that, in the finite difference case, the frequency-module is defined modulo $\mathbb{Z}$

As a conclusion, let us mention that we become interested in this question partly because it had been repeatly mentioned at various opportunities that the rotation number has no good analogue in the discrete case, and that counterexamples could exist to the gap labelling theorem in this case. These are the main justifications in our mind to publish these notes that we find simple and natural.

I. The Rotation Number and the Density of States

In this section we define a rotation number associated to Eq. (1), the general Eq. (2) being treated in Sect. III. We then establish its relation with the integrated density of states. Finally we note an expression of the density of states of Eq. (1) with a random potential.

Let H be defined by Eq. (1):

$$(Hu)(n) = -u(n+1) - u(n-1) + V(n)u(n), \tag{I.1}$$

and let us consider the solution of the equation $Hu = \lambda u$ for $n \geqq 0$ with initial condition $u(0) = \cos\theta$, $u(1) = \sin\theta$. We will denote by $u(x)$ the linear interpolation of the sequence $u(n)$. Let us first consider the number $N_L(\lambda)$, whose dependence on θ is left implicit, defined by:

Definition I.1. Let $N_L(\lambda)$ be the number of changes of sign of $u(n)$, for $1 \leqq n \leqq L$, adding 1 if $u(L) = 0$. Alternatively $N_L(\lambda)$ is the number of zeroes of $u(x)$ in $[1, L]$.

Remark. In the previous definition $u(n)$ may be zero for some n, $1 < n < L$, but the total number of changes of sign is well defined since $u(n) = 0$ implies $u(n-1) = -u(n+1) \neq 0$.

Definition I.2. Let $k_L(\lambda)$ be the integrated density of states [i.e. $(L-1)^{-1}$ times the number of eigenvalues less than or equal to λ] for the restriction H_L of the operator H to the set $\{1, \ldots, L-1\}$ with boundary conditions $\dfrac{u(0)}{u(1)} = \operatorname{cotg}\theta$, $u(L) = 0$.

The basic lemma in our future definition of the rotation number is the following one, which represents a finite difference analogue to the Sturm-Liouville theory, although its proof follows different lines.

Lemma II.3. $k_L(\lambda) = (L-1)^{-1} N_L(\lambda)$.

Proof. We consider the case $\operatorname{cotg}\theta = 0$, the other ones being obtained by replacing $V(1)$ by $V(1) + \operatorname{cotg}\theta$, except the cases where $\operatorname{cotg}\theta$ is infinite, in which case one studies the operator on $\{2, \ldots, L-1\}$ and boundary condition $\dfrac{u(2)}{u(1)} = 0$. The number of eigenvalues less than or equal to λ is also the number of negative or null diagonal elements of the quadratic form associated to $H - \lambda$ when put in diagonal form. We use the Lagrange method in order to construct a basis in which the quadratic form $H - \lambda$ is diagonal: in the initial basis the quadratic form has coefficients $\{\gamma_{ij}\}$, where i and j run in the set $\{1, \ldots, L-1\}$, and $\gamma_{ii} = V(i) - \lambda$, $\gamma_{ij} = -1$ if $|i-j| = 1$, $\gamma_{ij} = 0$ otherwise. We first orthogonalize at 1, and the new coefficients of the quadratic form become $\{\gamma'_{ij}\}$ with $\gamma'_{11} = \gamma_{11}$, $\gamma'_{12} = \gamma'_{21} = 0$,

$\gamma'_{22}=\gamma_{22}-\frac{1}{\gamma_{11}}$, and $\gamma'_{ij}=\gamma_{ij}$ otherwise. We can then proceed by induction at least if none of the diagonal term appearing is zero. We first suppose that such is the situation, and this amounts to excluding only a finite number of values of λ as will become clear below. Thus proceeding by induction we get a diagonal quadratic form $\{\delta_i\}_{i=1}^{L-1}$ whose diagonal terms have been constructed by induction according to

$$\delta_i=\gamma_{ii}-\frac{1}{\delta_{i-1}}=V(i)-\lambda-\frac{1}{\delta_{i-1}}. \tag{I.2}$$

