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On the Invariant Sets of a Family of Quadratic Maps

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Abstract. The Julia set B_λ for the mapping $z \to (z-\lambda)^2$ is considered, where λ is a complex parameter. For $\lambda \geqq 2$ a new upper bound for the Hausdorff dimension is given, and the monic polynomials orthogonal with respect to the equilibrium measure on B_λ are introduced. A method for calculating all of the polynomials is provided, and certain identities which obtain among coefficients of the three-term recurrence relations are given. A unifying theme is the relationship between B_λ and λ-chains $\lambda \pm \sqrt{(\lambda \pm \sqrt{(\lambda \pm \ldots)}}$, which is explored for $-\frac{1}{4} \leqq \lambda \leqq 2$ and for $\lambda \in \mathbb{C}$ with $|\lambda| \leqq \frac{1}{4}$, with the aid of the Böttcher equation. Then B_λ is shown to be a Hölder continuous curve for $|\lambda| < \frac{1}{4}$.

1. Introduction

In this paper we consider the Julia set B_λ for the mapping

$$T_\lambda z = (z-\lambda)^2, \quad z \in \mathbb{C},$$

of the complex plane into itself, where λ is a parameter which may be real or complex. Here T_λ is equivalent to $z \to 1 - \lambda z^2$ which has been studied in the context of iterated maps of intervals, see [10, 13], and also to $z \to z^2 + \lambda$, see [11].

B_λ was first studied by Fatou [12] and Julia [19] in the context of arbitrary rational transformations. With the notation

$$T_\lambda^0 z = z, \quad \text{and} \quad T_\lambda^{n+1} z = T_\lambda(T_\lambda^n z) \quad \text{for} \quad n \in \{1, 2, 3, \ldots\},$$

B_λ can be defined to be those points in $\mathbb{C}$ where $\{T_\lambda^n z\}$ is not normal. This is the starting point of the survey by Brolin [8]. Equivalently B_λ can be defined to be the closure of the set of all repulsive k-cycles, $k \in \{1, 2, 3, \ldots\}$, [12]. This shows at once the relevance of B_λ to the corresponding iterated real map where $B_\lambda \cap \mathbb{R}$ plays a central role.

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Our approach is to consider the set $\tilde{B}_\lambda$ of formal objects which we call λ-chains, $\{\lambda \pm \sqrt{(\lambda \pm \sqrt{(\lambda \pm \ldots\}}}$, where all half-infinite sequences of plus and minus signs are included, and where the branch cut is fixed, for example, on the negative real axis. We use λ-chains as a unifying idea in our discussion of B_λ. For $\lambda > 2$ we easily work out a one-to-one correspondence between the elements of $\tilde{B}_\lambda$ and the points of B_λ. For $-\frac{1}{4} \leqq \lambda \leqq 2$ the correspondence is exhibited via the Böttcher equation and conformal mapping, and for some values of λ we show only that almost all λ-chains correspond to individual points in B_λ.

In Sect. 2.1 we examine the case $2 \leqq \lambda < \infty$, where we give a direct construction of B_λ which displays its connection with $\tilde{B}_\lambda$. Use is made of a distance function, natural to B_λ, which yields a simple demonstration that the Lebesgue measure of B_λ is zero for $\lambda > 2$ and which provides a new upper bound on its Hausdorff dimension. We also construct the equilibrium measure σ on B_λ by means of a special sequence of approximating measures. The latter are related to the monic orthogonal polynomials with respect to σ which are considered in Sect. 2.2. A method for calculating all of the polynomials is provided, and certain identities which obtain among the coefficients of the three-term recurrence relations are discovered. Interrelations between the polynomials reflect the structure of B_λ. In particular, it is found possible to describe completely an infinite subsequence of Padé approximants to the generating function. The polynomials generalize those of Tchebycheff, to which they are simply related when $\lambda = 2$.

In Sect. 3.1, we present for the case $-\frac{1}{4} \leqq \lambda \leqq 2$ two constructions for B_λ; one from the "outside" and one from the "inside". Each involves a sequence of functions which converges to the solution of the functional equation $F_\lambda(z) = \lambda + \sqrt{F_\lambda(z^2)}$ with $F_\lambda(z) = z + O(1)$ at ∞. Here F_λ maps the exterior of the unit disk conformally onto a region bounded by B_λ, and by means of F_λ we relate λ-chains to B_λ. The first construction involves the formation of an increasing sequence of domains, successive inverse images under T_λ of a neighborhood of ∞, as suggested by Fatou and by Julia. We include it for completeness, and to shorten the proof of the second construction which involves a decreasing sequence of domains. Of interest are the complements of the domains which form an increasing sequence of trees and describe B_λ from the interior. We have recently proved [4] that this sequence of trees converges to B_λ itself for infinitely many values of $\lambda \in (0, 2)$.

In Sect. 3.2 we begin by restricting attention to $0 \leqq \lambda \leqq 0.2$ and we make explicit calculations with λ-chains to show that they are well-defined and that $F_\lambda(z)$ can be extended continuously to the unit circle. Next we make analytic extensions to λ in the set $L = \{\lambda \in \mathbb{C} \,|\, |\lambda| < 1/4\}$ and show that the mappings $G_\lambda(\theta) = F_\lambda(e^{i\theta})$ are uniformly Hölder continuous for λ in any compact subset of L.

In Sect. 4 we give a few pictures in connection with B_λ.

2. The Case $2 \leqq \lambda < \infty$

2.1. Construction of B_λ and of an Invariant Measure

Throughout this section we assume $\lambda \in [2, \infty)$. We begin with a construction of the Julia set B_λ which displays its connection with $\tilde{B}_\lambda$. Thus we obtain a concrete example of the correspondence apparently first introduced by Fatou [12, Sect. 23] for arbitrary rational transformations. Our approach uses a new distance function which allows us to obtain a significant improvement on Brolin's estimate [8, Theorem 12.2] of the Hausdorff dimension of B_λ.

We conclude this section with the construction of an invariant measure, supported on B_λ, by means of an approximating sequence of measures, different from the ones used by Brolin [8, p. 126], with the advantage that they are related to the measures which come from associated orthogonal polynomials. A related reference is [16].

Let Ω denote the set of all semi-infinite sequences of numbers from $\{-1,+1\}$. Then $\omega\in\Omega$ if and only if

$$\omega=(e_1,e_2,e_3,\ldots), \qquad e_i\in\{-1,+1\}.$$

Let

$$a=\lambda+\tfrac{1}{2}+\sqrt{\lambda+1/4} \quad \text{and} \quad I=[\lambda-\sqrt{a},a].$$

Note that a is the unique nonnegative real solution of $a=\lambda+\sqrt{a}$ and that $\lambda-\sqrt{a}\geqq 0$, where the inequality is strict when $\lambda>2$.

We define for $n\in\{0,1,2,\ldots\}$

$$s_0(\omega,x)=x,$$
$$s_n(\omega,x)=\lambda+e_1\sqrt{(\lambda+e_2\sqrt{(\lambda+\ \ldots\ +e_n\sqrt{(x)}\ldots))}}.$$

Then it is readily proved that $s_n:\Omega\times I\to I$.

We introduce a measure μ on I by

$$\mu E=\int_E \frac{dw}{\sqrt{w(2\lambda-w)}}$$

for all Lebesgue measurable subsets E of I. Then μ is absolutely continuous with respect to Lebesgue measure on I because $\int_I \frac{dw}{\sqrt{w(2\lambda-w)}}<\infty$. Also, Lebesgue measure is absolutely continuous with respect to μ on I because $\int_E dx=\int_E \sqrt{w(2\lambda-w)}\,d\mu(\omega)$. The corresponding distance function is

$$d(x,y)=|F(x)-F(y)| \quad \text{for} \quad x,y\in I,$$

where we define

$$F(x)=\int_0^x \frac{dw}{\sqrt{w(2\lambda-w)}}.$$

Lemma 1. *Let* $\lambda\in[2,\infty)$, $\varrho=\frac{1}{2}\cdot\sqrt{\frac{\lambda+\sqrt{a}}{\lambda^2-\lambda+\sqrt{a}}}$, *and* $n\in\{0,1,2,\ldots\}$. *Then*

$$d(s_n(\omega,x),s_n(\omega,y))\leqq\varrho^n d(x,y),$$

where $\varrho\leqq 1/2$, *the latter inequality being strict for* $\lambda>2$.

Proof. For $x<y$ and $e\in\{-1,+1\}$ we have by Cauchy's mean value theorem

$$\frac{d(\lambda+e\sqrt{x},\lambda+e\sqrt{y})}{d(x,y)}=\frac{|F(\lambda+e\sqrt{x})-F(\lambda+e\sqrt{y})|}{|F(x)-F(y)|}$$
$$=\left|\frac{1}{2}\cdot\frac{e}{\sqrt{C}}\cdot\frac{F'(\lambda+e\sqrt{C})}{F'(C)}\right|=\frac{1}{2}\sqrt{\frac{2\lambda-C}{\lambda^2-C}}$$

for some $C\in(x,y)$. Since $0\leqq\lambda-\sqrt{a}\leqq x<C<y\leqq a<2\lambda\leqq\lambda^2$, the right-hand-side is bounded above by ϱ. This establishes the lemma for $n=1$ and induction completes the proof. Q.E.D.

