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Supergravity, Complex Geometry and G-Structures

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Abstract. The geometry of supergravity is studied. New formulations of supergravity are given. The equivalence of different approaches to supergravity is analyzed.

Introduction

The subject of the present work is the geometrical aspect of supergravity. (We mean the $N=1$ supergravity everywhere.) There are various geometrical approaches to the supergravity theory. The most elegent one is due to V. Ogievetsky and E. Sokatchev [1], [2]. In that approach the role of the field is played by a (4,4)-dimensional surface in $\mathbb{C}^{4,2}$ (the complex superspace of the complex dimension (4,2)). Another approach, that by Wess and Zumino, is based upon the concept of the frame[1] fields in the (4,4)-dimensional superspace; the frame field in this approach is determined up to a transformation belonging to the Lorentz group. The equivalence between the Ogievetsky–Sokatchev approach and the Wess–Zumino approach was established previously [2] by means of a rather cumbersome consideration and its geometrical background is not always quite clear.

A purpose of the present investigation is to analyze the internal geometry of (4,4)-dimensional surfaces embedded into the space $\mathbb{C}^{4,2}$. The analysis results first of all in a simpler construction of the action functional than the Ogievetsky–Sokatchev approach [2, 3]. Namely, it was found that the action is simply expressed in terms of the so-called Levi form for the surface. Moreover, the analysis of the geometry in the surface provides manifest way to establish a correspondence between the Ogievetsky–Sokatchev and Wess–Zumino methods.

Our construction is based on the theory of G-structures[4]. The meaning of this statement is that the surface geometry is given by a frame field which is determined not up to a local Lorentz rotation, as it was in the Wess–Zumino approach, but by the frame field determined up to a transformation belonging to an arbitrary linear group G. In other words, two frames $\tilde{E}_A^M(x)$ and $E_A^M(x)$ are considered as corresponding to the same geometry, if $\tilde{E}_A^M(x)=g_A^B(x)E_B^M(x)$, where $g_A^B(x)$ is a function taking its values in the group G. The frame fields determining the

1 We prefer the term "frame" instead of "supertetrade" and "Vielbein" used in the physics literature

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considered G-structure will be called admissible, as well as the frames which are values of the fields at a point. In this work it is sufficient to use the simplest geometrical characteristics of the G-structure, the so-called structure function (to be more precise, the first structure function). The structure function may be defined as follows.

Write the (anti) commutator $[\partial_A, \partial_B\}$ in the form $\tau_{AB}^C \partial_C$. (We use the notation $\partial_A = E_A^M(x)\partial/\partial x^M$, where $E_A^M(x)$ is an admissible frame field.) Then the structure function may be defined as a part of the function τ_{AB}^C depending on the values of the frame field E_A^M at the point x, but not on the derivatives of the field. In fact, if an admissible frame is fixed at a point x, it may be included into the frame field in various manners; this freedom must be excluded imposing appropriate constraints upon τ_{AB}^C. In other words, one should consider τ_{AB}^C as an element of a subspace $\mathscr{C}$. Then the value of τ_{AB}^C is determined by the frame itself, but not by its derivatives, and it is identical to the structure function for the G-structure.

If a G-structure is introduced in the space, a G'-structure may be introduced in a surface in the space, where the group G' contains the linear transformations of the tangent plane to the surface, which are generated by those transformations from the group G leaving the tangent plane invariant. (Such a G'-structure is called induced.) If the structure function of the original G-structure is known, the corresponding constraints for the structure function of the induced structure may be found.

The described mathematical construction is applied to the investigation of the supergravity. The induced structure is considered for a (4, 4)-dimensional surface in $\mathbb{C}^{4,2}$. The action functional in the supergravity is expressed in terms of the structure function of the induced structure. Thus the Ogievetsky–Sokatchev formalism is replaced with an equivalent formalism based upon the internal geometry of the surface. In order to get a relation to the Wess–Zumino formalism, one should apply the reduction procedure for the structure group well known in mathematics.

In the present paper we are not concerned with the application of the exposed general construction to the non-minimal $N = 1$ supergravity and to the extended supergravity. Further development of the general methods and some of their applications will be published in a forthcoming paper by A. Rosly and the author [5]. In particular, constraints for the second structure function of the induced structure will be studied as well as the problem of indicating the sufficient constraints on the structure functions, under which the G'-structure in view is induced by the trivial G-structure. These results are applied in [5] to analyze the non-minimal $N = 1$ supergravity [6] and an alternative minimal supergravity [7].

The article is organized as follows. Section 2 contains the definition of the superspace and other basic concepts of the superspace theory. This section may be of interest by itself, since the proposed definitions are different from the conventional ones. It is our opinion that this system of definitions is more appropriate as, on one hand, they are closer to the relevant physical concepts and, on the other hand, they are quite rigorous. A reader for whom the supergravity by itself is of the most interest may skip this section and still understand the subsequent exposition. The only thing to hold in mind is that the mappings, surfaces and the vector fields involved may have the Grassmann coefficients. In the terminology of Sect. 2 we consider Λ-mappings, Λ-surfaces etc. The term "transformation" always denotes an invertible mapping. The word "plane" is used as a synonym of "linear subspace of linear space." The prefix "super-" is omitted sometimes.

The expression for the supergravity action functional in terms of the Levi form known in the complex geometry is given in Sect. 3. Section 4 presents an exposition of the basic concepts of the theory of G-structures. Induced structures in surfaces are defined and investigated in Sect. 5. (The induced structures have appeared in various concrete mathematical problems; however, the presented general definition of the induced structure is probably new.) Sections 6 and 7 contain investigation of the supergravity by means of the methods of the G-structure theory. These sections are written in such a manner that they may be understood with only minimal information on G-structures, given in the Introduction. Thus at first reading a physicist reader may omit Sect. 4 and 5.

Note that the exposition in Sect. 3–7 is not mathematically rigorous from the formal point of view, but a reader who is interested would be able to reconstruct the necessary proofs himself.

2. Superspaces

Let us denote by Λ^q the real Grassmann algebra having q generators $\varepsilon^1, \dots, \varepsilon^q$, and by $\Lambda^{p,q}$ the algebra of smooth functions on $\mathbb{R}^p$ taking values in Λ^q. The elements of $\Lambda^{p,q}$ may be represented in the form

$$\omega = \sum f_{\alpha_1 \dots \alpha_r}(u^1, \dots, u^p)\varepsilon^{\alpha_1} \dots \varepsilon^{\alpha_r}, \tag{1}$$

where $f_{\alpha_1 \dots \alpha_r}(u^1, \dots, u^p)$ are smooth real functions of the real variables $u^1, \dots, u^p$. The algebra $\Lambda^{p,q}$ can be considered as having p even generators $u^1, \dots, u^p$ and q odd generators $\varepsilon^1, \dots, \varepsilon^p$; the element ω is even (odd) if every term in (1) contains an even (odd) number of the generators ε^α. Any element of $\Lambda^{p,q}$ can be represented as the sum $\omega = m(\omega) + n(\omega)$, where $m(\omega)$ is a smooth real function and $n(\omega)$ is a nilpotent element of $\Lambda^{p,q}$ (i.e. $n(\omega)^N = 0$ for a sufficiently large N). In fact, $n(\omega)$ is the sum of terms with $r \geqq 1$ in (1). The algebras $\Lambda^{p,q}$ were introduced by F. Berezin; in the following they will be called the Berezin algebras.

It is important to note that one can substitute in (1) even elements $x^1, \dots, x^p$ and odd elements $\theta^1, \dots, \theta^q$ of an arbitrary Berezin algebra Λ, instead of the elements $u^1, \dots, u^p$, $\varepsilon^1, \dots, \varepsilon^q$. The expression

$$\sum f_{\alpha_1 \dots \alpha_r}(x^1, \dots, x^p)\theta^{\alpha_1} \dots \theta^{\alpha_r}$$

may be interpreted as an element of Λ. (In order to define $f_{\alpha_1 \dots \alpha_r}(x^1, \dots, x^p)$ one must represent x^a in the form $x^a = m(x^a) + n(x^a)$ and use the Taylor expansion with respect to $n(x^a)$. The elements $n(x^a)$ are nilpotent and therefore the Taylor expansion is finite.)

