Commun. Math. Phys. 89, 227–234 (1983)

Communications in Mathematical Physics

Kotani Theory for One Dimensional Stochastic Jacobi Matrices*

Barry Simon

Departments of Mathematics and Physics, California Institute of Technology, Pasadena, CA 91125, USA

Abstract. We consider families of operators, H_ω, on ℓ_2 given by $(H_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n)$, where V_ω is a stationary bounded ergodic sequence. We prove analogs of Kotani's results, including that for a.e. $\omega, \sigma_{ac}(H_\omega)$ is the essential closure of the set of E where $\gamma(E)$ the Lyaponov index, vanishes and the result that if V_ω is non-deterministic, then σ_{ac} is empty.

1. Introduction

In a beautiful paper, Kotani [10] has proved three remarkable theorems about one-dimensional stochastic Schrödinger operators, i.e. operators of the form $-d^2/dx^2 + V_\omega(x)$ on $L^2(-\infty,\infty)$, where V_ω is a stationary bounded ergodic process. It is not completely straightforward to extend his proofs to the case where $-d^2/dx^2$ is replaced by a finite difference operator, and that is our goal in this note.

Explicitly, let (Ω,μ) be a probability measure space, T a measure preserving invertible ergodic transformation, and f a bounded measurable real-valued function. We define $V_\omega(n) = f(T^n\omega)$. We let H_ω be the operator on $\ell^2(Z)$

$$(H_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n).$$

Integrals over ω will be denoted by $E(\,\cdot\,)$.

Given a subset, J, of Z, we let Σ_J be the sigma–algebra generated by $\{V_\omega(n)\}_{n\in J}$. We say that the process is *deterministic* if $\Sigma_{-\infty} \equiv \bigcap_{j=1}^{\infty} \Sigma_{(-\infty,-j)}$ is up to sets of measure zero, $\Sigma_{(-\infty,\infty)}$; equivalently if $V_\omega(n)$ is a.e., a measurable function of $\{V_\omega(n)\}_{n\leq 0}$. Otherwise it is *non-deterministic*. Almost periodic sequences are deterministic. Independent, identically distributed random variables are non-deterministic.

The Lyaponov index $\gamma(E)$ is defined, for example, in [1, 4]. It can be characterized as follows: For each complex E, for a.e. ω, any solution of $H_\omega u = Eu$ (in sequence sense) has $\lim\limits_{n\to\infty} \frac{1}{n} \ln[|u(n)|^2 + |u(n+1)|^2]^{1/2}$ exists and it is either γ or $-\gamma$. It is an

* Research partially supported by USNSF under Grant MCS-81-20833

old result of Pastur [11] and Ishii [7] (see also Casher–Lebowitz [3]) that $\gamma(E)$ on the real axis is related to absolutely continuous spectrum.

Theorem 0 ([7, 11]). *If $\gamma(E)>0$ on some set A in R, then $E_\omega^{ac}(A)=0$ for* a.e. ω, *where E_ω^{ac} is the absolute component of the spectral projection for H_ω.*

Here we will prove the following:

Theorem 1. *If $\gamma(E)=0$ on a subset, A, of R with positive Lebesgue measure, then $E_\omega^{ac}(A)\neq 0$ for* a.e. ω.

Theorem 2. *If $\gamma(E)=0$ on an open interval, I, of R, then for* a.e. ω, *the spectral measures are purely absolutely continuous on I.*

Theorem 3. *If the hypotheses of Theorem 1 hold, then V_ω is deterministic.*

Theorems 0 and 1 show that σ_{ac} is for a.e. ω the essential closure of the set where $\gamma(E)=0$. Theorem 3, which can be viewed as a kind of generalized Furstenberg theorem, says Thms. 1 and 2 aren't applicable very often. Theorems 0 and 3 imply that if V is non-deterministic, $\sigma_{ac}=\varnothing$. Theorems 1 and 2 are related to recent results of Carmona [2].