Lemma 2. *Let $\omega\in\Omega$, $x\in I$ and $\lambda\in[2,\infty)$. Then $s(\omega)=\lim_{n\to\infty}s_n(x,\omega)$ exists, belongs to I, and is a constant independent of $x\in I$. For $\lambda\in(2,\infty)$, $s:\Omega\to I$ is one-to-one.*

Proof. We have from Lemma 1

$$d(s_{n+1}(x,\omega),s_n(x,\omega))=d(s_n(\lambda+e_{n+1}\sqrt{x};\omega),s_n(x,\omega))$$
$$\leqq\varrho^n d(\lambda+e_{n+1}\sqrt{x},x)\leqq\varrho^n d(\lambda-\sqrt{a},a),$$

from which it follows that $\{s_n(x,\omega)\}_{n=1}^{\infty}$ is a Cauchy sequence. If

$$\lim_{n\to\infty}s_n(x,\omega)=s(\omega)\quad\text{and}\quad\lim_{n\to\infty}s_n(y,\omega)=\tilde{s}(\omega),$$

then

$$d(s(\omega),\tilde{s}(\omega))\leqq d(s(\omega),s_n(x,\omega))+d(\tilde{s}(\omega),s_n(y,\omega))+d(s_n(x,\omega),s_n(y,\omega)),$$

which can be made arbitrarily small by choosing n sufficiently large. The one-to-oneness follows at once from the fact that $0\notin I$ for $\lambda>2$. Q.E.D.

We now observe that the Julia set for $T_\lambda z=(z-\lambda)^2$, $\lambda\in[2,\infty)$ is precisely

$$B_\lambda=\{s(\omega)|\omega\in\Omega\}.$$

This follows from the fact that the Julia set is the set of all limit points of all finite order preimages of any point in the plane, with at most two exceptions, [8]. The relationship between B_λ and the collection of formal objects $\tilde{B}_\lambda$ is clear from this construction, and we will sometimes use the "λ-chain" notation

$$s(\omega)=\lambda+e_1\sqrt{}(\lambda+e_2\sqrt{}(\lambda+e_3\sqrt{}(\ldots))).$$

The known properties of B_λ, given by Brolin [8] and summarized in the following theorem, are now straightforwardly deduced.

Theorem 1. *For $\lambda\in[2,\infty)$, B_λ is perfect and totally invariant under T_λ, with*

$$T_\lambda s(\omega)=s(T\omega)\quad\textit{for all}\quad\omega\in\Omega,$$

where $T:\Omega\to\Omega$ is the left-shift operator. For $\lambda\in(2,\infty)$, B_λ is of Lebesgue measure zero.

Proof. Since for any $\omega \in \Omega$ and $x \in I$ we have

$$T_\lambda^{-1} s(\omega) = \left\{ \operatorname*{Lim}_{n\to\infty} s_n(\sigma, x) | \sigma \in T^{-1}\omega \right\} = s(T^{-1}\omega),$$

and since $T^{-1}\Omega = \Omega$, it follows that both $T_\lambda^{-1} B_\lambda = B_\lambda$ and $T_\lambda s(\omega) = s(T\omega)$.

Every element of B_λ is a limit point of other distinct elements of B_λ because if $\sigma_n \in \Omega$ agrees with $\sigma \in \Omega$ in the first n components then $\operatorname*{Lim}_{n\to\infty} s(\sigma_n) = s(\sigma)$. (We can ensure $\{s(\sigma_n)\}$ contains infinitely many distinct elements even when $\lambda = 2$ since for $\lambda = 2$, $s: \Omega \to B_\lambda$ is at most two-to-one.) Similarly, B_λ is compact because any infinite sequence in Ω contains a subsequence $\{\sigma_n\}$ and an element σ such that σ_n agrees with σ in the first n components. Hence B_λ is perfect.

Since $B_\lambda \subset I$ we have $B_\lambda \subset T_\lambda^{-n} I = I_n$, where I_n consists of 2^n intervals, each of which can be written $s_n(\omega, I)$ for some $\omega \in \Omega$. Using Lemma 1 we readily calculate

$$\mu(B_\lambda) \leqq \mu(I_n) \leqq 2^n \varrho^n \mu(I),$$

whence when $\lambda > 2$, so that $\varrho < 1/2$, we have $\mu(B_\lambda) = 0$. Q.E.D.

Using our alternative measure μ we obtain a new estimate for the Hausdorff dimension of B_λ.

Theorem 2. *For $\lambda \in (2, \infty)$ the Hausdorff dimension of B_λ is bounded above by the number*

$$\frac{\ln 2}{\ln\left(\dfrac{\sqrt{2\lambda^2 - 2\lambda + a}}{\sqrt{a}}\right)}.$$

Proof. This follows the same lines as [8, Theorem 12.2] except that here we use the distance function $d(x, y)$ in place of $|x - y|$, these being equivalent metrics when $\lambda \in (2, \infty)$. Consider the sequence of coverings $\{I_n\}$ introduced above and write I_n as a union of disjoint closed intervals, $I_n = \bigcup_{m=1}^{2^n} I_n^m$. Let

$$H_n(\alpha) = \sum_{m=1}^{2^n} (\mu(I_n^m))^\alpha \quad \text{for} \quad 0 < \alpha < \infty.$$

Then using Lemma 1 we readily calculate

$$H_{n+1}(\alpha) \leqq (2\varrho^\alpha) H_n(\alpha)$$

from which it follows that $\operatorname*{Lim}_{n\to\infty} H_n(\alpha)$ will be finite if $2\varrho^\alpha < 1$, which corresponds to $\alpha > (\ln \frac{1}{2})/\ln \varrho$. This implies that $(\ln \frac{1}{2})/\ln \varrho$ is an upper bound to the Hausdorff dimension of B_λ. Q.E.D.

Brolin has given the following upper bounds for the Hausdorff dimension of B_λ:

(i) $\dfrac{\ln 2}{\ln 2(2\lambda - a)^{1/2}}$, valid for $\lambda > 2 + \sqrt{2}$,

(ii) $\dfrac{\ln 2}{\exp\left\{-60\left(\ln\left(\dfrac{(2\lambda - a)^{1/2}}{5}\right)\right)^2\right\} + \ln 2}$, valid for $2 < \lambda \leqq 6$.

Our bound improves over both of these, where they apply. We note that for $\lambda=5$, Theorem 2 yields the upper bound 0.564 whilst (i) gives 0.636. Thus our bound is good enough, at $\lambda=5$, to distinguish B_λ from the classical ternary set of Cantor, whose Hausdorff dimension is $\ln 2/\ln 3=0.631$, see [18].

We next give a construction, involving a particular sequence $\{\sigma_n(x)\}$ of approximating distributions, for an invariant distribution $\sigma(x)$ of T_λ, supported upon B_λ. Here $\sigma(x)$ is an example of the equilibrium distributions described by Brolin [3, Chap. III], and the $\sigma_n(x)$'s are related to the orthogonal polynomials given in the next section.

Let $K_n=T_\lambda^{-n}\lambda$, which consists of the 2^n real points $\lambda\pm\sqrt{(\lambda\pm\sqrt{(\lambda\pm\ldots\pm\sqrt{(\lambda)}\ldots))}}$ where there are n plus – or – minus signs. Let

$$\sigma_n(x)=\frac{1}{2^n}\sum_{y\in K_n}\theta(x-y),$$

where $\theta(x)=0$ when $x\leqq 0$ and $\theta(x)=1$ when $x>0$. Thus, $\sigma_n(x)$ equals the proportion of members of K_n which are less than x.

It is straightforward to prove that $\{\sigma_n(x)\}$ converges uniformly to a continuous distribution $\sigma(x)$, for $x\in\mathbb{R}$. It is also straightforward to show, and in any case it follows from Brolin, that $\sigma(x)$ provides an invariant measure under T_λ, according to

$$\int_E f(x)d\sigma(x)=\int_{T_\lambda^{-1}E} f(Tx)d\sigma(x)$$

for all Borel measurable subsets E of $\mathbb{R}$ and all measurable functions f. When $\lambda=2$, $T_2z=(z-2)^2$, and we have [23]

$$d\sigma(x)=\begin{cases}0 & \text{for}\quad x\leqq 0,\\ \dfrac{1}{\pi}\dfrac{dx}{\sqrt{x(4-x)}} & \text{for}\quad 0<x<4,\\ 0 & \text{for}\quad x>4.\end{cases}$$

Let F denote the set of all Borel measurable subsets of B_λ. Then $(B_\lambda, F, \sigma, T_\lambda)$ is a *system* as defined by Billingsley [6]. It is readily proved to be isomorphic to the system formed by the left-shift on Ω with the usual uniform measure. Consequently $(B_\lambda, F, \sigma, T_\lambda)$ is mixing with entropy $\ln 2$. The system is also isomorphic to the one formed by $z\to z^2$ on the unit circle in $\mathbb{C}$, with circular Lebesgue measure. This is one way to see the connection between the system which exists when $\lambda\geqq 2$ and that which exists when $\lambda=0$.