Lemma 1. [8] *If $x^1, \dots, x^p$ are even elements and $\theta^1, \dots, \theta^q$ are odd elements of an arbitrary Berezin algebra Λ, one can construct a parity preserving homomorphism ρ of $\Lambda^{p,q}$ into Λ, satisfying the conditions $\rho(u^a) = x^a$, $\rho(\varepsilon^\alpha) = \theta^\alpha$. The homomorphism is defined by the formula*

$$\rho(\omega) = \sum f_{\alpha_1 / \dots \alpha_r}(x^1, \dots, x^p)\theta^{\alpha_1} \dots \theta^{\alpha_r}, \tag{2}$$

where ω is given by (1).

In the following we consider only the homomorphisms of the Berezin algebras described in Lemma 1.

Let us define a Λ point in the (p,q)-dimensional superspace $\mathbb{R}^{p,q}$ as a row $(x^1,\dots,x^p,\ \theta^1,\dots,\theta^q)$, where x^a are even elements and θ^α are odd elements of the Berezin algebra Λ; the set of all the Λ-points will be denoted by $\mathbb{R}^{p,q}_\Lambda$. (In other words, the point in the superspace $\mathbb{R}^{p,q}$ has p even coordinates and q odd coordinates.) If $\theta^1=\dots=\theta^q=0$ we say that the Λ-point is even, if $x^1=\dots=x^p=0$ we say that the Λ-point is odd. Instead of the term "Λ-point" we use the term "(p,q)-dimensional Λ-vector" in some cases.

A homomorphism ρ of a Berezin algebra Λ into a Berezin algebra Λ' generates a mapping $\tilde{\rho}$ of $\mathbb{R}^{p,q}_\Lambda$ into $\mathbb{R}^{p,q}_{\Lambda'}$. (The mapping $\tilde{\rho}$ transforms $(x^1,\dots,x^p,\ \theta^1,\dots,\theta^q)$ into $(\rho(x^1),\dots,\rho(x^p),\ \rho(\theta^1),\dots,\rho(\theta^q))$.) It is easy to verify that $\rho_1\rho_2=\tilde{\rho}_1\tilde{\rho}_2$ for arbitrary homomorphisms $\rho_1:\Lambda'\to\Lambda''$, $\rho_2:\Lambda\to\Lambda'$.

Suppose that for every Berezin algebra Λ a set $\mathscr{E}_\Lambda$ is defined, and that for every homomorphism ρ of Λ into Λ' a map $\tilde{\rho}$ of $\mathscr{E}_\Lambda$ into $\mathscr{E}_{\Lambda'}$. If $\rho_1\rho_2=\tilde{\rho}_1\tilde{\rho}_2$ for arbitrary homomorphisms ρ_1,ρ_2, we say that the collection of the sets $\mathscr{E}_\Lambda$ and maps $\tilde{\rho}$ determines the superspace[2] $\mathscr{E}=(\mathscr{E}_\Lambda,\tilde{\rho})$; the set $\mathscr{E}_\Lambda$ will be called the set of the Λ-points of $\mathscr{E}$. The set $\mathscr{E}_\Lambda$ corresponding to the algebra $\Lambda=\Lambda^{0,0}=\mathbb{R}$ will be called the underlying space of the superspace $\mathscr{E}$ and is denoted by $\mathscr{E}_0$.

Let us consider a set $U\subset\mathscr{E}_0$. We say that a Λ-point e lies over U if for every homomorphism ρ of Λ into $\mathbb{R}$ the point $\tilde{\rho}(e)$ belongs to U. The set of the Λ-points lying over U will be denoted by $\mathscr{E}^U_\Lambda$. It is easy to verify that $\tilde{\rho}(\mathscr{E}^U_\Lambda)\subset\mathscr{E}^U_{\Lambda'}$ for any homomorphism ρ of Λ into Λ'. Therefore the sets $\mathscr{E}^U_\Lambda$ and the maps $\tilde{\rho}$ determines a superspace $\mathscr{E}^U$, we will say that this superspace is a subsuperspace of $\mathscr{E}$ over U.

The simplest example of the superspace is $\mathbb{R}^{p,q}$; the underlying space of $\mathbb{R}^{p,q}$ is $\mathbb{R}^p$. If U is a domain in $\mathbb{R}^p$ then the subsuperspace $(\mathbb{R}^{p,q})^U$ of $\mathbb{R}^{p,q}$ over U is called a (p,q)-dimensional superdomain.

A morphism τ of the superspace $\mathscr{E}=(\mathscr{E}_\Lambda,\tilde{\rho})$ into the superspace $\mathscr{E}'=(\mathscr{E}'_\Lambda,\tilde{\rho}')$ can be defined as a collection of the maps τ_Λ of $\mathscr{E}_\Lambda$ into $\mathscr{E}'_\Lambda$, satisfying the condition $\tilde{\rho}'\tau_\Lambda=\tau_{\Lambda'}\tilde{\rho}$ for an arbitrary homomorphism ρ of Λ into Λ'.

The morphisms of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$ can be described easily. Let us consider the $\Lambda^{p,q}$-point $a=(u^1,\dots,u^p,\ \varepsilon^1,\dots,\varepsilon^q)$ of $\mathbb{R}^{p,q}$ (here $u^1,\dots,u^p,\ \varepsilon^1,\dots,\varepsilon^p$ are the generators of $\Lambda^{p,q}$). From Lemma 1 one can deduce that every Λ-point b of $\mathbb{R}^{p,q}$ can be represented in the form $b=\tilde{\rho}(a)$, where ρ is a homomorphism of $\Lambda^{p,q}$ into Λ. It follows from this assertion that the morphism τ of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$ can be completely characterized by means of the $\Lambda^{p,q}$-point $a'=\tau_{\Lambda^{p,q}}(a)$ of $\mathbb{R}^{p',q'}$. We see that there exists a one-to-one correspondence between the morphisms of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$ and the $\Lambda^{p,q}$-points of $\mathbb{R}^{p',q'}$. Using Lemma 1 we obtain the following

Lemma 2. *There exists a one-to-one correspondence between the morphisms of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$ and the homomorphisms of $\Lambda^{p',q'}$ into $\Lambda^{p,q}$.*

The morphism τ is an isomorphism if all τ_Λ are one-to-one maps.

We say that the superspace $G=(G_\Lambda,\tilde{\rho})$ is a supergroup if all sets G_Λ are provided

2 Using the concept of the functor, we define the superspace as a functor acting from the category of the Berezin algebras to the category of sets

with a group structure, and all the maps $\tilde{\rho}$ are homomorphisms. Similar definitions can be given for a linear superspace, the Lie superalgebras, etc.

It is useful to note that for every Z_2-graded space $E = E_0 + E_1$ one can construct a linear superspace, assuming that the set of the Λ-points $\mathscr{E}_\Lambda$ consists of the linear combinations $\sum \lambda_i e_i + \sum \mu_j f_j$, where λ_i and μ_j are even (odd) elements of Λ, $e_i \in E_0$, $f_j \in E_1$. If in the Z_2-graded space $E = E_0 \dotplus E_1$ the operation $[x, y\}$ is determined, one can define naturally the corresponding operation in $\mathscr{E}_\Lambda$. In the case when the axioms of the Lie algebra are satisfied for the operation in $\mathscr{E}_\Lambda$, we say that E is a Z_2-graded Lie algebra with respect to the operation $[x, y\}$.

The supergroup G acts in the superspace $\mathscr{E}$ if for every Λ the group G_Λ acts in $\mathscr{E}_\Lambda$ and

$$\tilde{\rho}\varphi_g(e) = \varphi_{\tilde{\rho}(g)}(\tilde{\rho}(e))$$

for every homomorphism ρ of the Berezin algebra Λ into the Berezin algebra Λ'. (Here φ_g denotes the transformation of $\mathscr{E}_\Lambda$ corresponding to $g \in G_\Lambda$, the maps of $\mathscr{E}_\Lambda$ into $\mathscr{E}_{\Lambda'}$ and of G_Λ into $G_{\Lambda'}$ corresponding to ρ are denoted by the same symbol $\tilde{\rho}$.)