The number of eigenvalues less than λ is the number of negative terms in the sequence δ_i. Let us now note that if $u(n)\neq 0$ for all n, $0<n<L$, the ratio $\frac{u(n)}{u(n-1)}$ satisfies the same induction relation (I.2). [Incidently, this justifies our claim that for all but a finite number of λ, the diagonal terms δ_i constructed by iteration do not vanish: otherwise λ is an eigenvalue of the matrix (I.1) restricted to some subset $\{1,\ldots,i\}$, $i<L$.] We have thus established Lemma I.3 for all λ but a finite number. Finally it is also easy to check that Lemma I.3 holds also for those exceptional values of λ, because $k_L(\lambda)$ and $N_L(\lambda)$ are right continuous: $k_L(\lambda)$ is right continuous by definition, $N_L(\lambda)$ is because of the continuity of u as a function of λ, because $u(n)=0$ for $1\leqq n\leqq L-1$ implies $u(n+1)=-u(n-1)\neq 0$ and because of the convention of adding 1 in the definition of $N_L(\lambda)$ when $u(L)=0$. $\square$

We are now in a position to state the definition of the rotation number:

Definition I.3. We define the rotation number as

$$\alpha(\lambda)=\lim_{L\to\infty}\frac{1}{2}\frac{1}{L}N_L(\lambda) \tag{I.3}$$

whenever the limit exists.

Remark. The $\alpha(\lambda)$ may depend on the parameter θ governing the boundary condition. However we will see that, in the cases of interest for us, $\alpha(\lambda)$ will turn out to be independent of θ.

Let us now restrict ourselves to the case where the potential $\{V(n)\}_{n\in\mathbb{Z}}$ is a random variable on some probability space $(\Omega,\mathscr{B},\mathbb{P})$, the probability $\mathbb{P}$ being ergodic with respect to the action of the shift on $\mathbb{Z}$. This general setting includes for example the two cases of particular interest: the first is the case of an almost periodic potential V, for which the space of configuration Ω is the hull of V and the probability $\mathbb{P}$ is the Haar measure on this hull, the second is the case when the $V(n)$ are independent random variables with a common distribution. Under the above general hypothesis, it follows from the work of Benderskii and Pastur [4] and of Pastur [12] and its adaptation to the almost periodic case [9, 1b], that the density of states $k(\lambda)$ of H satisfies:

Lemma I.5. *For $\mathbb{P}$-almost all V, the limit of $k_L(\lambda)$ as $L\to\infty$ exists, is independent of the boundary condition θ and*

$$\lim_{L\to\infty}k_L(\lambda)=k(\lambda). \tag{I.4}$$

In the case of almost periodic potentials, this result holds for all V in the hull.

As a consequence of Definition I.4 and Lemmas I.3 and I.5, we have obtained the

Theorem I.6. *For $\mathbb{P}$-almost all V, the rotation number $\alpha(\lambda)$ exists, is independent of the boundary condition θ and*

$$2\alpha(\lambda)=k(\lambda). \tag{I.5}$$

In the case of an almost periodic potential, these results hold for all V in the hull.

Finally we want to remark how our approach in Lemma I.3 to these problems, indicates a simple way to get an explicit expression for the density of states, or equivalently to the rotation number. For simplicity we restrict ourselves to the case of a potential $V(n)$ which is a set of random independent variables with a common distribution which will be absolutely continuous with a density $r(\cdot)$, and we will suppose $r(\cdot)$ to be in L_∞ with some weight $(1+|x|)^{1+\alpha}$, for some $\alpha>0$. We then have:

Proposition I.7. *Under the above hypothesis,*

$$k(\lambda)=2\alpha(\lambda)=\int_{-\infty}^{0}\mu_\lambda(x)dx\,, \tag{I.6}$$

where $\mu_\lambda(x)$ is the unique L_1 function satisfying

$$\mu_\lambda(x)=\int\mu_\lambda(y)r\left(\lambda-x-\frac{1}{y}\right)dy\,. \tag{I.7}$$