Theorems 1–3 are precise analogs of the main results of Kotani [10] in the continuous case. Kotani uses functions $h_\pm(\omega,E)$ defined for $\operatorname{Im}E>0$ by the following: If $\operatorname{Im}E>0$, there are unique (up to factor) solutions, $u_\pm(x,\omega,E)$, of $-u''+(V-E)u=0$ which are L^2 at $\pm\infty$. Define

$$h_\pm(\omega,E)=\pm\frac{u'_\pm(0,\omega,E)}{u_\pm(0,\omega,E)}.$$

As is well-known, the Green's function obeys

$$G^\omega(0,0;E)=-(h_++h_-)^{-1}. \tag{1.1}$$

Since $E(G)$ is the Borel transform of the density of states and the Thouless formula relates γ to this density of states (see e.g. [1]), one has:

$$E(\operatorname{Im}([h_++h_-]^{-1}))=-\partial\gamma(E)/\partial(\operatorname{Im}E). \tag{1.2}$$

Using the formula of Johnson and Moser [8]

$$E(\operatorname{Re}h_+)=E(\operatorname{Re}h_-)=-\gamma(E), \tag{1.3}$$

Kotani then proves:

$$E((\operatorname{Im}h_\pm)^{-1})=2\gamma(E)/\operatorname{Im}E. \tag{1.4}$$

Equations (1.2) and (1.4) then imply

$$\begin{aligned}&E([(\operatorname{Im}h_+)^{-1}+(\operatorname{Im}h_-)^{-1}]\{(\operatorname{Im}h_+-\operatorname{Im}h_-)^2+(\operatorname{Re}h_++\operatorname{Re}h_-)^2\}/|h_++h_-|^2)\\&\quad=4[(\operatorname{Im}E)^{-1}\gamma(E)-\partial\gamma(E)/\partial\operatorname{Im}E].\end{aligned} \tag{1.5}$$

The three theorems then follow from (1.4), (1.5).

The initial stages of extending Kotani's analysis are obvious. The proper analog of $h_\pm$ are:

$$m_\pm(\omega,E) = -u_\pm(\pm 1)/u_\pm(0),$$

where $u_\pm$ are the solutions l^2 at $\pm\infty$. The analog of (1.2) which will come from an analog of (1.1) is

$$E(\mathrm{Im}([m_+ + m_- + E - V(0)]^{-1})) = -\partial\gamma(E)/\partial(\mathrm{Im}\,E). \tag{1.6}$$

The analog of (1.3) is also easy:

$$E(\ln|m_+|) = E(\ln|m_-|) = -\gamma(E). \tag{1.7}$$

The analog of (1.4) is more subtle because Kotani's proof does not seem to extend. However, our first proof of (1.8) was by using the idea of Delyon–Souillard [5] to use linear interpolation to force the discrete case to look like the continuum case. By a more direct proof we will show, in Sect. 2, that

$$E(\ln[1 + (\mathrm{Im}\,E/\mathrm{Im}\,m_\pm)]) = 2\gamma(E). \tag{1.8}$$

It is not completely trivial to get an analog of (1.5). The key is the inequality

$$\ln(1+x) \geqq x/(1+\tfrac{1}{2}x).$$

From this and (1.8), we will get, in Sect. 2, two inequalities which are close enough to the equalities (1.4), (1.5) to prove Thms. 1–3 in Sect. 3. In Sect. 4, we make a remark on the connection of these results and the work of Carmona [2].

2. The *m* Functions

Given E with $\mathrm{Im}\,E > 0$ and ω, it is easy to show that the difference equation

$$u(n+1) + u(n-1) + V_\omega(n)u(n) = Eu(n) \tag{2.1}$$

has unique solutions $u_\pm(n)$ which are ℓ^2 at $\pm\infty$. Moreover,

$$2i\,\mathrm{Im}(\overline{u_\pm(0)u_\pm(\pm 1)}) = \overline{u_\pm(0)}u_\pm(\pm 1)) - \overline{u_\pm(\pm 1)}u_\pm(0).$$