2.2. *Orthogonal Polynomials*

One way of characterizing the invariant measure σ when $2\leqq\lambda<\infty$ is by means of the associated set of monic orthogonal polynomials. We denote this set by $\{P_n(x)\}_{n=-1}^{\infty}$, where $P_{-1}(x)\equiv 0$. For $n\geqq 0$, $P_n(x)$ has degree n and the coefficient of x^n is unity. The polynomials obey

$$\int_I P_n(x)P_m(x)d\sigma(x)=0\quad\text{for}\quad n\neq m.$$

These polynomials provide an interesting generalization of the Tchebycheff polynomials $\{T_n(x)=\cos(n\cos^{-1}x)\}$ to which, in view of the explicit formula for the measure given at the end of the last section, they must be related by

$$P_n(x)=2T_n(\tfrac{1}{2}x-1) \quad \text{when} \quad \lambda=2.$$

One would like to know how the invariance of the measure under T_λ relates to the structure of the polynomials. Also, what can be said about the associated three-term recurrence relations?

Let us introduce a second set of monic orthogonal polynomials $\{Q_n(x)\}_{n=-1}^{\infty}$, where $Q_{-1}(x)\equiv 0$. For $n\geqq 0$, $Q_n(x)$ has degree n and the coefficient of x^n is unity. They obey

$$\int_I xQ_n(x)Q_m(x)d\sigma(x)=0 \quad \text{for} \quad n\neq m.$$

Then we have, for $n\geqq 0$ and $2\leqq\lambda<\infty$, [9, 14, 22]

$$Q_n(x)=\frac{1}{x}[P_{n+1}(x)-P_{n+1}(0)P_n(x)/P_n(0)]. \tag{1}$$

We also define a set of polynomials $\{S_n(x)\}_{n=-1}^{\infty}$ by $S_{-1}(x)\equiv 0$ and

$$\left.\begin{aligned} S_{2m}(x)&=P_m(x^2), \\ S_{2m+1}(x)&=xQ_m(x^2), \end{aligned}\right\} \quad \text{for} \quad m=0,1,2,\ldots. \tag{2}$$

Theorem 3. *For* $\lambda\in[2,\infty)$,

$$S_n(x-\lambda)=P_n(x). \tag{3}$$

Proof. Clearly $S_n(x-\lambda)$ is a monic polynomial of degree n, when $n\geqq 0$. It remains only to prove that $\{S_n(x-\lambda)\}_{n=-1}^{\infty}$ is a set of polynomials orthogonal with respect to σ. Consider first for $n\neq m$

$$\begin{aligned}\int_I S_{2n+1}(x-\lambda)S_{2m+1}(x-\lambda)d\sigma(x)&=\int_I (x-\lambda)^2Q_n((x-\lambda)^2)Q_m((x-\lambda)^2)d\sigma(x)\\ &=\int_I Q_n(x)Q_m(x)xd\sigma(x)=0,\end{aligned}$$

where we have exploited the invariance of the measure σ under T_λ. Next consider

$$\begin{aligned}\int_I S_{2n}(x-\lambda)S_{2m}(x-\lambda)d\sigma(x)&=\int_I P_n((x-\lambda)^2)P_m((x-\lambda)^2)d\sigma(x)\\ &=\int_I P_n(x)P_m(x)d\sigma(x)=0,\end{aligned}$$

where we have again used the invariance of the measure. Finally consider, for m and n not necessarily distinct,

$$\int_I S_{2n}(x-\lambda)S_{2m+1}(x-\lambda)d\sigma(x)=\int_I (x-\lambda)P_n((x-\lambda)^2)Q_m((x-\lambda)^2)d\sigma(x).$$

This is zero because the integrand is antisymmetric about the midpoint λ of I. Q.E.D.

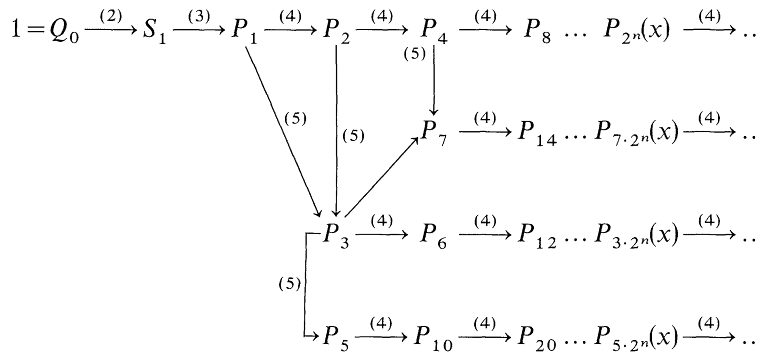
Upon combining (1), (2), and (3) one finds

$$P_{2n}(x+\lambda)=P_n(x^2), \tag{4}$$

and

$$P_{2n+1}(x+\lambda)=\frac{1}{x}\left[P_{n+1}(x^2)-\frac{P_{n+1}(0)}{P_n(0)}P_n(x^2)\right]. \tag{5}$$

These two equations provide a bootstrap procedure for calculating the $P_n(x)$'s, as indicated in the following scheme:



The numbers in parentheses indicate the equation to be used to travel along the arrows. Some examples of the resulting formulas are

$$P_1(x)=x-\lambda,$$
$$P_{2^n}(x)=(\ldots((x-\lambda)^2-\lambda)^2\ldots-\lambda)^2-\lambda, \qquad n\geqq 1,$$
$$P_3(x)=(x-\lambda)((x-\lambda)^2-\lambda-1),$$
$$P_{3\cdot 2^n}(x)=P_{2^n}(x)(P_{2^{n+1}}(x)-1), \qquad m\geqq 0,$$
$$P_5(x)=(x-\lambda)\left[(x-\lambda)^4-\frac{(2\lambda^2+2\lambda-1)}{\lambda-1}(x-\lambda)^2+\frac{\lambda^3-2\lambda^2+2\lambda+1}{\lambda-1}\right].$$

Notice that the set of zeros of $P_{2^n}(x)$ is precisely the set K_n which was used in the construction σ. Similarly one can give expressions for all the zeros of $P_{3\cdot 2^n}(x)$ and $P_{5\cdot 2^n}(x)$.

We next consider the three-term recurrence relation satisfied by the polynomials.

Theorem 4. *For $2\leqq\lambda<\infty$ there exists a unique set of real numbers $\{a_m\}_{m=1}^{\infty}$ such that for $n\in\{1,2,3,\ldots\}$*

$$P_{n+1}(x)=(x-\lambda)P_n(x)-a_n^2P_{n-1}(x), \tag{6}$$
$$\lambda-a_{2n+1}^2-a_{2n}^2=0, \tag{7}$$

and

$$a_n^2=a_{2n}^2a_{2n-1}^2. \tag{8}$$

Proof. Since the measure $\sigma(x)$ is symmetric about $x=\lambda$ it follows at once from the theory of orthogonal polynomials that there exists a unique set of real numbers $\{a_m\}_{m=1}^{\infty}$ such that the three-term recurrence formula (6) holds. To prove (7) and (8) it is convenient to work in terms of the shifted polynomials $S_n(x)=P_n(x+\lambda)$: we consider

$$S_{2n+2}(x)=xS_{2n+1}(x)-a_{2n+1}^2S_{2n}(x).$$

Eliminating $S_{2n+1}(x)$ from this expression by using the recurrence relation we obtain

$$S_{2n+2}(x)=(x^2-a_{2n+1}^2)S_{2n}(x)-xa_{2n}^2S_{2n-1}(x).$$

Herein we reexpress $S_{2n-1}(x)$ in terms of $S_{2n}(x)$ and $S_{2n-2}(x)$, again with the aid of the recurrence equation, to obtain

$$S_{2n+2}(x)=(x^2-a_{2n+1}^2-a_{2n}^2)S_{2n}(x)-a_{2n}^2a_{2n-1}^2S_{2n-2}(x).$$

We now set $x'=x^2-\lambda$ and use the fact that

$$S_{2m}(x)=S_m(x^2-\lambda) \quad \text{for} \quad m\in\{0,1,2,\ldots\};$$

which yields

$$S_{n+1}(x')=(x'+\lambda-a_{2n+1}^2-a_{2n}^2)S_n(x')-a_{2n}^2a_{2n-1}^2S_{n-1}(x').$$

Equations (7) and (8) follow at once upon comparing with the recurrence relation $S_{n+1}(x)=xS_n(x)-a_n^2S_{n-1}(x)$. Q.E.D.