Proof. We just sketch the proof, which is not technically difficult to complete; it follows from the ideas developed in the proof of Lemma I.3 that the rotation number or the integrated density of states are linked to the proportion of n such that $\delta_n\leqq 0$ in the sequence (I.2), and in fact

$$k(\lambda)=2\alpha(\lambda)=\lim_{L\to\infty}\frac{1}{L}\,\#\{n,0<n\leqq L|\delta_n\leqq 0\}\,, \tag{I.8}$$

from which it follows easily that

$$k(\lambda)=2\alpha(\lambda)=\lim_{n\to\infty}\mathbb{P}\{\delta_n\leqq 0\}\,. \tag{I.9}$$

But the probability distribution of δ_n is deduced from the one of δ_{n-1} by the following relationship between their densities:

$$\mu_\lambda^{(n)}(x)=\int r\left(\lambda-x-\frac{1}{y}\right)\mu_\lambda^{(n-1)}(y)dy\,. \tag{I.10}$$

It is not hard to verify that this integral operator is compact in appropriate spaces and has its maximal eigenvalue 1 nondegenerate, with an eigenvector $\mu_\lambda(x)$ [which is the density of the invariant measure of the Markov process (I.2)] and that $\mu_\lambda^{(n)}(x)\to\mu_\lambda(x)$ as $x\to\infty$. $\square$

II. The Gap Labelling Theorem

We assume throughout this section that $V(n)$ is an almost periodic sequence, and we will prove that the integrated density of states $k(\lambda)=2\alpha(\lambda)$ of the operator H is in the frequency-module of V, for all real λ outside the spectrum of H.

We first recall the definition of the frequency-module $\mathfrak{M}(V)$ of an almost periodic sequence on $\mathbb{Z}$. It is the $\mathbb{Z}$-module of the real numbers modulo 1, generated by the θ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0}^{N-1} e^{2i\pi n\theta} V(n) \neq 0. \tag{II.1}$$

Then V can be expressed as a uniform limit of a Fourier series, the (countably many) frequencies of which are in $\mathfrak{M}(V)$.

Let us now recall a few basic facts concerning the Green's functions that we will need later. For λ in the resolvent set, the Green's functions $G_\lambda(n,m)$ are the matrix elements of $(H-\lambda)^{-1}$; it can be written when $n \leqq m$, as

$$G_\lambda(n,m) = \frac{u^-(n)u^+(m)}{[u^+,u^-]}, \tag{II.2}$$

where u^- and u^+ are the solutions of $Hu=\lambda u$ which go to zero respectively at $-\infty$ or at $+\infty$, and

$$[u^+,u^-] = u^+(n+1)u^-(n) - u^+(n)u^-(n+1) \tag{II.3}$$

is the Wronskian of u^+ and u^- and is independent of n. In the following, we will always normalize u^+ and u^- in such a way that their Wronskian will be equal to one.

The main result of this section is:

Theorem II.1. *For any real λ in the resolvent set*

$$k(\lambda) = 2\alpha(\lambda) \in \mathfrak{M}(V).$$

Proof. As in the proof of Johnson and Moser for the continuous case, we are going to rely on the following lemma [9] which is a direct consequence of a lemma on almost periodic functions (e.g. Lemma 6.7, p. 104 of [7]):

Lemma II.2. *Let $f(x)$ be an almost periodic function on $\mathbb{R}$, with frequency-module $\mathcal{M}$ and such that $f'(x)=\frac{df}{dx}$ is also almost periodic with frequency module included in $\mathcal{M}$. Also suppose that any $\tilde{f}$ in the hull of f has only simple zeroes. Then the number $N(x)$ of zeroes of $f(t)$ in $[0,x]$ satisfies $\lim_{x\to\infty} \frac{1}{2}\frac{N(x)}{x} \in \mathcal{M}$.*

In fact we are going to construct a function $\tilde{g}_\lambda(x)$ satisfying the hypothesis of the above lemma, and for which $N(x)$ will be associated to the number of changes of sign of the two solutions u^+ and u^- and hence to the rotation number $\alpha(\lambda)$ of our operator. In [9] the analogue function was elegantly found as the Green's function, $G_\lambda(x,x)$, of the continuous problem. Our search for a function $\tilde{g}_\lambda(x)$ will also begin by a study of our Green's functions, and we state first:

Lemma II.3. *For λ real in the resolvent set, the sequences $\{G_\lambda(n+p,m+p)\}_{p\in\mathbb{Z}}$ are almost periodic for all n and m in $\mathbb{Z}$ and their frequency module is included in $\mathfrak{M}(V)$.*

Proof. The proof is elementary, using the criterion of Bohr (see e.g. [7]): it is sufficient to prove that for all sequences t_n of integers such that $V_{t_n} \xrightarrow{\ell_\infty} V$, then $G_{t_n} \xrightarrow{\ell_\infty} G$, V_{t_n} and G_{t_n} denoting the translated of V and G by the translation t_n. But the ℓ_∞ convergence of the potential implies the norm convergence of $(H-\lambda)^{-1}$ and hence the ℓ_∞ convergence of the Green's functions. □

We come back now to the proof of Theorem II.1. Let us first introduce the linear interpolations $u^+(x)$ and $u^-(x)$ of $u^+(n)$ and $u^-(n)$:

$$u^\pm(n+y)=(1-y)u^\pm(n)+yu^\pm(n+1), \qquad 0\leqq y\leqq 1, \tag{II.4}$$

and note that

$$u^+(x)\frac{du^-(x)}{dx}-u^-(x)\frac{du^+(x)}{dx}\equiv[u^+,u^-]=1 \tag{II.5}$$

on $\mathbb{R}\backslash\mathbb{N}$ and also on the integers for the right or left derivatives, hence these linear interpolations also satisfy a Wronskian condition. This leads us to introduce a continuous interpolation $g_\lambda(x)$ of $G_\lambda(n,n)$ as:

$$g_\lambda(n+y)=u^-(n+y)u^+(n+y) \tag{II.6}$$

$$=(1-y)^2G_\lambda(n,n)+2y(1-y)G_\lambda(n,n+1)+y^2G_\lambda(n+1,n+1)-y(1-y) \tag{II.7}$$

for $0\leqq y\leqq 1$.

This interpolation $g_\lambda(x)$ satisfies:

Lemma II.4. *The function $g_\lambda(x)$ has the following properties:*

i) *it is an almost periodic function on $\mathbb{R}$, and its frequency-module $\mathfrak{M}'$ is included in the module generated by $\mathfrak{M}(V)$ and $\mathbb{Z}$.*

ii) $\#\{t, 1<t\leqq x|g_\lambda(x)=0\}$

$$=\#\{t,1<t\leqq x|u^+(x)=0\}+\#\{t,1<t\leqq x|u^-(x)=0\}.$$

iii) *$g'_\lambda(x)$ and $g''_\lambda(x)$ exist on $\mathbb{R}\backslash\mathbb{N}$ and are uniformly bounded.*

iv) *$\exists\delta>0$, such that any two zeroes of g_λ have distance larger than δ.*

Proof. Part i) follows directly from Lemma II.3 and from the formula (II.7) (see e.g. Theorem 4.5 of [7]). Part ii) is the same as to say that $u^+(x)$ and $u^-(x)$ cannot vanish simultaneously; this is implied by the property (II.5). In order to verify iii) it is sufficient to look at the explicit expressions of g'_λ and g''_λ, which in view of formula (II.7) will depend only on the sequences $G_\lambda(n,n)$ and $G_\lambda(n,n+1)$; the uniform boundedness will follow from the one of these sequences which is due to their almost periodicity (Lemma II.3). Let us come now to the proof of point iv), and let us begin by noting the following expression for the discontinuity of the derivative of $g_\lambda(x)$ at an integer n: in view of (II.4) and (II.6), we have

$$\begin{aligned} g'_\lambda(n-0)-g'_\lambda(n+0) &= (u^+(n)-u^+(n-1))u^-(n)+(u^-(n)-u^-(n-1))u^+(n) \\ &\quad -[(u^+(n+1)-u^+(n))u^-(n)+(u^-(n+1)-u^-(n))u^+(n)] \\ &= 4u^+(n)u^-(n)-u^-(n)(u^+(n+1)+u^+(n-1))-u^+(n)(u^-(n-1)+u^-(n+1)) \\ &= [4+2(\lambda-V(n))]u^+(n)u^-(n)=[4+2(\lambda-V(n))]g_\lambda(n). \end{aligned} \tag{II.8}$$