The coset superspace $\mathscr{E}/G$ (the space of orbits of G in $\mathscr{E}$) is determined as a collection of the sets $\mathscr{E}_\Lambda/G_\Lambda$ and maps $\tilde{\rho}$ of $\mathscr{E}_\Lambda/G_\Lambda$ into $\mathscr{E}_{\Lambda'}/G_{\Lambda'}$ induced by maps $\tilde{\rho}$ of $\mathscr{E}_\Lambda$ into $\mathscr{E}_{\Lambda'}$.

The concept of the superspace is too general for our purpose. The most interesting superspaces are supermanifolds. The superspace $\mathscr{E} = (\mathscr{E}_\Lambda, \tilde{\rho})$ will be called a (p, q)-dimensional supermanifold if there exists such a regular covering $\mathscr{U} = \{U_\nu\}$ of $\mathscr{E}_0$ that any subsuperspace $\mathscr{E}_\Lambda^{U_\nu}$ of $\mathscr{E}_\Lambda$ over U_ν is isomorphic to a (p, q)-dimensional superdomain for $U_\nu \in \mathscr{U}$. (We say that the collection $\mathscr{U}$ of the sets U_ν is a regular covering of $\mathscr{E}_0$ if every point $e \in \mathscr{E}_0$ belongs to a finite number of the sets U_ν, the intersection $U_\nu \cap U_\mu \in \mathscr{U}$ for every $U_\nu \in \mathscr{U}$, $U_\mu \in \mathscr{U}$, and for arbitrary points e_1, $e_2 \in \mathscr{E}_0$ one can find non-intersecting sets U_ν, $U_\mu \in \mathscr{U}$ containing e_1 and e_2, respectively.)

A supergroup $G = (G_\Lambda, \tilde{\rho})$ is called the (p, q)-dimensional Lie supergroup if the superspace G is a (p, q)-dimensional supermanifold.

One can check that the definition of the supermanifold and of the Lie supergroup given above are equivalent to the standard ones [8].

There exist various modifications of the definition for the superspace. For example, one can assume that the sets $\mathscr{E}_\Lambda$ are defined only in the case when Λ is a Grassmann algebra. One can impose the condition that the sets $\mathscr{E}_\Lambda$ are topological spaces or smooth manifolds; then the maps $\tilde{\rho}$ must be continuous or smooth, respectively[3]. For every modification of the definition of the superspace we can give the corresponding definitions of the morphism, the supergroup, the supermanifold, etc. It is assumed further that in the definition of the superspace Λ runs over the Grassmann algebras, $\mathscr{E}_\Lambda$ is a smooth manifold, and $\tilde{\rho}$ is a smooth map. A correspondence between the present definition of the superspace and other definitions will be studied in a separate paper.

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3 Considering the superspace as a functor, we can assume that this functor is defined on the category of the Grassmann algebras or on the category of Z_2-graded supercommutative algebras and takes values in the category of the topological spaces, in the category of smooth manifolds, etc.

Every element $\omega \in \Lambda^{p,q}$ determines a Λ-valued function on $\mathbb{R}_\Lambda^{p,q}$. (If ω is given by (1) the value of this function at the point $z=(x^1,\dots,x^p,\ \theta^1,\dots,\theta^q)\in\mathbb{R}_\Lambda^{p,q}$ can be obtained by means of the substitution of $x^1,\dots,x^p$, $\theta^1,\dots,\theta^q$ in place of $u^1,\dots,u^p$, $\varepsilon^1,\dots,\varepsilon^q$.) Therefore a formal linear combination $\sum\lambda_i\omega_i$ where $\lambda_i\in\Lambda$, $\omega_i\in\Lambda^{p,q}$ determines a Λ-valued function on $\mathbb{R}_\Lambda^{p,q}$ as well. The functions on $\mathbb{R}_\Lambda^{p,q}$ obtained by means of this construction will be called the Λ-functions. (There exists a one-to-one correspondence between the Λ-functions and the elements of the tensor product $\Lambda\otimes\Lambda^{p,q}$.) For every homomorphism ρ of the algebra Λ into the algebra Λ' one can define a map of the set of Λ-functions into the set of Λ'-functions replacing the coefficients λ_i in the linear combination $\sum\lambda_i\omega_i$ by $\rho(\lambda_i)$. One can define naturally the concepts of even Λ-functions and odd Λ-functions.

The map φ of $\mathbb{R}_\Lambda^{p,q}$ into $\mathbb{R}_\Lambda^{p',q'}$ will be called a Λ-map if the even coordinates (odd coordinates) of the point $\varphi(z)$ $z\in\mathbb{R}_\Lambda^{p,q}$ are even Λ-functions (odd Λ-functions, respectively) on $\mathbb{R}_\Lambda^{p,q}$. The set of all Λ-maps of $\mathbb{R}_\Lambda^{p,q}$ into $\mathbb{R}_\Lambda^{p',q'}$ will be denoted by $\mathscr{U}_\Lambda(p,q|p',q')$. Every homomorphism ρ of an algebra Λ into an algebra Λ' generates a map $\tilde\rho$ of $\mathscr{U}_\Lambda(p,q|p',q')$ into $\mathscr{U}_{\Lambda'}(p,q|p',q')$. The sets $\mathscr{U}_\Lambda(p,q|p',q')$ and the maps $\tilde\rho$ determine a superspace $\mathscr{U}(p,q|p',q')$ the superspace of the maps of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$.

It is noteworthy that a composition of the Λ-maps is also a Λ-map. (The proof of this assertion is based essentially on the fact that Λ is the Grassmann algebra. Every element $\lambda\in\Lambda$ can be represented in the form $m(\lambda)+n(\lambda)$, where $m(\lambda)$ is a real number, and $n(\lambda)$ is a nilpotent element of Λ; we apply the Taylor expansion with respect to the nilpotent elements. If we consider the Berezin algebras Λ, one can find such Λ-maps that their composition is not a Λ-map.)

The set of invertible Λ-maps of $\mathbb{R}^{p,q}$ onto $\mathbb{R}^{p,q}$ will be denoted by $\mathscr{R}_\Lambda^{p,q}$; this set can be considered as an infinite-dimensional Lie group with respect to the composition of the Λ-maps. The groups $\mathscr{R}_\Lambda^{p,q}$ determine the supergroup $\mathscr{R}^{p,q}$ (the supergroup of transformations of $\mathbb{R}^{p,q}$).

A linear Λ-map of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$ can be represented in the form

$$x'^b = x^a M_a^b + \theta^\alpha M_\alpha^b,$$

$$\theta'^\beta = x^a M_a^\beta + \theta^\alpha M_\alpha^\beta,$$

where M_a^b, M_α^β are even elements of Λ and M_α^b, M_a^β are odd elements. We say that the linear map is regular if the rank of $m(M_a^b)$ equals $\min(p,p')$ and the rank of $m(M_\alpha^\beta)$ equals $\min(q,q')$ (i.e. these ranks have the maximal possible values). The regular Λ-maps of $\mathbb{R}^{p,q}$ into itself are invertible; the groups $\mathrm{GL}_\Lambda(p,q,\mathbb{R})$ of the invertible linear Λ-maps determine the supergroup $\mathrm{GL}(p,q,\mathbb{R})$. A composition of the maps determines the action of the groups $\mathrm{GL}(p,q,\mathbb{R})$ and $\mathrm{GL}(p',q',\mathbb{R})$ on the superspace $\mathrm{RL}(p,q|p',q')$ of the linear regular maps of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$ (the Λ-point of $\mathrm{RL}(p,q|p',q')$ is a regular Λ-map of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$). If $p\leqq p', q\leqq q'$ one can define the superspace $G(p,q|p',q')$ of (p,q) planes in $\mathbb{R}^{p',q'}$ as the coset space $\mathrm{RL}(p,q|p',q')/\mathrm{GL}(p,q,\mathbb{R})$. One can identify this superspace with $\mathrm{RL}(p',q'|p'-p,$ $q'-q)/\mathrm{GL}(p'-p,q'-q,\mathbb{R})$. (The Λ-point of $G(p,q|p',q')$ is the (p,q)-dimensional Λ-plane; it can be defined by means of the map of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$ or by means of a set of $(p'-p)$ even and $(q'-q)$ odd equations.) A straightforward generalization of the standard definition is used to define the differential of the Λ-map at a Λ-point. This differential can be considered as a linear Λ-map. If the differentials of the Λ-map at all the Λ-points are regular, we say that the Λ-map is regular. A regular Λ-map φ of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$, with $p \leqq p', q \leqq q'$, determines a (p, q)-dimensional Λ-surface in $\mathbb{R}^{p',q'}$; the Λ-maps φ_1 and φ_2 determine the same Λ-surface if φ_2 can be obtained from φ_1 by means of a reparametrization. (In other words, the superspace of (p,q)-dimensional surfaces is the space of orbits of the supergroup $\mathscr{R}^{p,q}$ acting on the superspace of regular maps of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$.) Our definition of the Λ-surface is local; the standard considerations permit one to get a global definition of the Λ-surface. The Λ-map of $\mathbb{R}^{p,q}$ determining a (p,q)-dimensional Λ-surface can be considered as a parametric equation of the Λ-surface. From another point of view, a (p,q)-dimensional Λ-surface in $\mathbb{R}^{p',q'}$ can be determined (locally) by means of $p'-p$ even and $q'-q$ odd equations $f^k(z)=0, f^\varkappa(z)=0$, where $z \in \mathbb{R}_\Lambda^{p',q'}$. (The Λ-functions $(f^k, f^\varkappa)$ determine a map of $\mathbb{R}_\Lambda^{p',q'}$ into $\mathbb{R}_\Lambda^{p'-p,q'-q}$; the differential of this map must be regular at the points of the surface.) Of course, the equations $\hat{f}^k(z)=0, \hat{f}^\varkappa(z)=0$, where