With the aid of (7) and (8) we readily calculate

$$a_1^2=\lambda, \quad a_2^2=1, \quad a_3^2=\lambda-1,$$

$$a_4^2=\frac{1}{\lambda-1}, \quad a_5^2=\frac{\lambda^2-\lambda-1}{\lambda-1}, \quad a_6^2=\frac{\lambda^2-2\lambda+1}{\lambda^2-\lambda-1}.$$

We also obtain the following continued fractions representation for a_n: for $n\geqq 2$

$$a_{2n}^2=\frac{a_n^2|}{|\lambda}-\frac{a_{n-1}^2|}{|\lambda}\cdots-\frac{a_2^2|}{|\lambda-1},$$

and

$$a_{2n+1}^2=\lambda-\frac{a_n^2|}{|\lambda}-\frac{a_{n-1}^2|}{|\lambda}\cdots-\frac{a_2^2|}{|\lambda-1}$$

In particular, when $\lambda=2$ we have $a_n^2=1$ for $n>2$ and $a_1^2=2$, and (6) becomes exactly the three-term recursion relation for the Tchebycheff polynomials $\{T_n(\frac{1}{2}x-1)\}_{n=-1}^{\infty}$. From this it follows that the zeros of $T_{2^n}(\frac{1}{2}x-1)$ are precisely the set of numbers

$$\underbrace{2\pm\sqrt{(2\pm\sqrt{(2\pm\cdots\pm\sqrt{2})\cdots})}}_{n \text{ times}}.$$

The densification of the latter set of numbers on $[0,4]$ can thus be seen as an example of Blumenthal's theorem [7] on the distribution of zeros of orthogonal polynomials upon the support of the measure.

Further information, which relates in particular to the sequence of approximating measures $\{\sigma_n(x)\}$ given in Sect. 2.1, is obtained by examining the polynomials of the second kind, $\{P_n^1(x)\}_{n=0}^{\infty}$, which are defined by

$$P_n^1(x)=\int_I \frac{P_{n+1}(x)-P_{n+1}(y)}{x-y}\,d\sigma(y) \quad \text{for} \quad n\in\{-1,0,1,2\ldots\}. \tag{9}$$

Theorem 5. *For all* $n\in\{0,1,2,\ldots\}$,

$$P_{2n+1}^1(x)=(x-\lambda)P_n^1((x-\lambda)^2) \tag{10}$$

and

$$P_{2n}(x)=\frac{1}{(x-\lambda)}\left[P_{2n+1}^1(x)-\frac{P_{2n+2}(\lambda)}{P_{2n}(\lambda)}P_{2n-1}^1(x)\right]. \tag{11}$$

Proof. From (9) one has

$$P_n^1((x-\lambda)^2)=\int_I \frac{P_{n+1}((x-\lambda)^2)-P_{n+1}((y-\lambda)^2)}{(x-\lambda)^2-(y-\lambda)^2}\,d\sigma(y),$$

where the invariance of the measure under T_λ has been exploited. We now split up the denominator and use (4), which yields

$$P_n^1((x-\lambda)^2)=\frac{1}{2(x-\lambda)}\left[\int_I \frac{P_{2n+2}(x)-P_{2n+2}(y)}{(x-\lambda)-(y-\lambda)}\,d\sigma(y)\right.$$
$$\left.+\int_I \frac{P_{2n+2}(x)-P_{2n+2}(y)}{(x-\lambda)+(y-\lambda)}\,d\sigma(y)\right].$$

In the second integral here we make the change of variable $y-\lambda\to-(y-\lambda)$, use the symmetry of the measure and I about λ, and again exploit (4), to provide

$$P_n^1((x-\lambda)^2)=\frac{1}{(x-\lambda)}\int_I \frac{P_{2n+2}(x)-P_{2n+2}(y)}{(x-y)}\,d\sigma(y).$$

From this (10) is immediate.

Equation (9) also implies the recursion relation

$$P_{n+1}^1(x)=(x-\lambda)P_n^1(x)-a_{n+1}^2P_{n-1}^1(x) \quad \text{for} \quad n\in\{0,1,2,\ldots\},$$

with $P_{-1}^1(x)=0$, and $P_0^1(x)=1$. This implies (11) when a_{n+1}^2 is eliminated with the aid of (6) wherein $\lambda=x$. Q.E.D.

From (10) it is apparent, in contrast to the previous case, that the odd polynomials of the second kind are easily calculated from the even ones. Some examples of the polynomials are

$$P_0^1(x)=1,$$
$$P_1^1(x)=x-\lambda=P_1(x),$$
$$P_3^1(x)=(x-\lambda)((x-\lambda)^2-\lambda)=P_1(x)P_2(x)=\frac{1}{4}\frac{d}{dx}P_4(x),$$
$$P_{2^n-1}^1(x)=\prod_{k=0}^{n-1}P_{2^k}(x)=\frac{1}{2^n}\frac{d}{dx}P_{2^n}(x), \qquad n\geqq 1$$
$$P_{3\cdot 2^n-1}^1(x)=\left(\prod_{k=0}^{n-1}P_{2^k}(x)\right)(P_{2^{n+1}}(x)+\lambda-1).$$

Now consider the moment functions

$$G_K(x) = \int_I \frac{d\sigma_K(y)}{x-y},$$

and

$$G(x) = \int_I \frac{d\sigma(y)}{x-y}.$$

From the theory of Padé approximants [2], one has that the $[n-1/n](x)$ approximant to $G(x)$ is

$$[n-1/n](x) = \frac{P^1_{n-1}(x)}{P_n(x)}, \quad \text{for} \quad n \in \{1, 2, \ldots\}.$$

Using (4) and the fact that $S_{2m}(x) = S_m(x^2 - \lambda)$ we discover the remarkable result

$$[2n-1/2n](x) = (x-\lambda)[n-1/n]((x-\lambda)^2) \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Also, for $n = 2^k$ where $k \in \{0, 1, 2, \ldots\}$, we find

$$[2^k - 1/2^k](x) = \frac{1}{2^k}\frac{d}{dx}\ln P_{2^k}(x) = \int_I \frac{d\sigma_K(x)}{x-y},$$

which makes contact with the sequence of approximating measures $\{\sigma_n(x)\}$.

Finally, we note that since $(B_\lambda, F, \sigma, T_\lambda)$ is a mixing system, so is $(B_\lambda, F, \sigma, T_\lambda^n)$ for $n \in \{1, 2, 3, \ldots\}$. Hence $P_{2^n}(x) + \lambda$ provides a mixing transformation on B_λ, with respect to σ. Shifting B_λ to the left by subtracting λ, and correspondingly adjusting the measure, this shows that each of the polynomials $P_{2^n}(x+\lambda)$ provides a mixing transformation upon the shifted system. This extends a result of Adler and Rivlin [1], and is itself a special case of a wide reaching theorem [5].

3. The Cases $-1/4 \leqq \lambda \leqq 2$ and $|\lambda| \leqq 1/4$ with $\lambda \in \mathbb{C}$

In this section, the Julia set B_λ is connected, and a central role is played by the Böttcher equation, see [12], for T_λ at ∞

$$T_0 \circ H_\lambda = H_\lambda \circ T_\lambda, \quad H_\lambda(z) = z + O(1) \text{ at } \infty. \tag{1a}$$

We actually use the equivalent equation in terms of inverses, where $F_\lambda = H_\lambda^{-1}$,

$$F_\lambda(z) = \lambda + \sqrt{F_\lambda(z^2)}, \quad F_\lambda(z) = z + O(1) \text{ at } \infty. \tag{1b}$$

We let $\mathbb{C}$ be the complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $D_0 = \{z \in \hat{\mathbb{C}} \,|\, |z| > 1\}$. Then we shall see that F_λ maps D_0 conformally onto a region bounded by B_λ, and by means of F_λ we can relate λ-chains to B_λ.

3.1. Two Constructions for B_λ

We present two constructions for B_λ; one from the "outside", and one from the "inside", when $-\frac{1}{4} \leqq \lambda \leqq 2$. The first is not new in principle: it involves the formation of an increasing sequence of domains, successive inverse images under T_λ of a neighborhood of ∞, as suggested by Fatou [12] and by Julia [19]. We include it both for completeness and for comparison with the second method.

The second construction, from the "inside", provides a decreasing sequence of domains and a corresponding sequence of functions, from D_0 to the domains, converging uniformly to $F_\lambda(z)$ on compact subsets of D_0. Of interest are the complements of the domains, which form an increasing sequence of trees, with fractal-like structure [20] and two-dimensional measure zero, which serve to describe B_λ from the interior. This construction turns out to be important: we have recently proved [4] that this sequence of trees converges to B_λ itself, for infinitely many values of $\lambda \in (0, 2)$.

For $\lambda \geqq -\frac{1}{4}$ we define

$$a = \sqrt{\lambda + 1/4} + \lambda + 1/2, \quad \text{and} \quad b = \sqrt{|\lambda| + 1/4} + |\lambda| + 1/2.$$

Then a is the unique positive real numbers which obeys $a = \lambda + \sqrt{a}$. Notice that $a \leqq b$, and $\lambda - \sqrt{a} \leqq 0$, for $-\frac{1}{4} \leqq \lambda \leqq 2$.

The following Lemma will allow us to make a concrete iterative solution of the Böttcher equation (1b). This construction is important later on in our discussion of Hölder continuity.

Lemma 3. *For $-\frac{1}{4} \leqq \lambda < 2$, we may start from $f_0(z) = \lambda + \sqrt{b}z$ and iteratively define analytic functions $f_n : D_0 \to \hat{\mathbb{C}}$ for $n \in \{1, 2, 3, \ldots\}$ by*

$$f_n(z) = \lambda + \sqrt{f_{n-1}(z^2)} = b^{2^{-(n+1)}} z + \lambda + O\left(\frac{1}{z}\right),$$

and they will satisfy

$$\hat{\mathbb{C}} \backslash [\lambda - \sqrt{a}, a] \supset f_n(D_0) \supset f_{n-1}(D_0).$$

Sketch of Proof. The lemma can be easily proved by induction. The exclusion of the interval $[\lambda - \sqrt{a}, a]$, which contains zero, follows from the facts that $a \leqq b$ and T_λ maps the excluded interval into itself. Thus, f_n is well defined because $0 \notin f_{n-1}(D_0)$. That $f_1(D_0) \supset f_0(D_0)$ is a simple calculation using the definition of b, and upon iteration we obtain the monotonicity of the images.