Recognizing this as a Wronskian of solutions of (2.1) for E and $\bar{E}$, and using the fact that $u_\pm \to 0$ at $\pm\infty$, one finds that

$$\mathrm{Im}(\overline{-u_\pm(0)}u_\pm(\pm 1)) = \mathrm{Im}\,E\left(\sum_{j=1}^{\infty} |u_\pm(\pm j)|^2\right), \tag{2.2}$$

so that $u_\pm(0) \neq 0$, and we can define

$$m_\pm(\omega,E) = -\frac{u_\pm(\pm 1)}{u_\pm(0)}, \tag{2.3}$$

and by (2.2), it obeys $\mathrm{Im}\,m_\pm > 0$. For later purpose we note that

$$m_\pm(T^{-n}\omega) = -u_\pm(n \pm 1)/u_\pm(n), \tag{2.4}$$

so that the equation of motion for u yields

$$m_\pm(T^{-n}\omega) = V(n) - E - [m_\pm(T^{-n\pm 1}\omega)]^{-1}, \tag{2.5}$$

and in particular

$$\frac{u_-(1)}{u_-(0)} = m_- + E - V(0). \tag{2.6}$$

As usual, $(H_\omega - E)^{-1}$ has an integral kernel $G_\omega(n,m;E)$ which is symmetric in n,m and for $n \leqq m$:

$$G_\omega(n,m;E) = u_-(n)u_+(m)/[u_+(1)u_-(0) - u_-(1)u_+(0)].$$

In particular, (2.3) and (2.6) yield

$$-G_\omega(0,0;E)^{-1} = m_+ + m_- + E - V(0). \tag{2.7}$$

Now, $G_\omega(0,0;E)$ is related to the density of states by [1, 8]

$$E(G_\omega(0,0;E)) = \int \frac{dk(E')}{E' - E}. \tag{2.8}$$

The Thouless formula [1] says that

$$\gamma(E) = \int \ln|E - E'|\, dk(E'). \tag{2.9}$$

Equations (2.7), (2.8) and (2.9) immediately imply:

Proposition 2.1. $E(\mathrm{Im}([m_+ + m_- + E - V_\omega(0)]^{-1})) = -\partial\gamma(E)/\partial(\mathrm{Im}\,E)$.

We let H_ω^+ be the operator on $\ell_2(1,\infty)$ which is obtained from H_ω by imposing the boundary condition (bc) $u(0) = 0$. If $w(n)$ obeys (2.1) with the bc $w(0) = 0$, $w(1) = 1$, then for $n \leqq m$:

$$(H_\omega^+ - E)^{-1}(n,m) = w(n)u_+(m)/[u_+(1)w_+(0) - w(1)u_+(0)],$$

and in particular

$$m_+(\omega,E) = (H_\omega^+ - E)^{-1}(1,1). \tag{2.10a}$$

By the spectral theorem, the right side of (2.10) has the form

$$\int \frac{d\rho(x)}{x - E}, \tag{2.10b}$$

where $\int d\rho \equiv 1$ and ρ is supported on $[-\|f\|_\infty - 2,\ \|f\|_\infty + 2]$. From this representation one easily obtains an upper bound on $|m_+|$ and a lower bound on $|\mathrm{Im}\,m_+|$ and so:

Proposition 2.2. *For any fixed E with* $\mathrm{Im}\,E > 0$, *there are constants* $c_1(E)$, $c_2(E)$, $d_1(E)$, $d_2(E)$ *in* $(0,\infty)$ *with*

$$c_1(E) \leqq |m_+(\omega,E)| \leqq c_2(E),$$
$$d_1(E) \leqq \mathrm{Im}\,m_+(\omega,E) \leqq d_2(E),$$

for all ω.

From the bounds on $|m_+|$ and the fact that for a.e. ω every solution either "decays" as $e^{-\gamma|n|}$ or grows as $e^{+\gamma|n|}$, we see that

$$\lim_{n\to\infty} \frac{1}{n} \ln|u_+(n)/u_+(0)| = -\gamma.$$

Since $\ln|u_+(n)/u_+(0)| = \sum_{j=0}^{n-1} \ln|m_+(T^{-j}\omega)|$ (by (2.4)), we can apply the individual ergodic theorem ($\ln|m_+(\omega)|$ is bounded and so in L^1 by Prop. 2.2) to find

Proposition 2.3. $E(\ln|m_+(\omega,E)|) = -\gamma(E)$.

Now, we come to the first result of this note that is essentially new.