$$\hat{f}^k(z) = f^{k'}(z)\eta^k_{k'}(z) + f^{\varkappa'}(z)\eta^k_{\varkappa'}(z), \tag{3}$$
$$\hat{f}^\varkappa(z) = f^{k'}(z)\eta^\varkappa_{k'}(z) + f^{\varkappa'}(z)\eta^\varkappa_{\varkappa'}(z),$$

and the matrix

$$\begin{pmatrix} \eta^k_{k'}(z) & \eta^k_{\varkappa'}(z) \\ \eta^\varkappa_{k'}(z) & \eta^\varkappa_{\varkappa'}(z) \end{pmatrix}$$

is invertible, determine the same Λ-surface as the equations $f^k(z)=0, f^\varkappa(z)=0$. It is convenient to write the equations for the surface in the form $f^K(z)=0$, where $f^K=(f^k, f^\varkappa)$ and $K=(k,\varkappa)$; then the relations (3) take the form $f^K(z)=f^{K'}(z)\eta^K_{K'}(z)$.

One can easily define the tangent Λ-plane to the Λ-surface at a Λ-point. If the Λ-surface is determined by means of a Λ-map of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$, one can define the tangent Λ-plane using the differential of this map; if the Λ-surface is determined by equations one has to linearize these equations at the point under consideration. The points of the tangent Λ-plane will be called the tangent Λ-vectors.

If the Λ-surface is determined by equations $f^K(z)=0$, then the tangent Λ-vector E^M satisfies the condition

$$E^M \frac{\partial f^K}{\partial z^M} = 0.$$

The Λ-basis E_A^M in the tangent Λ-plane will be called the tangent frame for the Λ-surface. We assume that $E_A^M = (E_a^M, E_\alpha^M)$, where E_a^M are even vectors, E_α^M are odd vectors, $a = 1, \ldots, p$, $\alpha = 1, \ldots, q$.

If the Λ-surface is defined by means of a parametric equation $z^M = \varphi^M(v)$ (i.e. by means of a Λ-map φ of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p',q'}$), then every (p,q)-dimensional Λ-vector E^L can be interpreted as a tangent vector to the surface. (This vector determines an element in the tangent plane by the relation $\hat{E}^M = E^L(\partial\varphi^M/\partial v^L)$). The tangent vector E^L transformers under a reparametrization $\tilde{v} = \lambda(v)$ in the standard way: $\tilde{E}^{\tilde{L}} = E^L(\partial\tilde{v}^{\tilde{L}}/\partial v^L)$. The tangent co-vector can be defined according to the transformation law $E_L = (\partial\tilde{v}^{\tilde{L}}/\partial v^L)\tilde{E}_{\tilde{L}}$.

Let us consider an automorphism J of the superspace $\mathbb{R}^{2p,2q}$ transforming a Λ-point $(x^1,\dots,x^{2p}, \theta^1,\dots,\theta^{2q})$ into the Λ-point $(-x^2, x^1,\dots,-x^{2p}, x^{2p-1}, -\theta^2, \theta^1,\dots,-\theta^{2q}, \theta^{2q-1})$. Instead of the coordinates x^a, θ^α taking values in the real Grassmann algebra Λ we can characterize the point of $\mathbb{R}^{2p,2q}$ by means of coordinates $w^1 = x^1 + ix^2,\dots, w^p = x^{2p-1} + ix^{2p}, v^1 = \theta^1 + i\theta^2, v^q = \theta^{2q-1} + i\theta^{2q}$ taking values in the complex Grassman algebra. The morphism J transforms the point with the coordinates $(w^1,\dots,w^p, v^1,\dots,v^q)$ into the point $(iw^1,\dots,iw^p, iv^1,\dots,iv^q)$. Hence we can consider the superspace $\mathbb{R}^{2p,2q}$ with the automorphism J as a complex superspace $\mathbb{C}^{p,q}$; the space $\mathbb{C}^{p,q}$ has the complex dimension (p,q) (i.e. the point of $\mathbb{C}^{p,q}$ has p complex even and q complex odd coordinates). In the complex Grassmann algebra an involution $(A \to \bar{A})$ is defined, satisfying the conditions

$$\overline{(AB)} = \bar{B}\bar{A},\ \overline{(\lambda A)} = \bar{\lambda}\bar{A},\ \varepsilon^{\bar{\alpha}} = \varepsilon^\alpha$$

(here ε^α are generators of the Grassmann algebra, λ is a complex number, A and B are elements of the algebra). Using this involution, we define the involution in $\mathbb{C}^{p,q}$ transforming any point $(w^1,\dots,w^p, v^1,\dots,v^q)$ into $(\overline{w^1},\dots,\overline{w^p},\overline{v^1},\dots,\overline{v^q})$.

The notions introduced above for the superspace $\mathbb{R}^{p,q}$ can be defined also for the superspace $\mathbb{C}^{p,q}$. It is important to note that one can introduce the notions of the complex linear Λ-map φ of $\mathbb{C}^{p,q}$ into $\mathbb{C}^{p',q'}$ (the real linear Λ-map φ of $\mathbb{R}^{2p,2q}$ into $\mathbb{R}^{2p',2q'}$ is a complex linear map of $\mathbb{C}^{p,q}$ into $\mathbb{C}^{p',q'}$ if $\varphi J = J\varphi$.) The Λ-map of $\mathbb{R}^{2p,2q}$ into $\mathbb{R}^{2p',2q'}$ can be considered as an analytic map of $\mathbb{C}^{p,q}$ into $\mathbb{C}^{p',q'}$ if the differentials of this map at all points are complex linear Λ-maps.

Later we will consider real Λ-surfaces in $\mathbb{C}^{p,q}$. The Λ-surface in $\mathbb{C}^{p,q}$ having a real dimension (m,n) can be determined by means of $2p - m$ even equations $f^k(z,\bar{z}) = 0$, and $2q - m$ odd equations $f^\varkappa(z,\bar{z}) = 0$ (f^k and $f^\varkappa$ are real Λ-functions here).

Let us consider the complexification $\tilde{\Lambda}^{p,q}$ of the algebra $\Lambda^{p,q}$ (the elements of $\tilde{\Lambda}^{p,q}$ can be represented in the form (1) where $f_{\alpha_1\dots\alpha_r}(u^1,\dots,u^p)$ are smooth complex functions of real variables $u^1,\dots,u^p$). Using, as before, the Taylor expansion with respect to the nilpotent elements, we can replace $u^1,\dots,u^p$, $\varepsilon^1,\dots,\varepsilon^q$ in (1) by $w^1,\dots,w^p$, $v^1,\dots,v^q$, where $m(w^1),\dots,m(w^p)$ are real. We will say that the point $(w^1,\dots,w^p, v^1,\dots,v^q)$ of $\mathbb{C}^{p,q}$ is simple if $m(w^1),\dots,m(w^p)$ are real. The set of p' even and q' odd elements of $\tilde{\Lambda}^{p,q}$ generates the map of $\mathbb{C}^{p,q}_\Lambda$ into $\mathbb{C}^{p',q'}_\Lambda$, defined on simple points of $\mathbb{C}^{p,q}$. It is easy to check that this map is analytic.