Theorem 6. *Let $-\frac{1}{4} \leqq \lambda \leqq 2$. The sequence $\{f_n\}$ of Lemma 3 converges uniformly on compact subsets of D_0 to a function F_λ which obeys the inverse Böttcher equation* (1b).

Remark. We denote the boundary of $F_\lambda(D_0)$ by B_λ. The theorem says $T_\lambda^{-1} B_\lambda = B_\lambda$. Then B_λ turns out to be the Julia set for T_λ, see [12] for example.

Proof. The theorem follows at once from Lemma 3 and Caratheodory's theorem on domain convergence, see Goluzin [15, p. 53]. Q.E.D.

We can now set up a correspondence between λ-chains and points on B_λ. Corresponding to $\omega = (e_1, e_2, e_3, \ldots) \in \Omega$ we define $S_0(\omega, z) = z$ and, for $n \in \{1, 2, 3, \ldots\}$,

$$S_n(\omega, z) = \lambda + e_1 \sqrt{(\lambda + e_2 \sqrt{(\lambda + \ldots + e_{n-1} \sqrt{(\lambda + e_n \sqrt{z}) \ldots}))}}.$$

The value of the square root $\sqrt{w}$ for $w \in \mathbb{C}$ is fixed by writing $w = \Gamma e^{i\theta}$ with $0 \leqq \theta < 2\pi$, and then

$$\sqrt{z} = \sqrt{\Gamma} e^{i\theta/2}, \quad \text{and} \quad -\sqrt{z} = -\sqrt{\Gamma} e^{i\theta/2} = \sqrt{\Gamma} e^{i(\theta+2\pi)/2}.$$

We will say that we have a *positive axis* branch cut. Thus $S_n(\omega, z)$ is clearly a well defined function of its arguments.

Theorem 7. *Let* $-\frac{1}{4} \leqq \lambda \leqq 2$ *and* $0 \leqq \theta < 2\pi$. *Write* $\theta = 2\pi \sum_{j=1}^{\infty} d_j/2^j$, *where* $d_j \in \{0, 1\}$. *Let* $e_j = +1$ *if* $d_j = 0$, $e_j = -1$ *if* $d_j = 1$, *and* $\omega = (e_1, e_2, e_3, \ldots)$. *Then*

i) $\lim_{n \to \infty} F_\lambda^{-1}(S_n(\omega, z)) = e^{i\theta}$,

and

(ii) $\lim_{n \to \infty} S_n(\omega, z)$ *exists for almost all* θ, *independent of* $z \in F_\lambda(D_0)$.

Remark. We define the λ-chain $S(\omega)$ to be the limit of $S_n(\omega, z)$ when it exists.

Proof. Let $z = F_\lambda(\Gamma e^{i\alpha})$, $0 \leqq \alpha < 2\pi$, $\Gamma > 1$. From symmetry observe that F_λ maps each of the upper half plane, the lower half plane, the positive axis and the negative axis into itself. Hence when we use the Böttcher equation (1b) we can choose the appropriate branches of the successive square roots and find

$$F_\lambda^{-1}(S_n(\omega, z)) = \Gamma^{2^{-n}} e^{i\alpha/2^n} e^{2\pi i} \sum_{j=1}^{n} d_j/2^j.$$

From this (i) follows.

It is easy to show that in (i) the limit $e^{i\theta}$ on the unit circle is approached nontangentially. Since $F_\lambda(z) - z$ is regular and bounded on D_0, it has nontangential limits almost everywhere on the unit circle, Goluzin [15, p. 343], and we find that $\{S_n(\omega, z)\}$ has a unique limit point for almost every θ. Q.E.D.

The power series coefficients of the unique solution of the Böttcher equation (1b) can be calculated recursively. For example, after several iterations we obtain

$$F_\lambda(z) = z + \lambda + \frac{\lambda}{2z} + \frac{\lambda(2-\lambda)}{8z^2} - \frac{\lambda^2(2-\lambda)}{16z^5} + O\left(\frac{1}{z^7}\right).$$

When $\lambda = 2$ this reduces to $F_2(z) = z + 2 + \frac{1}{z}$.

In order to obtain more detailed information about B_λ it is useful to construct a second sequence of functions $\{f_n^*(z)\}$ which is convergent to $F_\lambda(z)$. Whereas the sequence $\{f_n(z)\}$ provides a sequence of images which increase to $F_\lambda(D_0)$ (whose boundary is B_λ), the sequence $\{f_n^*(z)\}$ yields decreasing images, and we get convergence to B_λ from the "inside". The proof is similar to that of Lemma 3.

Lemma 4. *For* $-\frac{1}{4} \leqq \lambda \leqq 2$ *we may start with* $f_0^*(z) = \lambda + \sqrt{a/4}\left(z + \frac{1}{z}\right)$ *and iteratively define analytic functions* $f_n^* : D_0 \to \hat{\mathbb{C}}$ *for* $n \in \{1, 2, 3, \ldots\}$ *by*

$$f_n^*(z) = \lambda + \sqrt{f_{n-1}^*(z^2)} = (a/4)^{2^{-(n+1)}} z + \lambda + O\left(\frac{1}{z}\right),$$

and they will satisfy

$$f_n^*(D_0) \subset f_{n-1}^*(D_0) \subset \hat{\mathbb{C}} \setminus [\lambda - \sqrt{a}, a].$$

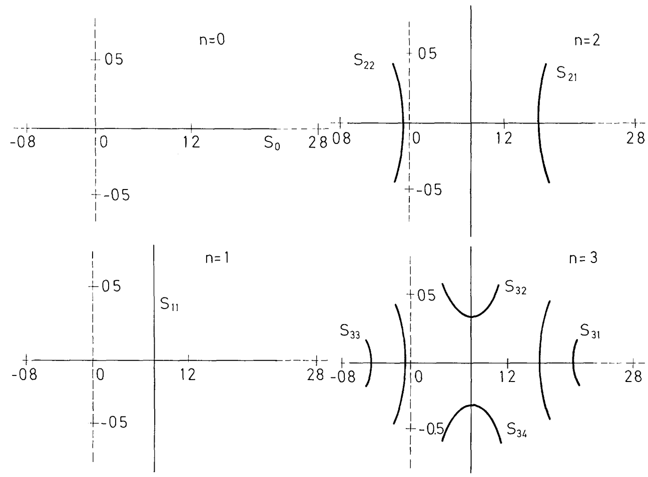


Fig. 1. The boundaries Γ_n for $n=0$, 1, 2, and 3, are drawn unbroken; the dots lie upon the coordinate axes. In general, Γ_{n+1} contains 2^n new arcs not in Γ_n, and $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \ldots$

It turns out to be very useful to give a fairly complete characterization of the sets $\Gamma_n = f_n^*(\partial D_0)$. Let $S_0 = [\lambda - \sqrt{a}, a] = \Gamma_0$. Then the image of S_0 under the principal branch of $\lambda + \sqrt{z}$ is $[\lambda, a] \cup$[the closed line segment from λ to $\lambda + i\sqrt{\sqrt{a} - \lambda}$]. Since $f_1^*(z) - \lambda$ is an odd function of z we can reflect the latter set about λ to obtain the rest of Γ_1. In this way we find

$$\Gamma_1 = S_0 \cup S_{11} \supset \Gamma_0,$$

where S_{11} is the closed line segment connecting $\lambda \pm \sqrt{\sqrt{a} - \lambda}$. It will be found helpful here to look at Fig. 1. Here Γ_2 is the image of Γ_1 under $\lambda \pm \sqrt{z}$. Then S_0 will generate $S_0 \cup S_{11}$, but S_{11} will produce two new analytic arcs denoted by S_{21} and S_{22}. In general Γ_n will contain 2^{n-1} new arcs not in Γ_{n-1}:

$$\Gamma_n = S_0 \cup \left\{ \bigcup_{j=1}^{n} \left\{ \left(\bigcup_{k=1}^{2^{j-1}} S_{jk} \right) \right\} \right\},$$

where each S_{jk} is an analytic arc. The endpoints of S_0 are $\lambda \pm \sqrt{a} = F_\lambda(\pm 1)$, and by repeatedly using the Böttcher equation (1b) and the definition of f_n^* we find that the endpoints of the arcs composing Γ_n are $F_\lambda(e^{j\pi i/2^n})$ for all $j \in \{1, 2, \ldots, 2^{n+1}\}$. Equivalently, the endpoints of Γ_n are all numbers expressible in the form

$$F_\lambda(\underbrace{\pm\sqrt{}(\pm\sqrt{}(\pm \ldots \sqrt{1}}_{n \text{ times}}) \ldots))) = \lambda \pm \underbrace{\sqrt{}(\lambda \pm \sqrt{}(\lambda \pm \ldots \sqrt{a}}_{n \text{ times}}) \ldots)),$$

where $\pm$ indicates that both branches of $\sqrt{}$ can be chosen.