Proposition 2.4. $E(\ln(1+[\operatorname{Im}E/\operatorname{Im}m_+(\omega,E)])) = 2\gamma(E)$.

Proof. We start with (2.5). Taking imaginary parts, then dividing by $\operatorname{Im}m_+$ and taking logs we find

$$\ln(1+[\operatorname{Im}E/\operatorname{Im}m_+(\omega,E)]) = \ln(-\operatorname{Im}[m_+(T\omega,E)]^{-1}) - \ln(\operatorname{Im}m_+(\omega,E)).$$

But $-\operatorname{Im}[m_+^{-1}] = \operatorname{Im}m_+/|m_+|^2$, so taking expectations of both sides and using the invariance of μ under T, we find that the expectation of the right side is $-E(\ln|m_+|^2)$ which is 2γ by Prop. 2.3. ■

Lemma 2.5. *For* $x \geqq 0$, $\log(1+x) \geqq x/(1+\frac{1}{2}x)$.

Proof. Both sides are equal at $x=0$. The derivative of the left hand side is $(1+x)^{-1}$ and that of the right is $(1+\frac{1}{2}x)^{-2} = (1+x+\frac{1}{4}x^2)^{-1}$, so we get the inequality by integrating. ■

Theorem 2.6. *Let* $b(\omega,E) = m_+ + m_- + E - V(0)$ *and* $n_\pm = \operatorname{Im}m_\pm + \frac{1}{2}\operatorname{Im}E$. *Then*:

(a) $E((n_\pm)^{-1}) \leqq 2\gamma(E)/\operatorname{Im}E$,

(b) $E([n_+^{-1}+n_-^{-1}]\{(n_+-n_-)^2+(\operatorname{Re}b)^2\}/|b|^2) \leqq 4[(\operatorname{Im}E)^{-1}\gamma(E) - \partial\gamma(E)/\partial\operatorname{Im}E]$.

Proof. (a) follows immediately from Prop. 2.4 and the inequality in the lemma. To get (b), we write $(n_+-n_-)^2 = (n_++n)^2 - 4n_+n_-$, and using the fact that $n_+ + n_- = \operatorname{Im}b$, we see the argument in the expectation is $n_+^{-1}+n_-^{-1} - 4(n_++n_-)/b^2 = n_+^{-1}+n_-^{-1}+4\operatorname{Im}(1/b)$. We use Prop. 2.1 to get $E(\operatorname{Im}(1/b))$ and (a) to bound $E(n_\pm^{-1})$ to get the required inequalities.

3. Proofs of the Theorems

Given Thm. 2.6, the proof below follows the strategy of Kotani [10] with some changes of tactics. We begin by recalling without proofs some basic facts about Herglotz functions. As remarked by Kotani [10], these are proven most easily by mapping the upper half plane to the disc, taking logs and using the theory of H_2 functions (see e.g. [6, 9]).

(1) $F(z)$ defined in $\operatorname{Im}z > 0$ is called Herglotz if it is analytic and has $\operatorname{Im}F(z) > 0$ there. A typical example (indeed, up to linear factors, every example) is the Steiltjes

transform of a measure, μ, on R, viz:

$$F(z)=\frac{1}{\pi}\int\frac{d\mu(x)}{x-z}\cdot \tag{3.1}$$

(2) $\lim_{\varepsilon\downarrow 0} F(x+i\varepsilon)\equiv F(x+i0)$ exists (and is finite and non-zero) for a.e. $x\in R$.

(3) If F comes from μ, then $d\mu_{\mathrm{ac}}$, the absolutely continuous part of μ, obeys

$$d\mu_{\mathrm{ac}}(x)=[\operatorname{Im}F(x+i0)]\,dx. \tag{3.2}$$

(4) If F comes from μ, $d\mu_{\mathrm{sing}}\equiv d\mu-d\mu_{\mathrm{ac}}$ is supported on $\{x|\lim_{\varepsilon\downarrow 0}\operatorname{Im}F(x+i\varepsilon)=\infty\}$.

(5) If $F(x+i0)=G(x+i0)$ for $x\in A$, a set with positive Lebesgue measure and F and G are Herglotz, then $F=G$.