3. Supergravity and the Complex Geometry

The action functional in the supergravity can be considered as a functional defined on the superspace of (4,4)-dimensional surfaces in the complex superspace $\mathbb{C}^{4,2}$. This functional can be represented as a surface integral; the integrand (the action density) is expressed in terms of the first and the second derivatives of the functions present in the equations determining the surface. The density is invariant with respect to analytic transformations of $\mathbb{C}^{4,2}$ preserving the supervolume. These properties determine the density uniquely (up to a constant factor) [9], [3].

We will show how one can get a simple expression for the action functional of the supergravity using some notions of the complex geometry.

Let us denote by $x^1, \ldots, x^p$ the even coordinates in $\mathbb{C}^{p,q}$, and by $\theta^1, \ldots, \theta^q$ the odd coordinates; we use also the notation $z^A = (x^a, \theta^\alpha)$, when the even and odd coordinates are considered simultaneously. The symbol $\bar{z}^{\dot{A}} = (\bar{x}^{\dot{a}}, \bar{\theta}^{\dot{\alpha}})$ will be also used instead of $\overline{z^A} = (\overline{x^a}, \overline{\theta^\alpha})$. The derivatives $\partial/\partial x^a$, $\partial/\partial \overline{x^a}$, $\partial/\partial\theta^\alpha$, $\partial/\partial\overline{\theta^\alpha}$, $\partial/\partial z^A$, $\partial/\partial\overline{z^A}$ will be denoted by ∂_a, $\bar{\partial}_{\dot{a}}$, ∂_α, $\bar{\partial}_{\dot{\alpha}}$, ∂_A, $\bar{\partial}_{\dot{A}}$, respectively. The (m,n)-dimensional surface Ω in $\mathbb{C}^{p,q}$ will be described by means of $2p-m$ even equations $f^k(z^A, \bar{z}^{\dot{A}}) = 0$ and $2q-n$ odd equations $f^\varkappa(z^A, \bar{z}^{\dot{A}}) = 0$, where $\overline{f^k} = f^k$, $\overline{f^\varkappa} = f^\varkappa$. These equations can be represented in the form

$$f^K(z^A, \bar{z}^{\dot{A}}) = 0, \tag{1}$$

where $f^K = (f^k, f^\varkappa)$ is the map of $\mathbb{C}^{p,q}$ into the real superspace $\mathbb{R}^{2p-m, 2q-m}$. The equations of the tangent plane (tangent subspace) $T_z(\Omega)$ to the surface Ω at a point z are

$$dz^A \partial_A f^K + d\bar{z}^{\dot{A}} \bar{\partial}_{\dot{A}} f^K = 0.$$

The plane $T_z(\Omega)$ is a real (m,n)-dimensional plane. The maximal complex plane contained in $T_z(\Omega)$ will be denoted by $C_z(\Omega)$. The equations of the plane $C_z(\Omega)$ are

$$dz^A \partial_A f^K = 0. \tag{2}$$

(The complex plane is invariant with respect to the multiplication by i, i.e. with respect to the automorphism J transforming $(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ into $(ix^1, \ldots, ix^p, i\theta^1, \ldots, i\theta^q$.) The maximal complex plane contained in a real plane K is described as the intersection of the planes K and JK.) We assume that the equations (2) are linearly independent; the complex plane $C_z(\Omega)$ has the complex dimension $(m-p, n-q)$. In the case of the supergravity $m=4$, $n=2$, $p=4$, $q=2$ and the complex dimension of $C_z(\Omega)$ equals $(0,2)$.

The Levi form for the surface Ω is called the expression[4]

$$\omega^K = \pm\, dz^A \frac{\partial^2 f^K}{\partial z^A \partial \bar{z}^{\dot{B}}}\, d\bar{z}^{\dot{B}}, \tag{3}$$

considered on the complex subspace $C_z(\Omega)$. (In other words, we assume that the differentials in the Levi form are constrained by the relations (2).) The Levi form may be considered as a Hermitean form on $C_z(\Omega)$ taking values in the superspace $\mathbb{R}^{2p-m, 2q-n}$. It is easily seen that the expression (3) is invariant with respect to analytical transformations $z \to \Phi(z)$. As it was mentioned above, the surface Ω may be fixed by means of various sets of equations. The equations $f^K(z, \bar{z}) = 0$ and the equations $\hat{f}^K(z, \bar{z}) = 0$ determine the same surface in the case where $\hat{f}^K = f^{K'} \eta^K_{K'}$, $\eta^K_{K'}$ being an invertible matrix. If the functions f^K are replaced by $\hat{f}^K$ the expression (3) is transformed into

$$\begin{aligned}\hat{\omega}^K = \omega^{K'} \eta^K_{K'} + dz^A(&\pm \partial_A f^{K'} \bar{\partial}_{\dot{B}} \eta^K_{K'} \\ &\pm \bar{\partial}_B f^{K'} \partial_A \eta^K_{K'} \pm f^{K'} \partial_A \bar{\partial}_B \eta^K_{K'})\, d\bar{z}^{\dot{B}}.\end{aligned}$$

Hence it is clear that under the conditions $f^K = 0$, $dz^A \partial_A f^K = 0$ the transformation

4 The sign $\pm$ (which emerges in this section) denotes that we omit inessential factors (-1) in the sums

law for the forms ω^K is quite simple

$$\hat{\omega}^K = \omega^{K'} \eta^K_{K'}. \tag{5}$$

Thus the expression (3) is transformed according to (5) in the complex tangent subspace $C_z(\Omega)$. By definition, the expression (3) in the subspace $C_z(\Omega)$ is the Levi form for the surface Ω. So (5) gives the rule according to which the Levi form is changed at variations of the equations determining the surface Ω.

Now we turn to the supergravity. In this case the surface Ω is determined by a set of four even equations $f^k(z, \bar{z}) = 0$. The equation of the complex tangent plane $C_z(\Omega)$ is written as

$$dx^a \partial_a f^k + d\theta^\alpha \partial_\alpha f^k = 0.$$

This equation enables one to express dx^a via $d\theta^\alpha$, so $d\theta^\alpha$ may be considered as coordinates in the complex tangent plane. (Recall that it was assumed that the equations determining the complex tangent plane are linearly independent; this is true for a generic surface.) Of course, another choice of the coordinates is possible in the subspace: one may take the coordinates v^λ related to $d\theta^\alpha$ by means of the relation $d\theta^\alpha = v^\lambda R^\alpha_\lambda$. Then $dx^a = v^\lambda R^a_\lambda$ where R^a_λ and R^α_λ satisfy the condition $R^a_\lambda \partial_a f^k + R^\alpha_\lambda \partial_\alpha f^k = 0$. The vectors $R^A_1 = (R^a_1, R^\alpha_1)$ and $R^A_2 = (R^a_2, R^\alpha_2)$ provide a basis in the plane $C_z(\Omega)$. In terms of the coordinates v^λ the Levi form is written as follows

$$\omega^k = \Gamma^k_{\lambda\dot{\mu}} v^\lambda \bar{v}^{\dot{\mu}},$$

where

$$\Gamma^k_{\lambda\dot{\mu}} = R^A_\lambda \frac{\partial^2 f^k}{\partial z^A \partial \bar{z}^{\dot{B}}} \bar{R}^{\dot{B}}_{\dot{\mu}}. \tag{6}$$

This expression will be called the Levi matrix in the basis R^A_λ.

It is appropriate to introduce a matrix $\Gamma^k_i = \Gamma^k_{\lambda\dot{\mu}} \sigma^{\lambda\dot{\mu}}_i$, where $\sigma^{\lambda\dot{\mu}}_i$ are elements of the Pauli matrices for $i = 1, 2, 3$, and of the unity matrix for $i = 0$, and to consider a quantity $\Gamma = \det(\Gamma^k_i)$. Using the Levi form one can get the action for the supergravity. If the Levi matrix is constructed with the coordinates $d\theta^\alpha$ in the complex tangent plane (i.e. $R^\alpha_\lambda = \delta^\alpha_\lambda$) then the action is

$$S(\Omega) = C \int \left| \det\left(\frac{\partial f^k}{\partial x^a} \right) \right|^{4/3} |\Gamma|^{-1/3} \prod_k \delta(f^k(z, \bar{z}))\, dz\, d\bar{z}, \tag{7}$$

where C is an arbitrary constant (see [10]).