Since $g_\lambda(x)$ is also derivable on $\mathbb{R}\backslash\mathbb{N}$, it follows from (II.8) that $g_\lambda(x)$ is derivable at all its zeroes. It then follows from the Wronskian condition (II.5) that $g'_\lambda(x)=\pm 1$ at any zero of g_λ, and it has to change sign at any two consecutive zeroes because of the continuity of g_λ. Hence if x_1 and x_2 or two consecutive zeroes, we have

$$\pm 2 = g'_\lambda(x_1)-g'_\lambda(x_2)=\int_{x_1}^{x_2} g''_\lambda(x)dx+\Delta g'_\lambda, \tag{II.9}$$

where $\Delta g'_\lambda$ is the discontinuity of g'_λ in the case where the open interval $]x_1,x_2[$ contains an integer n. On the other hand, using (II.8) we get:

$$\Delta g'_\lambda=[4+2(\lambda-V(n))]\int_{x_1}^{n} g'_\lambda(x)dx=[4+2(\lambda-V(n))]\int_{n}^{x_2} g'_\lambda(x)dx. \tag{II.10}$$

It follows now from (II.9), (II.10) and from the uniform boundedness of g'_λ, g''_λ, and $V(n)$ that $|x_1-x_2|$ is uniformly bounded from below. □

Let us now come back to the proof of Theorem II.1, and consider now the following function on $\mathbb{R}$:

$$\tilde{g}_\lambda(x)=\frac{2}{\delta}(\chi_{\delta/2}*g_\lambda)(x), \tag{II.11}$$

where $\chi_{\delta/2}$ is the characteristic function of the interval $\left[-\frac{\delta}{4},+\frac{\delta}{4}\right]$. It follows directly from Lemma II.4 that $\tilde{g}_\lambda(x)$ is an almost periodic function with frequency-module $\mathfrak{M}'$; furthermore it is differentiable and its derivative $\tilde{g}'_\lambda(x)=g_\lambda\left(x+\frac{\delta}{4}\right)$ $-g_\lambda\left(x+\frac{\delta}{4}\right)$ is an almost periodic function with the same frequency-module which is $\mathfrak{M}'$. When δ is chosen as in Lemma II.4 iv), $\tilde{g}_\lambda(x)$ has the same number of zeroes as g_λ, and they are simple as can be seen from the above expression of $\tilde{g}'_\lambda(x)$. All functions in the hull of $\tilde{g}_\lambda(x)$ also have simple zeroes, and we can hence apply Lemma II.2 to $\tilde{g}_\lambda(x)$, and we obtain that the corresponding number of zeroes $\tilde{N}(x)$ satisfies $\lim_{x\to\infty}\frac{1}{2}\frac{\tilde{N}(x)}{x}\in\mathfrak{M}'$. But since $\tilde{g}_\lambda$ has the same number of zeroes as g_λ, this limit is also $2\alpha(\lambda)$ in view of ii) in Lemma II.4, and in view of the results in Sect. I concerning the existence and independence with respect to the boundary conditions of the rotation number. So $2\alpha(\lambda)\in\mathfrak{M}'$, but $2\alpha(\lambda)=k(\lambda)$ in view of Sect. I, and hence belongs to $[0,1]$, which implies that $2\alpha(\lambda)\in\mathfrak{M}$. This ends the proof of Theorem II.1. □