(6) If $\operatorname{Re}F(x+i0)=0$ a.e. $x\in I$, an open interval, then F has an analytic continuation through I and $F(x+i0)\neq 0$ for *any* x in I.

(7) By (4) and (6), if F is a Steiltjes transform and $\operatorname{Re}F(x+i0)=0$ on I, then $\mu=\mu_{\mathrm{ac}}$ on I.

Proof of Theorem 1. By (2.8), (2.9), $-\gamma(E)$ is the real part of a function whose derivative $\int\ (dh(E')/E'-E)$ is a Steiltjes transform. Thus, by (2) above, $\lim_{\varepsilon\downarrow 0} d\gamma(E^0+i\varepsilon)/d\varepsilon$ exists for a.e. E_0. For any such E_0 where also $\gamma(E_0)=0$, we have that

$$\lim_{\varepsilon\downarrow 0}\gamma(E_0+i\varepsilon)/\varepsilon=\lim_{\varepsilon\downarrow 0}d\gamma(E_0+i\varepsilon)/d\varepsilon, \tag{3.3}$$

and in particular the limit is finite. Thus, by Thm. 2.6(a),

$$\varlimsup_{\varepsilon\downarrow 0} E\left(\frac{1}{\operatorname{Im}m_\pm(\omega,E_0+i\varepsilon)}\right)<\infty\cdot \tag{3.4}$$

By (2.10b), for every $\omega, m_\pm(\omega,E+i0)$ exists for a.e. E so for a.e. $E, m_\pm(\omega,E+i0)$ exists for a.e. ω. Thus, for a.e. E_0 for which $\gamma(E_0)=0$, we have by (3.4) and Fatou's lemma that

$$E\left(\frac{1}{\operatorname{Im}m_\pm(\omega,E_0+i0)}\right)<\infty. \tag{3.5}$$

So, for a.e. ω,E_0, $\operatorname{Im}m_\pm(\omega,E_0+i0)>0$. Since $m_++m_-+E-V(0)$ has a finite limit for a.e. ω,E, $\operatorname{Im}G>0$ a.e. E_0,ω which implies μ_{ac} has a positive component on such E_0 by (3.2). ■

Proof of Theorem 2. By (3.3), (3.5) and Thm. 2.6(b) and Fatou again, we learn that, for a.e. pair $\{(\omega,E)|\gamma(E)=0\}$, we have that

$$\operatorname{Im}m_+(\omega,E_0+i0)=\operatorname{Im}m_-(\omega,E_0+i0), \tag{3.6}$$

$$\operatorname{Re}(m_++m_-+E_0-V(0))(\omega,E_0+i0)=0. \tag{3.7}$$

By (6) above, $m_+ + m_- + E - V(0)$ is analytic on I and nonzero, so (by (2.7)) G is analytic through I which, by (4) above, implies $d\mu_{\text{sing}} = 0$ on I. ■

Proof of Theorem 3. Suppose that $\gamma(E) = 0$ on a set A with positive measure. Suppose we know $V_\omega(n)$ on $n \leqq 0$. Then, $\{V_\omega(n)\}_{n \leqq -1}$ determines m_- and so by (3.6), (3.7), m_+ is determined for a.e. $E_0 \in A$ (and a.e. ω) by $\{V_\omega(n)\}_{n \leqq 0}$ and then by (5) above, m_+ is determined for all E. Thus the lemma below (which we learned from P. Deift) shows that $\{V_\omega(n)\}_{n \leqq 0}$ determines $\{V_\omega(n)\}_{n \geqq 1}$. ■

Lemma 3.1. *$\{V_\omega(n)\}_{n \geqq 1}$ can be constructed from $m_+(\omega,E)$.*

Proof. By (2.10), $m_+(\omega,E)$ determines $(H_\omega^+)^k(1,1)$. But it is easy to see that $(H_\omega^+)^{2k+1}(1,1) = V_\omega(k+1) + a$ function of $\{V_\omega(j)\}_{1 \leqq j \leqq k}$, so that inductively $(H_\omega^+)^k(1,1)$ determines $V_\omega(j)$. ■

4. A Connection with some work of Carmona

In [2], Carmona proved an interesting deterministic theorem showing that certain conditions on $\{V(n)\}_{n \geqq 0}$ imply $H = H_0 + V$ has only absolutely continuous spectrum in some interval. Here we give another condition which is clearly closely connected to his which yields the same conclusion. For any V yielding a limit point situation at $\pm\infty$, say $|V(n)| \geqq -Cn^2$, we still have functions $m^\pm(E)$ and $m^\pm$ depend only on $\{V(n)\}_{\pm n \geqq 1}$.