If the Levi matrix is constructed for an arbitrary basis R^A_λ, the expression for the action functional is somewhat more complicated. Let us consider the forms $\sigma^K = dz^A \sigma^K_A$ satisfying the conditions $\mathrm{Ber}(\sigma^K_A) = 1$ (the unimodularity) and $\sigma^k_A = \partial_A f^k$. (In other words, $\sigma^k = dz^A \partial_A f^k$ and the forms $\sigma^\varkappa$ complete the system σ^k so that a unimodular system is obtained.) The basis R^A_λ in $C_z(\Omega)$ is used to construct the matrix $W^\varkappa_\lambda = R^A_\lambda \sigma^\varkappa_A$; its determinant will be denoted by W. It is noteworthy that W is determined completely by the functions f^k and the basis R^A_λ (i.e. W is independent of the choice of the forms $\sigma^\varkappa$). Actually, the forms $\sigma^\varkappa$ may be substituted by $\tilde{\sigma}^\varkappa = \sigma^\nu Q^\varkappa_\nu + \sigma^k Q^\varkappa_k$; the unimodularity is intact if $\det(Q^\varkappa_\nu) = 1$. The form $\sigma^\varkappa_A$ is transformed into $\tilde{\sigma}^\varkappa_A = \sigma^\nu_A Q^\varkappa_\nu + \partial_A f^k Q^\varkappa_k$, and the matrix $W^\varkappa_\lambda$ is replaced by $\tilde{W}^\varkappa_\lambda = W^\nu_\lambda Q^\varkappa_\nu$ so that $\tilde{W} = \det(Q^\varkappa_\nu) \cdot W = W$.

At an arbitrary choice of the basis R^A_λ in $C_z(\Omega)$ the action may be rewritten in the form

$$S(\Omega) = C\int |W|^{4/3}|\Gamma|^{-1/3}\prod_k \delta(f^k(z,\bar{z}))\,dz d\bar{z}. \tag{8}$$

Note that the expression (7) is obtained from (8) putting

$$\sigma^\varkappa = (\det(\partial_a f^k))^{1/2}\, d\theta^\varkappa.$$

First of all, we verify that (8) is determined by the surface Ω, i.e. is independent of the choice of the equations $f^k = 0$, as well as of the choice of the basis R^A_λ. If one replaces the equations $f^k = 0$ by other equations $\hat{f}^k = 0$, where $\hat{f}^k = \eta^k_{k'} f^{k'}$, the matrix $\Gamma^k_{\lambda\dot{\mu}}$ must be replaced by $\hat{\Gamma}^k_{\lambda\mu} = \eta^k_{k'}\Gamma^{k'}_{\lambda\mu}$, hence $\hat{\Gamma} = \det\eta\cdot\Gamma$. The matrices σ^k are replaced by $\hat{\sigma}^k = \eta^k_{k'}\sigma^k$; in order to preserve the unimodularity, one should put $\hat{\sigma}^\varkappa = (\det\eta)^{1/2}\sigma^\varkappa$, so that $\hat{W}^\varkappa_\lambda = (\det\eta)^{1/2} W^\varkappa_\lambda$, and $\hat{W} = \det\eta\cdot W$. The product $\prod_k \delta(f^k(z,\bar{z}))$ is divided by $\det\eta$ at the substitution in view. The substitution does not change the integrand in (8), as $4/3 + (-1/3) = 1$. If one would like to change the basis, $\hat{R}^A_\lambda = S^\rho_\lambda \hat{R}^A_\rho$, the matrices $W^\varkappa_\lambda$ and $\Gamma^k_{\lambda\mu}$ are transformed as follows: $\hat{W}^\varkappa_\lambda = S^\rho_\lambda W^\varkappa_\rho$ and $\hat{\Gamma}^k_{\lambda\dot{\mu}} = \Gamma^k_{\rho\dot{\sigma}} S^\rho_\lambda \bar{S}^{\dot{\sigma}}_{\dot{\mu}}$.
Hence $\hat{W} = \det(S^\rho_\lambda)$. W and $\hat{\Gamma} = (\det(S^\rho_\lambda))^4$. Since $4/3 + 4(-1/3) = 0$, the integrand in (8) is not changed.

Let us show that the expression in (8) may be considered as the Lagrangian for the supergravity. To this end one has to verify that the integrand is not changed by any analytical transformation $\Phi(z)$ preserving the supervolume. Because of the indicated above invariance of the Levi form under the analytical transformations, the Levi matrix $\Gamma^k_{\lambda\dot{\mu}}$, constructed for the surface Ω and for a basis R^A_λ in $C_z(\Omega)$, coincides with the Levi matrix constructed for a surface $\tilde{\Omega}$, fixed by equations $\tilde{f}^k(z,\bar{z}) = 0$ and for a basis $\tilde{R}^A_\lambda$ in the plane $C_{\tilde{z}}(\Omega)$. (The functions $\tilde{f}^k$ and the basis R^A_λ are related to f^k and R^A_λ as follows:

$$f^k(z,\bar{z}) = \tilde{f}^k(\Phi(z),\overline{\Phi(z)}), R^A_\lambda = \tilde{R}^B_\lambda \frac{\partial\Phi^A}{\partial z^B}, z = \Phi(\tilde{z}).)$$

Because of the supervolume conservation, the Berezinian of the matrix $\partial_B\Phi^A$ is unity, so the unimodular system of the forms σ^K is transformed into another unimodular system of the forms $\tilde{\sigma}^K$ at the change of the variables $z \to \Phi(z)$. Having this fact in mind, one sees that $W(\tilde{f}^k, \tilde{R}^A_\lambda) = W(f^k, R^A_\lambda)$. To conclude, the density of the action functional (8) is invariant with respect to analytical transformations preserving the supervolume, so it may be considered as the Lagrangian density for the supergravity. (Of course, this result may be checked also by means of a direct comparison with other known forms of the action functional for the supergravity.)

It is remarkable that the above approach also enables one to construct a multidimensional extension of the supergravity Lagrangian. Let us consider a surface Ω of the real dimension (m,n) in the complex superspace $\mathbb{C}^{M,N}$. Suppose the surface is fixed by the equations $f^K(z^A, \bar{z}^{\dot{A}}) = 0$. Introducing a basis R^A_L in the complex tangent subspace, one gets the Levi matrix in the form

$$\Gamma^K_{P\dot{Q}} = \pm R^A_P \frac{\partial^2 f^K}{\partial z^A \partial \bar{z}^{\dot{B}}} \bar{R}^{\dot{B}}_{\dot{Q}}.$$

The Levi matrix may be considered as the matrix of an operator acting from the $((m-M)^2+(n-N)^2, 2(m-M)(n-N))$-dimensional superspace into a $(2M-m, 2N-n)$-dimensional superspace. We restrict ourselves to the case where $(m-M)^2+(n-N)^2=2M-m$, $2(m-M)(n-N)=2N-n$, so that the dimension of the space in which the operator $\hat{\Gamma}$ is defined coincides with the dimension of the space in which it takes its values. In this case the determinant, Γ, of the operator $\hat{\Gamma}$ is meaningful. In the situation in view it is possible to extend the arguments, used for the supergravity, to the multidimensional case and to construct an action functional defined on (m,n)-dimensional surfaces in $\mathbb{C}^{M,N}$ and invariant with respect to the analytical transformations preserving the volume. (The supergravity itself is obtained at $m=4$, $n=4$, $M=4$, $N=2$.) The action functional looks like

$$S(\Omega)=C\int|W|^{\rho}|\Gamma|^{\tau}\prod_{k}\delta(f^K(z,\bar{z}))\,dz\,d\bar{z}, \tag{9}$$

where $\tau=[(m-n)-(M-N)]/[3(M-N)-(m-n)]$, $\rho=\tau-1$, the symbol Γ stands for the Berezinian of the Levi matrix, constructed for the basis R_L^A in the complex tangent subspace. As for W, it is defined as follows. We complete the system of the forms $\sigma^K=dz^A\partial_A f^K$ with the forms $\sigma^Q=dz^A\sigma_A^Q$, so that a unimolar system of the forms is obtained. Then W is the Berezinian of the matrix $W_L^Q=R_L^A\sigma_A^Q$. As in the supergravity, it is possible to verify that W is independent of the choice of the system of the forms σ^Q, while the integrand in (9) does not depend neither on the choice of the equations for the surface Ω, nor on the basis R_L^A. Thus the functional $S(\Omega)$ is determined completely by the surface Ω, so it may be considered as the action functional. The integrand in (9) (the action density) depends on the first and second derivatives and is not changed by analytical transformations preserving the supervolume. By means of the methods used in [9], [3], one can see that any functional having these properties equals that given in (9) times a constant factor. (In the work [3] the case $n=2N$, $m=M=N^2$ was investigated.)