III. The General Second Order Difference Equation

We study now the operator $H(J)$:

$$[H(J)u](n) = -J_{n,n+1}u(n+1) - J_{n,n-1}u(n-1) + V(n)u(n), \qquad \text{(III.1)}$$

for which we assume $J_{n,n+1} = J_{n+1,n}$ for all $n \in \mathbb{Z}$. We suppose that the operator $H(J)$ is almost periodic, which means that it satisfies the Bohr criterion, in the operator norm; it implies that the sequences $\{J_{n,n+1}\}_{n\in\mathbb{Z}}$ and $\{V(n)\}_{n\in\mathbb{Z}}$ are almost periodic. We denote by $\mathfrak{M}$ the frequency module of the operator $H(J)$; it is also the module generated by the frequency modules $\mathfrak{M}_J$ and $\mathfrak{M}_V$ of the sequences J and V. It follows from the same references as for Lemma I.5 above, that the integrated density of states $k(\lambda)$ of $H(J)$ exists, is equal to the limit as $L \to \infty$ of the density of states of the restriction of $H(J)$ to $\{1, \ldots, L\}$ with arbitrary boundary conditions at 0, and is continuous in λ.

We are going to prove the following:

Theorem III.1. *For any real λ in the resolvent set, the integrated density of states $k(\lambda)$ satisfies*

$$k(\lambda) \in \mathfrak{M}.$$

Proof. The spirit of the proof is very similar to Sect. I and II above, and we will just sketch the differences. First let us consider the following unitary transformation

$$\begin{aligned} \psi(n) &= u(n) \sum_{i=0}^{n-1} \operatorname{sgn}(J_{i,i+1}) \quad \text{if} \quad n > 0, \\ \psi(n) &= u(n) \sum_{i=0}^{n+1} \operatorname{sgn}(J_{i-1,i}) \quad \text{if} \quad n < 0, \\ \psi(0) &= u(0), \end{aligned} \qquad \text{(III.2)}$$

where $\operatorname{sgn}(J_{i,i+1})$ denotes the sign of $J_{i,i+1}$ and is taken, by convention, as $+1$ when $J_{i,i+1} = 0$. By the transformation (IV.2), the operators $H(J)$ and $H(|J|)$ are unitarily equivalent and hence have the same integrated density of states. Let now λ be in the resolvent set of $H(J)$; it is also in the resolvent set of $H(|J|)$, and also of $H(|J| + \varepsilon)$ for all ε small enough, where $H(|J| + \varepsilon)$ is the operator deduced from $H(|J|)$ by adding $-\varepsilon$ to all its coefficients $-|J_{i,i+1}|$. By the operator convergence and the continuity of $k(\lambda)$, the integrated density of states of $H(|J| + \varepsilon)$ will converge to $k(\lambda)$ as $\varepsilon \to 0$; hence in order to prove Theorem III.1, it is sufficient to prove it in the case of an operator (III.1) with $J_{n,n+1} > \varepsilon > 0$ for all n. We can now relate our density of states to the number of changes of sign of a sequence $u(n)$ solution of $H(J)u = \lambda u$, as in the proof of Lemma I.3; Eq. (I.2) becomes now $\delta_i = V(i) - \lambda - J_{i-1,i}^2/\delta_{i-1}$, which is also the equation satisfied by the sequence $J_{n,n+1}\dfrac{u_{n+1}}{u_n}$. Theorem III.1 can then be obtained in a way parallel to Sect. II above; the Wronskian is now given as:

$$W(u^+, u^-) = J_{n,n+1}(u^+(n+1)u^-(n) - u^+(n)u^-(n+1))$$

and is still identical to 1. Equation (II.7) is now

$$g_\lambda(n+y)=(1-y)^2G_\lambda(n,n)+2y(1-y)G_\lambda(n,n+1)+y^2G_\lambda(n+1,n+1)-\frac{y(1-y)}{J_{n,n+1}},$$

and at zero of g_λ, its derivative now satisfies $g'_\lambda(x)=\pm\frac{1}{J_{n,n+1}}$ for $n<x<n+1$. The discontinuity (II.8) of the derivative of g_λ at integer points is now

$$g'_\lambda(n-0)-g'_\lambda(n+0)=2[-J_{n,n+1}-J_{n,n-1}+V(n)-\lambda]g_\lambda(n). \quad \square$$