We can replace the f_n of Theorem 6 by f_n^* to obtain the following result.

Theorem 8. *Let $-\frac{1}{4} \leqq \lambda \leqq 2$. Then the sequence of functions $\{f_n^*\}$ provided by Lemma 4 converges uniformly to F_λ on compact subsets of D_0.*

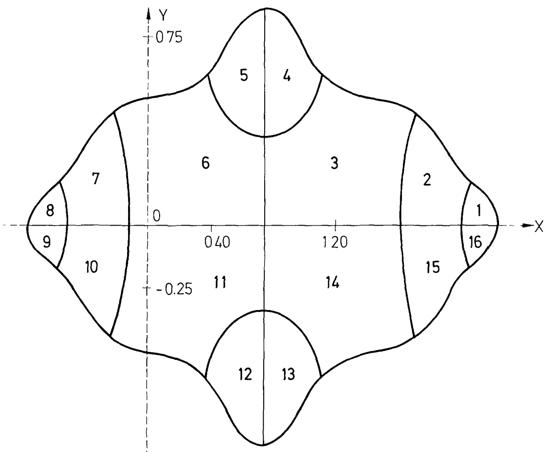


Fig. 2. The boundaries of $f_3(D_0)$ and $f_3^*(D_0)$ are superimposed, and sixteen components are thereby defined. The closure of the component labeled j contains the part of B_λ which corresponds to $\{e^{i\theta}|(j-1)\pi/16 \leqq \theta \leqq j\pi/16\}$

The proof is like that of Theorem 6. The uniqueness of the solution to the Böttcher equation ensures that the limit is F_λ.

If K is a compact subset of D_0, then the sequence of sets $\{f_n(K)\}_{n=0}^{\infty}$ and $\{f_n^*(K)\}_{n=0}^{\infty}$ increase and decrease respectively to $F_\lambda(K)$. Similarly $\{f_n(D_0)\}_{n=1}^{\infty}$ increases to $F_\lambda(D_0)=D_\lambda$. However, the decreasing sequence $\{f_n^*(D_0)\}_{n=0}^{\infty}$ does *not* converge to D_λ. This sequence illustrates a peculiar property of domain convergence. For example, we will show in the next section that when $0 \leqq \lambda \leqq 1/4$, D_λ^c contains the disk of radius 1/4 about 0. In this case the area of D_λ^c is strictly positive, yet the area of Γ_n is zero for all n.

The behavior of these image sets can be better understood by considering the endpoints of the analytic arcs in Γ_n. The set $\left\{\bigcup_{n=1}^{\infty} e^{j\pi i/2^n}|j=1,2,\ldots,2^{n+1}\right\}$ is dense in the unit circle $\partial D_0 = B_0$, and its image under F_λ is dense in B_λ. The latter image is precisely the set of endpoints of all of the analytic arcs in all of the Γ_n's. Thus, consistently with the general theory of domain convergence, we find $F_\lambda(D_0)$ is one component of the interior of $\operatorname{Lim}_{n\to\infty} f_n^*(D_0)$.

For $0<\lambda<2$ the numbers $\lambda \pm \sqrt{a}$ are on the boundary of both $f_0(D_0)$ and $f_0^*(D_0)$. Hence

$$f_n(z)=f_n^*(z)=F_\lambda(z) \quad \text{for} \quad z=e^{j\pi i/2^n},$$

for all $j\in\{1,2,\ldots,2^{n+1}\}$ and $n\in\{0,1,2,\ldots\}$. If we superimpose the boundaries of $f_n(D_0)$ and $f_n^*(D_0)$ then we separate the complex plane into many components. The portion of B_λ corresponding to $\{e^{i\theta}|(j-1)\pi/2^n \leqq \theta \leqq j\pi/2^n\}$ must lie in the closure of the j^{th} component counted counter-clockwise, starting from the one in the first quadrant which has a on its boundary. As an example we illustrate the case $n=3$ in Fig. 2 where the sixteen components are labeled.

3.2. The Relation Between B_λ and $\tilde{B}_\lambda$ when $|\lambda|<1/4$ with $\lambda\in\mathbb{C}$

First we restrict attention to $0\leqq\lambda\leqq 0.2$ and make explicit calculations with λ-chains which show that $S(\omega)$ is always well-defined and that F_λ can be extended continuously to the unit circle. Next we make analytic extensions to λ in the set $L=\{\lambda\in\mathbb{C}\,|\,|\lambda|<1/4\}$ and show that the mappings $G_\lambda(\theta)=F_\lambda(e^{i\theta})$ are uniformly Hölder continuous for λ in any compact subset of L.

We introduce the notation

$$S(+,z)=\lambda+\sqrt{z} \quad \text{and} \quad S(-,z)=\lambda-\sqrt{z},$$

where the positive axis branch cut is used. Then we have the following two simple computational Lemmas whose proofs we omit.

Lemma 5. *Let $\lambda\in\mathbb{R}$ and $z\in\mathbb{C}$ with $z\not> 0$. Then $S(+,z)=\overline{S(-,\bar{z})}$.*

For $\lambda\in L$ and $d\geqq b$ let

$$C=-|\lambda|+\tfrac{1}{2}+\sqrt{-|\lambda|+1/4}$$

and

$$R_d=\{z\in\mathbb{C}\,|\,C\leqq|z|\leqq d\}.$$

Lemma 6. *Let $\lambda\in L$ and $d\geqq b$. Then both $S(+,z)$ and $S(-,z)$ map R_d into itself.*

Let μ, σ, and ξ in Ω be defined by

$$\mu=(-1,-1,-1,-1,\ldots),$$

$$\sigma=(+1,-1,-1,-1,\ldots),$$

and

$$\xi=(-1,+1,+1,+1,\ldots).$$

Lemma 7. *For $0<\lambda\leqq 2$, $S(\sigma)$ and $S(\xi)$ both exist and $S(\sigma)=S(\xi)=S(-,a)$.*

Proof. We claim that $S(-,z)$ is a contraction mapping towards a on $\{z|\mathrm{Im}\,z<0\}$ and that $S(+,z)$ is a contraction mapping towards a on $\{z|\mathrm{Im}\,z>0\}$. By Lemma 5 these claims are equivalent, so we prove only the latter. For $\mathrm{Im}\,z>0$ we have

$$\left|\frac{S(+,z)-a}{z-a}\right|=\left|\frac{\lambda+\sqrt{z}-(\lambda+\sqrt{a})}{z-a}\right|=\left|\frac{\sqrt{z}-\sqrt{a}}{z-a}\right|$$

$$=\left|\frac{1}{\sqrt{z}+\sqrt{a}}\right|\leqq\frac{1}{\mathrm{Re}(\sqrt{z}+\sqrt{a})}$$

$$\leqq\frac{1}{\sqrt{a}}<1,$$

where we have used the fact that $\mathrm{Re}\sqrt{z}>0$. This proves our claim. It now follows that the sequence $\{S_n(\mu,z)\}_{n=1}^{\infty}$ approaches *a through the fourth quadrant*, which, in turn implies that $\{\sqrt{S_n(\mu,z)}\}_{n=1}^{\infty}$ converges to $-\sqrt{a}$. Hence $\{S_n(\sigma,z)\}_{n=1}^{\infty}$ converges to $\lambda-\sqrt{a}$. Q.E.D.

Lemma 8. *Let* $0<\lambda<0.2$, $d\geqq a$, $n\in\{1,2,\dots\}$, *and* $\omega\in\Omega$. *Then for any* $z_1,z_2\in R_d$,

$$|S_n(\omega,z_1)-S_n(\omega,z_2)|\leqq 2dr^{n-3},$$

where $r=0.87$.

Proof. First note that for any $w_1,w_2\in R_d$ we have

$$\begin{aligned}|\arg\sqrt{w_1}-\arg\sqrt{w_2}|&=|\arg(-\sqrt{w_1})-\arg(-\sqrt{w_2})|\\&=1/2|\arg w_1-\arg w_2|.\end{aligned}$$

Also, for any $w\in R_d$, $\sqrt{w}$ lies in the upper half plane and

$$\begin{aligned}0&\leqq\arg\sqrt{w}-\arg(\sqrt{w}+\lambda)\\&\leqq\sin^{-1}(\lambda/\sqrt{|w|})\leqq\sin^{-1}(0.2/\sqrt{0.5})<0.3.\end{aligned}$$

Here we have used the fact that when $0<\lambda<0.2$, then $C\geqq 0.5$ so that for all $w\in R_d$ we have $|w|>0.5$. Then we can show

$$|\arg S(\pm,w_1)-\arg S(\pm,w_2)|\leqq 1/2|\arg w_1-\arg w_2|+0.3.$$

Lemma 6 allows us to iterate the above result starting from $z_1,z_2\in R_d$. We find

$$|\arg S_m(\tilde{\omega},z_1)-\arg S_m(\tilde{\omega},z_2)|\leqq\beta=1.25<\tfrac{\pi}{2}\tag{1}$$

for all $m\geqq 3$, for any $\tilde{\omega}\in\Omega$.