4. G-Structures

Let G denote a supergroup consisting of linear transformations of the (p,q)-dimensional superspace $\mathbb{R}^{p,q}$. (In other words, G is a subgroup of $\mathrm{GL}(p,q,\mathbb{R})$.) A transformation matrix belonging to G will be denoted by g_A^B.

Two frames $\tilde{E}_A^M$ and E_A^M on a (p,q)-dimensional surface Ω will be called G-equivalent, if $\tilde{E}_A^M$ can be obtained from E_A^M by means of a transformation belonging to G:

$$\tilde{E}_A^M=g_A^B E_B^M. \tag{1}$$

(The frame E_A^M consists of p even vectors $E_a^M, a=1,\dots,p$, and q odd vectors $E_\alpha^M=1,\dots,q$. If a frame contains complex vectors, we assume that it contains also the complex conjugated vectors.) Suppose that a set of the G-equivalent frames is fixed at every point of Ω. Then we say that these sets determine the G-structure in Ω. The frames belonging to these sets will be called admissible (or compatible with the G-structure).

It is convenient to describe the G-structure in Ω by means of the frame field

$E_A^M(x)$. Two frame fields $\tilde{E}_A^M(x)$ and $E_A^M(x)$ determine the same G-structure if they are related by the formula

$$\tilde{E}_A^M(x) = g_A^B(x) E_B^M(x), \tag{2}$$

where $g_A^B(x)$ is a G-valued function.

Using the frame field $E_A^M(x)$, we can define the differential operators

$$\partial_A = E_A^M(x) \frac{\partial}{\partial x^M}. \tag{3}$$

(If the surface Ω is determined by a parametric equation, then x^M in (3) are coordinated in Ω and the components of E_A^M must be calculated with the aid of the coordinates x^M. If Ω is a surface in $\mathbb{R}^{p',q'}$ satisfying the equations $f^K(x) = 0$, then x^M must be considered as coordinates in $\mathbb{R}^{p',q'}$. The frame $E_A^M(x)$ satisfies in this case the equations $E_A^M(\partial f^K/\partial x^M) = 0$ on the surface Ω.)

Consider a function $\tau_{AB}^C(x)$ defined by the formula

$$[\partial_A, \partial_B\} = \tau_{AB}^C(x)\partial_C \tag{4}$$

on the surface Ω. (As usual, $[\partial_A, \partial_B\} = \partial_A\partial_B \pm \partial_B\partial_A$, the sign $+$ stands for both odd indices A, B; the sign is $-$ in all other cases.) If

$$\tilde{E}_A^M(x) = g_A^B(x) E_B^M(x),$$

$$\tilde{\partial}_A = \tilde{E}_A^M \frac{\partial}{\partial x^M} = g_A^B(x)\partial_B,$$

then

$$[\tilde{\partial}_A, \tilde{\partial}_B\} = \tilde{\tau}_{AB}^C(x)\tilde{\partial}_C,$$

where

$$\tilde{\tau}_{AB}^C(x) = \hat{\tau}_{AB}^C(x) + (g_A^D\partial_D g_B^{C'} \pm g_B^D\partial_D g_A^{C'})(g^{-1})_{C'}^C \tag{5}$$

(here $\hat{\tau}$ is connected with τ via tensor transformation law). Let us denote by $\mathscr{T}$ the space of the tensors τ_{AB}^C satisfying the conditions $\tau_{AB}^C = \pm\tau_{BA}^C$ and by $\mathscr{A}$ the subspace of $\mathscr{T}$ consisting of tensors having the form $l_{AB}^C \pm l_{BA}^C$, where the matrix l_{AB}^C belongs to the Lie superalgebra of G for a fixed B. The $\mathscr{T}$-valued function $\tau_{AB}^C(x)$ determines a function $\sigma_{AB}^C(x)$ taking values of $\mathscr{T}/\mathscr{A}$. The function $\sigma_{AB}^C(x)$ will be called the structure function for the G-structure. If we replace ∂_A by $\tilde{\partial}_A = g_A^B(x)\partial_B$ then it follows from (5) that σ_{AB}^C transforms covariantly.

A gauge field $(\omega_B^A)_M$ taking values in the Lie algebra $\mathscr{G}$ of the group G can be considered as the connection in the G-structure. The torsion T_{AB}^C for this connection is given by

$$T_{AB}^C = \tau_{AB}^C + \omega_{AB}^C \pm \omega_{BA}^C,$$

where $\omega_{AB}^C = E_B^M(\omega_A^C)_M$. The structure function σ_{AB}^C can be considered as a part of the torsion which does not depend on the choice of the connection.

One can say also that the structure function is a part of τ_{AB}^C which depends on the values of the vectors $E_A^M(x)$, but is independent of their derivatives. Let us consider the complement $\mathscr{C}$ to the linear subspace $\mathscr{A}$ in the space $\mathscr{T}$. The natural map of $\mathscr{T}$ onto $\mathscr{T}/\mathscr{A}$ generates a one-to-one correspondence between $\mathscr{C}$ and $\mathscr{T}/\mathscr{A}$. Hence we may consider the structure function as a $\mathscr{C}$-valued function.

The map of the surface Ω onto itself is called an automorphism of the G-structure if it transforms every admissible frame into an admissible one. The transformation of any surface into another one, at which admissible frames are transformed into admissible frames, is called the G-structure isomorphism.

Let us consider a few examples. If a Riemannian metrics is introduced in a p-dimensional surface, it is natural to consider orthonormalized frames as admissible. An orthonormal frame is defined up to a transformation from the group $\mathrm{O}(p)$, so the surface in view is provided with the $\mathrm{O}(p)$-structure. Inversely, if the $\mathrm{O}(p)$-structure is given, a Riemannian metrics is determined. Automorphisms of the $\mathrm{O}(p)$-structure are transformations conserving the Riemannian metrics (isometrices). The structure function for the $\mathrm{O}(p)$-structure vanishes. (An orthonormal frame at a point x may always be included into the orthonormal frame field, for which τ_{AB}^{C} vanishes at the point x. In other words, in this case $\mathscr{A}=\mathscr{T}$, so that $\mathscr{C}=0$.)

Now suppose $\mathscr{M}=\mathbb{C}^m$, the complex space. Let an admissible frame in $\mathbb{C}^m$ be the frame containing the vectors E_A^M of a complex basis in the space $\mathbb{C}^m$, and their complex conjugated vectors $\bar{E}_A^M$. The admissible frames are determined up to a non-degenerate complex linear transformation. So selecting such frames we introduce the $\mathrm{GL}(m,\mathbb{C})$-structure in the space $\mathbb{C}^m$. Evidentally, analytical transformations of the space $\mathbb{C}^m$ are automorphisms of this $\mathrm{GL}(m,\mathbb{C})$-structure. Note that it is not in every case that the frame of complex vectors and their conjugates is the most suitable; sometimes it is useful to consider systems where some vectors, or even all of them, are represented by their real and imaginary parts.

The above example admits an evident generalization. First of all, the complex space $\mathbb{C}^m$ may be replaced with a complex superspace $\mathbb{C}^{m,n}$. The supergroup of analytical transformations of the space $\mathbb{C}^{m,n}$ may be considered as the automorphism group of the $\mathrm{GL}(m,n,\mathbb{C})$-structure in $\mathbb{C}^{m,n}$.