As a matter of fact the above proof also suggests the definition of a rotation number $\alpha(\lambda)$ for the operator $H(J)$, which reduces to the one of Sect. I when $J_{i,i+1}=1$ for all i. One first introduces the number $\tilde{N}_L(\lambda)$ of changes of sign of the sequence $\psi(n)$, defined as in formula (III.2), where u is a solution of $H(J)u=\lambda u$ with given boundary conditions at 0 and 1. One then proves the analogue of Sturm-Liouville theory as in Lemma I.3, with the use of the modifications mentioned in the proof of Theorem III.1. We can then define the rotation number $\alpha(\lambda)$ as the limit, if it exists, of $\frac{1}{2}\frac{1}{L}N_L(\lambda)$ as $L\to\infty$. If the operator $H(J)$ is now a random variable on some probability space $(\Omega,\mathscr{B},\mathbb{P})$, $\mathbb{P}$ being ergodic with respect to the shift on $\mathbb{Z}$, which includes the case of almost periodic operators, but also of random operators with J and/or V random variables, we have:

Proposition III.2. *For $\mathbb{P}$ almost all $H(J)$, the rotation number $\alpha(\lambda)$ exists, is independent of the boundary conditions and*

$$2\alpha(\lambda)=k(\lambda).$$

In the case of an almost periodic operator, these results hold for all $H(J)$ in the hull.

Remark. In the above introduction of the rotation number we implicitly assumed that $J_{i,i+1}\neq 0$ for all J. Otherwise it is defined by continuity as was done in the proof of Theorem III.1.

Finally let us mention that for example in the case of a random $H(J)$, where $J_{i,i+1}$ and $V(n)$ are independent random variables with respective densities of probability $p(J)$ and $r(V)$, we have an analogue of Proposition I.7 for the density of states:

Proposition III.3.

$$k(\lambda)=2\alpha(\lambda)=\int_{-\infty}^{0}\mu_\lambda(x)dx,$$

where $\mu_\lambda(x)$ is the unique L_1 function satisfying

$$\mu_\lambda(x)=\int\mu_\lambda(y)r\left(\lambda-x-\frac{J^2}{y}\right)p(J)dy\,dJ.$$

Concluding Remarks

i) Theorem II.1 (or III.1) tells that for any gap of H, there exist a finite number of integers m_n such that $2\alpha(\lambda)=\sum_n m_n\omega_n$, where the ω_n are the frequencies, including the frequency 1 of $\mathbb{Z}$. As explained in [9] (see the remark after Theorem 4.7) these integers, which label the gaps, have a clear topological interpretation as winding numbers.

ii) As mentioned in the same remark of [9], it is possible to look only at the winding of one particular solution, for example the one which decays at $+\infty$, instead of looking at the winding of the Green's function. This variation of the same proof can also of course be readily adapted to the discrete case.

iii) The integers m_n are clearly constant if one varies λ and the operator H in such a way that λ remains in a gap.

iv) The $\alpha(\lambda)$ and $k(\lambda)$ are constant in a gap but are strictly increasing for λ in the spectrum of H. We suppressed the simple proof of this fact that we gave in the preprint of this note, since it was proved before in [1b] by another simple proof.

v) We have mentioned in the introduction that the spectrum for almost periodic potentials has a "tendency" to be a Cantor set. As a matter of fact there are already a certain number of results in this direction. First of all one may note that if $V(\cdot)$ is not periodic and if all gaps are opened, i.e. if for any y in the frequency-module there is a gap where $2\alpha(\lambda)=y$, then the spectrum is a Cantor set [15]; and actually it is known, in the special case of limit-periodic potentials, that the spectrum is a Cantor set [11, 1a, 5] in a certain number of situations, and it is also a Cantor set for the almost Mathieu equation in some generic sense [3]. Let us note that in the limit periodic case, all gaps allowed by gap labelling occur.

vi) Michel Herman has independently obtained a definition of a rotation number $\tilde{\alpha}(\lambda)$ for almost periodic potentials, and a proof that $2\tilde{\alpha}(\lambda)$ lies in the frequency-module whenever λ is in a gap, that is analogue of our Theorem II.1. His definition is however completely different from ours and is purely geometrical, whereas our approach is purely analytical, and both proofs of Theorem II.1 are also completely different. We do not know if there is a systematic link between his $\tilde{\alpha}(\lambda)$ and our $\alpha(\lambda)$.

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