Again let us suppose $w_1,w_2\in R_d$, but now with $|\arg w_1-\arg w_2|\leqq\beta$. Then

$$\begin{aligned}\left|\frac{S(\pm,w_1)-S(\pm,w_2)}{w_1-w_2}\right|&=\left|\frac{\sqrt{w_1}-\sqrt{w_2}}{w_1+w_2}\right|=\frac{1}{\sqrt{w_1}+\sqrt{w_2}}\\&\leqq\frac{1}{(\cos\beta/2)|\sqrt{w_1}|+(\cos\beta/2)|\sqrt{w_2}|}\\&\leqq\frac{1}{(\cos\beta/2)(2\sqrt{0.5})}\leqq 0.87=r<1.\end{aligned}\tag{2}$$

Since the diameter of R_d is $2d$ and $r<1$, the lemma is true for $n\leqq 3$. To obtain the result for larger n, we use (1) and then apply (2) repeatedly. Q.E.D.

Theorem 9. *Let* $0<\lambda<0.2$, $z\in R_d$, *and* $d\geqq a$. *Then* $S(\omega)$ *exists and lies in* R_a, *for all* $\omega\in\Omega$, *and*

$$|S_n(\omega,z)-S(\omega)|<2d(0.87)^{n-3}$$

Proof. Let $m\geqq n$ and note that

$$|S_n(\omega,z)-S_m(\omega,z)|=|S_n(\omega,z)-S_n(\omega,S_{m-n}(\tilde{\omega},z))|$$

for some $\tilde{\omega}\in\Omega$. Moreover $S_{m-n}(\tilde{\omega},z)\in R_d$ by Lemma 6. Hence Lemma 8 applies with $z_1=z$ and $z_2=S_{m-n}(\tilde{\omega},z)$, yielding

$$|S_n(\omega,z)-S_m(\omega,z)|\leqq 2dr^{n-3}.$$

Thus $\{S_n(\omega, z)\}_{n=1}^{\infty}$ is a Cauchy sequence in the closed set R_d, and by Lemma 8 the limit is independent of $z \in R_d$. If we take $z \in R_a$ we find by Lemma 6 the limit $S(\omega)$ also belongs to R_a. Q.E.D.

Next we relate $S(\omega)$ to F_λ and $\{f_n\}$.

Theorem 10. *Let* $0 < \lambda < 0.2$, *and let* θ *and* ω *be related as in Theorem 7. Then*

$$\lim_{n \to \infty} f_n(e^{i\theta}) = S(\omega), \tag{3}$$

and

$$\lim_{\substack{z \to e^{i\theta} \\ |z| > 1}} F_\lambda(z) = S(\omega). \tag{4}$$

Proof. Notice that for some values of θ there are two expansions $\theta = 2\pi \sum_{j=1}^{\infty} d_j/2^j$, one involving infinitely many zeros and one involving infinitely many ones. In these cases there correspond two distinct elements ω_1 and ω_2 in Ω, and for the theorem to make sense it must be true that $S(\omega_1) = S(\omega_2)$. But this is just what is implied by Lemma 7. Accordingly, without loss of generality, we can assume that ω contains infinitely many $+1$'s.

Let $z = \Gamma e^{i\theta}$. Then it follows that $0 \leqq \arg z^{2^{j-1}} < \pi$ precisely when $d_j = 0$. Using the definition of f_n we obtain

$$f_n(z) = S_n(\omega, f_0(z^{2^n})) \quad \text{for} \quad \Gamma \geqq 1 .$$

If $\Gamma = 1$ then $f_0(z^{2^n}) \in R_a$ and we can let $n \to \infty$ with the aid of Theorem 9, yielding (3).

To obtain (4), first let $\varepsilon > 0$ and choose $d = 2a$ in Lemma 8. Pick n so that $4a(0.87)^{n-3} < \varepsilon$. Since $B_\lambda \subset R_a$ there exists $\varrho > 1$ such that

$$F_\lambda(z) \in R_{2a} \quad \text{for} \quad 1 < |z| < \varrho . \tag{5}$$

Let $I = \left\{ \phi \,\middle|\, \frac{\phi}{2\pi} - \frac{1}{2^{n+1}} < \sum_{j=1}^{n} \frac{d_j}{2^j} \leqq \frac{\phi}{2\pi} \right\}$. From the Böttcher equation (1b) we find that

$$F_\lambda(\Gamma e^{i\phi}) = S_n(\omega, F_\lambda(\Gamma^{2^n} e^{2^n i\phi})) \quad \text{for} \quad \phi \in I .$$

Combining this with (5) and Theorem 9 we obtain

$$|F_\lambda(\Gamma e^{i\phi}) - S(\omega)| < \varepsilon \quad \text{for} \quad \phi \in I, \quad \text{and} \quad 1 < \Gamma < \varrho^{1/2^n} . \tag{6}$$

If θ is an interior element of I we are done. Otherwise θ is the left endpoint and $\theta/2\pi$ has a second binary representation which we can use similarly to show we can allow $\phi < \theta$ in (6). Q.E.D.

Next we extend the allowed values of λ from the interval $0 < \lambda < 0.2$ to the set $L = \{\lambda \in \mathbb{C} \,|\, |\lambda| < 1/4\}$. Observe that the inductive definition of $\{f_n\}$ of Lemma 3 applies also for $\lambda \in L$, and that we can establish the containment condition

$$\{z \in \mathbb{C} \,|\, |z| > C\} \supset f_n(D_0) \supset f_{n-1}(D_0)$$

for $n \in \{1, 2, \ldots\}$. As before, we prove the existence of the limit F_λ of the sequence $\{f_n\}$, and thus we obtain direct proof of the existence of a conformal mapping of D_0 which solves the Böttcher equation (1b).

In the following theorem we alter the notation slightly to emphasize the dependence upon λ.

Theorem 11. *For any fixed $\lambda \in L$ let $F(\lambda, z)$ be the conformal mapping of D_0 which solves the Böttcher equation* (1b). *Then F may be extended continuously to $L \times \overline{D_0}$, where it is analytic in λ. The functions $G_\lambda(\theta) = F(\lambda, e^{i\theta})$ are uniformly Hölder continuous for λ in any compact subset of L.*

Proof. Let $f_n(\lambda, z) = f_n(z)$. Observe that for $\lambda \in L$ and $z \in \overline{D_0}$ the functions in $\{f_n(\lambda, z)\}$ as functions of λ are analytic, and their images omit the disk $\{w \in \mathbb{C} \mid |w| < C\}$. Hence they constitute a normal family. This means that they have an infinite subsequence which converges to some function $H(\lambda, z)$ which, among its other properties, is analytic in λ for $\lambda \in L$, for each fixed $z \in \bar{D}_0$. Suppose that there are in fact two different subsequences convergent to two different functions H_1 and H_2. Since the *whole* sequence is convergent to a single limit for $0 < \lambda < 0.2$, H_1 and H_2 agree on a set which contains a limit point. Hence by Vitali's theorem $H_1 \equiv H_2$. Thus, we have the existence of

$$H(\lambda, z) = \operatorname*{Lim}_{n \to \infty} f_n(\lambda, z),$$

analytic in λ for $\lambda \in L$ for each fixed $z \in \bar{D}_0$. Moreover $H(\lambda, z) = F(\lambda, z)$ for $z \in D_0$.

Next consider $\operatorname*{Lim}_{\substack{z \to e^{i\theta} \\ |z| > 1}} F(\lambda, z)$. For $0 < \lambda < 0.2$ this limit exists and equals $\operatorname*{Lim}_{n \to \infty} f_n(\lambda, e^{i\theta}) = H(\lambda, e^{i\theta})$. Thus, by again applying Vitali's theorem, we have that the limit exists for all $\lambda \in L$ and

$$\operatorname*{Lim}_{\substack{z \to e^{i\theta} \\ |z| > 1}} F(\lambda, z) = \operatorname*{Lim}_{n \to \infty} f_n(\lambda, e^{i\theta}).$$

It now follows that as a function of z, $F(\lambda, z)$ is a one-to-one analytic function on D_0 with well-defined boundary values.