Theorem 4.1. *If* $\lim_{\varepsilon \downarrow 0} \operatorname{Im} m^+(E + i\varepsilon) > 0$ *for* ***all*** *E in a set A, then for the spectral measure $d\mu$ associated to δ_0, we have $\mu_{\text{sing}}(A) = 0$.*

Proof. By assertion (4) above (the theorem of de'Vallee Poussin), $\mu_{\text{sing}}(C) = 0$, where $C = \{E \mid \overline{\lim_{\varepsilon \downarrow 0}} |G(0,0;E + i\varepsilon)| < \infty\}$. But since $G = -(m_+ + m_- + E - V(0))^{-1}$, we have that $|G| \leqq (\operatorname{Im} m_+ + \operatorname{Im} m_- + \operatorname{Im} E)^{-1} \leqq (\operatorname{Im} m^+)^{-1}$ so the hypothesis implies $A \subset C$. ■

This is connected to the considerations of Kotani, in that:

Proposition 4.2. *In the stochastic context of Sect.* 1–3, *if $\gamma(E) = 0$ on an interval, I, then for* a.e. ω, $\operatorname{Im} m^+(E + i0,\omega) > 0$ *for all $E \in I$.*

Proof. As we saw in Sect. 3, $\operatorname{Im}(m_+ + m_-)$ is *everywhere* nonzero and $\operatorname{Im} m_+ = \frac{1}{2}\operatorname{Im}(m_+ + m_-)$. ■

This shows that the periodic example of Carmona [2] can be analyzed using Thm. 4.1. Similarly, these methods extend to the continuum case and it must be true that for the Stark problem $-d^2/dx^2 - x$, $\operatorname{Im} m^+ > 0$ for all E. This leaves us with an open question: Within the stochastic setting, if $\gamma(E) = 0$ for all $E \in I$, is it true that for all ω and every compact $K \subset I$, we have that $\sup_{E \subset K, x > 0} \|U_E(x,0)\| < \infty$, where $U_E(x,0)$ is the transfer matrix from 0 to x?

Acknowledgement. It is a pleasure to thank S. Kotani for making his preprint available, and W. Craig and P. Deift for useful discussions.

References

1. Avron, J., Simon, B.: Almost periodic Schrodinger operators, II. The integrated density of states. Duke Math. J. (to appear)
2. Carmona, R.: One dimensional Schrondinger operators with random or deterministic potentials: New spectral types. J. Funct. Anal. **51**, (1983)
3. Casher, A., Lebowitz, J.: Heat flow in disordered harmonic chains. J. Math. Phys. **12**, 8 (1971)
4. Craig, W., Simon, B.: Subharmonicity of the Lyaponov index. Duke Math. J. (submitted)
5. Delyon, F., Souillard, B.: The rotation number for finite difference operators and its properties. Commun. Math. Phys. (to appear)
6. Donoghue, W.: Distributions and Fourier transforms. New York: Academic Press 1969
7. Ishii, K.: Localization of eigenstates and transport phenomena in the one dimensional disordered system. Supp. Theor. Phys. **53**, 77–138 (1973)
8. Johnson, R., Moser, J.: The rotation number for almost periodic potentials. Commun. Math. Phys. **84**, 403–438 (1982)
9. Katznelson, Y.: An introduction to harmonic analysis. New York: Dover 1976
10. Kotani, S.: Lyaponov indices determine absolutely continuous spectra of stationary random one-dimensional Schrondinger operators. Proc. Kyoto Stoch. Conf., 1982
11. Pastur, L.: Spectral properties of disordered systems in the one body approximation. Commun. Math. Phys. **75**, 179–196 (1980)

Communicated by T. Spencer
Received December 15, 1982