Finally, we establish the Hölder continuity. Since $C > 1/4$ for all $\lambda \in L$ we can choose β with $0 < \beta < \frac{\pi}{2}$ and $R > 1$ such that for any compact subset M of L

$$2\sqrt{C}\cos(\beta/2) > R > 1 \quad \text{for all} \quad \lambda \in M.$$

Since $G_\lambda(\theta)$ is continuous in λ and θ we can choose $\gamma > 0$ such that

$$|\arg(G_\lambda(\theta)/G_\lambda(\phi))| < \beta \quad \text{if} \quad |\theta - \phi| < \gamma \quad \text{with} \quad \lambda \in M.$$

Let θ and ϕ be real with $|\theta - \phi| < \gamma$. Then there exists an integer $n \geqq 0$ such that

$$\gamma/2^{n+1} \leqq |\theta - \phi| < \gamma/2^n. \tag{7}$$

Hence

$$\begin{aligned} |G_\lambda(\theta) - G_\lambda(\phi)| &= |\sqrt{G_\lambda(2\theta)} - \sqrt{G_\lambda(2\phi)}| \\ &= \frac{|G_\lambda(2\theta) - G_\lambda(2\phi)|}{|\sqrt{G_\lambda(2\theta)} + \sqrt{G_\lambda(2\phi)}|} \\ &\leqq \frac{|G_\lambda(2\theta) - G_\lambda(2\phi)|}{R}. \end{aligned}$$

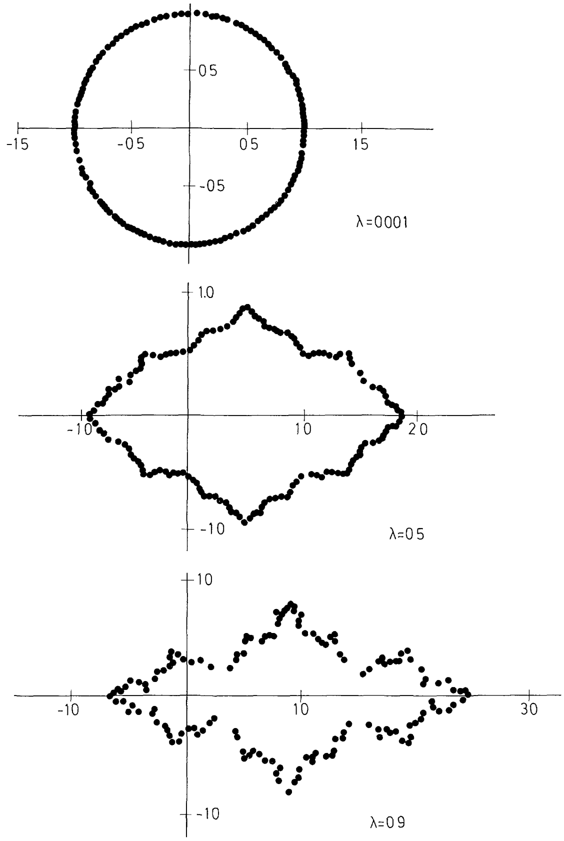


Fig. 3. Representation of B_λ for $\lambda=0.001$, 0.5, and 0.9. The crosses indicate dots in the complex plane which belong to B_λ. These drawings are based upon accurate plots of five hundred points in B_λ, picked at random

Iterating we find

$$|G_\lambda(\theta)-G_\lambda(\phi)| \leqq \frac{|G_\lambda(2^n\theta)-G_\lambda(2^n\phi)|}{R^n} \leqq \frac{K_1}{R^n},$$

where K_1 is the maximum diameter of B_λ for $\lambda \in M$. Using (7), with $\alpha=\ln R/\ln 2$ and $K_2=K_1(2/\gamma)$ we have

$$|G_\lambda(\theta)-G_\lambda(\phi)| \leqq K_2|\theta-\phi|^\alpha \quad \text{for} \quad |\theta-\phi|<\gamma .$$

To allow for $|\theta-\phi| \geqq \gamma$ we replace K_2 by

$$K=\operatorname{Max}\{K_2, K_1\gamma^{-\alpha}\} . \quad \text{Q.E.D.}$$

We can illustrate some of the boundary behavior by calculating $\frac{\partial F}{\partial \lambda}(0,z)$. We use the fact that $\partial F/\partial\lambda = \lim_{n\to\infty} \partial f_n/\partial\lambda$ and, for $n>0$,

$$\frac{\partial f_n}{\partial \lambda}(\lambda,z)=1+\frac{1}{\sqrt{f_{n-1}(z^2)}}\cdot\frac{1}{2}\left(1+\frac{1}{\sqrt{f_{n-2}(z^4)}}\frac{1}{2}\left(1+\dots\left(1+\frac{1}{2}\frac{\sqrt{b}}{\sqrt{f_0(z^{2^n})}}\right)\dots\right)\right).$$

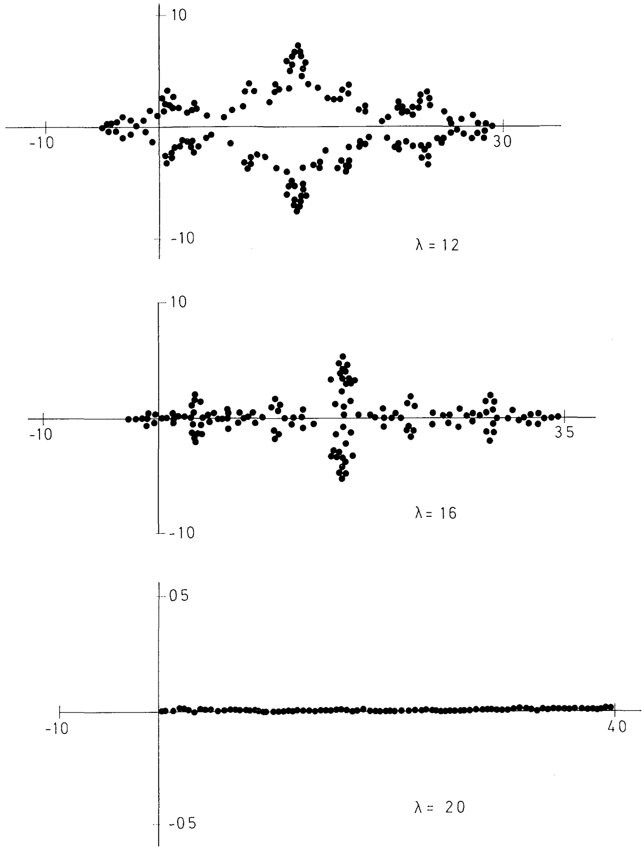


Fig. 4. Representation of B_λ for $\lambda=1.2$, 1.6, and 2.0. See the caption of Fig. 3

Then since $f_n(0,z)=f(0,z)=z$ (note that $b=1$ when $\lambda=0$) it follows that

$$\frac{\partial F}{\partial \lambda}(0,z)=1+\sum_{j=1}^{\infty}\frac{1}{2^j z^{2^j-1}}.$$

This series implies just the expected boundary behavior. It converges uniformly for $|z|\geqq 1$ and is analytic for $|z|>1$. On the other hand, it is a gap series and thus the unit circle is the natural boundary of the domain of the analytic function. In particular, we find that

$$\lim_{\Gamma\to 1+}\frac{\partial^2 F}{\partial z\partial \lambda}(0,\Gamma e^{i\theta})=\infty \quad \text{for} \quad \theta=2\pi/2^n.$$

4. Pictures Relating to B_λ

Here we present pictorially the results of some calculations which concern the structure of B_λ.

Figures 3 and 4 represent B_λ for $0\leqq\lambda\leqq 2$. (See also [21] where better pictures are given.) These drawings are not exact, but are based upon similar ones in which

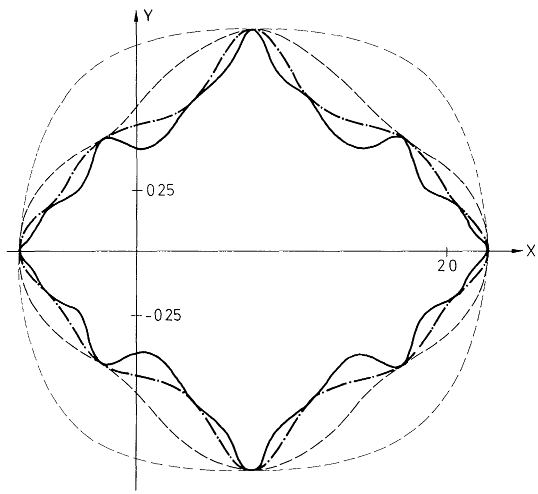


Fig. 5. Successive inclusion domains for B_λ when $\lambda = 0.75$. The boundary indicated by $n \in \{1, 2, 3, 4\}$ is the image of the unit circle under $f_n(x)$. The sequence of boundaries converges to B_λ in the manner described in Sect. 3.1

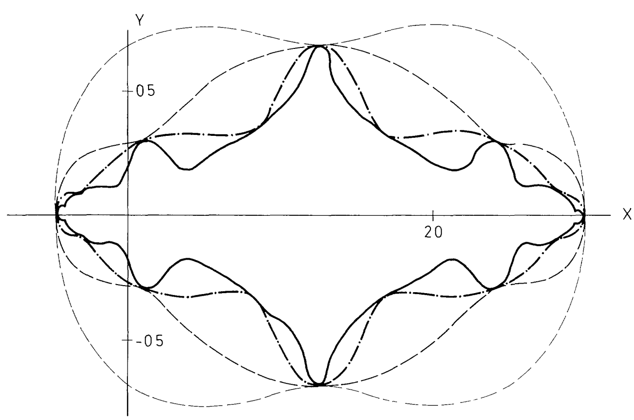


Fig. 6. Same as Fig. 5, but here $\lambda = 1.25$

about five hundred points in B_λ were plotted. The points were all of the form $S_{50}(\omega, a)$, and $\omega \in \Omega$ was chosen at random.

An alternative view of B_λ is in Figs. 5 and 6 where we have plotted the boundaries of the sets $f_n(D_0)$ for $n \in \{1, 2, 3, 4\}$, at two different values of λ. Recall that $f_1(D_0) \subset f_2(D_0) \subset f_3(D_0) \subset \ldots$, and that the boundary of the set obtained in the limit is B_λ.

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