If the class of the admissible frames in $\mathbb{C}^{m,n}$ is restricted by means of the additional constraint $\mathrm{Ber}(E_A^M)=1$, the frames are determined up to a complex linear transformation preserving the supervolume; the transformation group will be denoted by $\mathrm{SL}(m,n,\mathbb{C})$. The automorphisms of the $\mathrm{SL}(m,n,\mathbb{C})$ structure in $\mathbb{C}^{m,n}$ are the supervolume preserving analytical transformations.

In general, if G is a subgroup of the supergroup $\mathrm{GL}(p,q,\mathbb{R})$ then the G-structure may be introduced in the space $\mathbb{R}^{p,q}$ by means of the frame field $E_A^M(x)=\delta_A^M$. (The admissible frames for this structure are the frames of the type g_A^M, which are matrix elements of a matrix $g\in G$.) The G-structure isomorphic to the structure described here for the space $\mathbb{R}^{p,q}$ is called the trivial G-structure. The term "the flat G-structure" is used instead of "the trivial G-structure" more often in the mathematical literature. We prefer our version of the terminology, since the geometry of the flat superspace is described in terms of a non-trivial G-structure. The automorphisms of the trivial G-structure in $\mathbb{R}^{p,q}$ are transformations with the Jacobi matrix belonging to the group G.

The described $\mathrm{GL}(m,n,\mathbb{C})$-structure and $\mathrm{SL}(m,n,\mathbb{C})$-structure in $\mathbb{C}^{m,n}=\mathbb{R}^{2m,2n}$ are the trivial G-structures.

5. Induced Structure

Let $G \subset \mathrm{GL}(p', q', \mathbb{R})$ be a matrix group. The group G acts naturally in the set of (p, q)-dimensional subspaces of the space $\mathbb{R}^{p',q'}$. Let us fix a (p, q)-dimensional subspace E; with no loss of the generality one can assume that this subspace is spanned by the first p even vectors and q odd vectors belonging to the standard basis in $\mathbb{R}^{p',q'}$. Let $\tilde{G}$ be a subgroup of G containing the transformations under which the subspace E is invariant. The elements of $\tilde{G}$ generate transformations of E, the corresponding transformation group for the subspace will be denoted by G'. Identifying E with $\mathbb{R}^{p,q}$, one has $G' \subset \mathrm{GL}(p, q, \mathbb{R})$. The subspaces that are obtained from the fixed subspace E by means of transformations belonging to the group G will be called regular. Let us consider a (p', q')-dimensional manifold provided with G-structure. A surface $\Omega \subset \mathscr{M}$ of the dimension (p, q) will be called regular if all the tangent planes are regular. (Indication to an admissible frame in the tangent plane to $\mathscr{M}$ enables one to identify this plane with $\mathbb{R}^{p',q'}$. So a tangent plane to Ω may be identified with a (p, q)-dimensional subspace of the space $\mathbb{R}^{p',q'}$. Whether this subspace is regular is independent of the choice of the admissible frame.)

The frame in a regular surface Ω will be called admissible if it may be continued up to an admissible frame in the manifold $\mathscr{M}$. To be more precise, if even vectors $r_1, \ldots, r_p$ and odd vectors $s_1, \ldots, s_q$ form a basis in the tangent subspace of Ω this basis is called an admissible frame if there are such even vectors $r_{p+1}, \ldots, r_{p'}$ and odd vectors $s_{q+1}, \ldots, s_{q'}$ that the vectors $r_1, \ldots, r_{p'}, s_1, \ldots, s_{q'}$ form an admissible frame in $\mathscr{M}$. It is easily seen that two admissible frames in Ω are related by a transformation belonging to the group G'. Thus the defined concept of the admissible frame in Ω provides the regular surface Ω with a G'-structure.

Some examples illustrating the introduced notions are presented below. If $\mathscr{M}$ is a Riemannian manifold (i.e. $G = \mathrm{O}(p')$) then $G' = \mathrm{O}(p)$ and all the surfaces are regular. The induced $\mathrm{O}(p)$-structure is generated by the Riemannian metrics defined in the surface $\Omega \subset \mathscr{M}$ as usual.

If $\mathscr{M}$ is a m-dimensional complex space $\mathbb{C}^m$, then the $\mathrm{GL}(m, \mathbb{C})$-structure is introduced in it naturally. For a generic surface $\Omega \subset \mathscr{M}$ of a real dimension p, a G'-structure is induced naturally, where G' is a group described as follows. Consider a subspace E of $\mathbb{C}^m$ determined by the k_0 equations

$$\mathrm{Im}\, x^{m-k_0+1} = 0, \ldots, \mathrm{Im}\, x^m = 0,$$

where $x^M = u^M + iv^M$ $(1 \leqq M \leqq m)$ are coordinates in $\mathbb{C}^m$, and $k_0 = 2m - p$. The group $\tilde{G}$ contains complex linear transformations of the space $\mathbb{C}^m$, under which E is invariant. Clearly, these transformations are

$$x'^r = a^r_s x^s, \qquad (1)$$

$$x'^d = b^d_s x^s + c^d_e x^e,$$

where $1 \leqq d,\ e \leqq p - m,\ p - m < r,\ s \leqq m$ and the matrix elements a^r_s are real.

The group G' is described as the group of the transformations of the subspace E written as (1). It is suitable to describe a point of E by means of $k_0 = 2m - p$ real coordinates $x^{p-m+1}, \ldots, x^m$ and of $m = k_0 = p - m$ complex coordinates $x^1, \ldots, x^{p-m}$. Suppose a subapace $E' \subset E$ is fixed by means of the equations

$x^{p-m+1} = \cdots = x^m = 0$. The coordinates $x^1, \ldots, x^{p-m}$ in E' are complex, so it may be considered as a complex space. Elements of G' are linear transformations of the space E transforming E' into itself, which are complex linear transformations in E'.

The G'-structure in a surface embedded into a complex space, as it was described above, is called the CR (Cauchy–Riemann)-structure. The admissible frames for this structure are as follows. In every tangent subspace to the surface Ω a complex tangent subspace is indicated; in the general case its dimension is $p - m$. A frame is called admissible if it contains vectors forming a complex basis in the complex tangent subspace and the complex conjugated vectors (other vectors of the frame are arbitrary).

It is assumed that in the space $\mathbb{R}^{2m} = \mathbb{C}^m$ a basis is chosen, containing the complex vectors $\partial/\partial x^a$, $a = 1, \ldots, p-m$, the complex conjugated vectors $\partial/\partial \bar{x}^a$ and real vectors $\partial/\partial u^r$ and $\partial/\partial v^r$, $r = p-m+1, \ldots, m$. Then the complex vectors, their conjugates and the first $k_0 = 2p - m$ real vectors of this basis form a basis in the space E, its real dimension being p. Here and in the subsequent discussion of this example we shall assume that the indices a, b, c run from 1 up to $p - m$, the indices r, s, t, k run from $p - m + 1$ up to m, and the index M runs from 1 up to m.

If the surface Ω is given by $k_0 = 2m - p$ real equations $f^k(x) = 0$, then the admissible frame contains the complex vectors[5] $\partial_a = E_a^M \partial/\partial x^M$, satisfying the equation

$$E_a^M \partial_M f^k = 0, \tag{2}$$

the complex conjugated vectors $\bar{\partial}_{\dot{a}} = \bar{E}_{\dot{a}}^{\dot{M}} \partial/\partial \bar{x}^{\dot{M}}$ and arbitrary tangent real vectors $\partial_r = E_r^M \partial/\partial x^M + \bar{E}_r^{\dot{M}} \partial/\partial \bar{x}^{\dot{M}}$, i.e. the vectors satisfying the relation

$$E_r^M \partial f^k/\partial x^M + \bar{E}_r^{\dot{M}} \partial f^k/\partial x^{\dot{M}} = 0. \tag{3}$$

It is suitable to impose another constraint upon the arbitrary tangent vectors,

$$E_r^M \frac{\partial f^k}{\partial x^M} = i\delta_r^k. \tag{4}$$

Let us consider the structure function for the CR-structure in Ω. The commutator of the vector fields $\partial_a = E_a^M(x, \bar{x})\partial/\partial x^M$ and $\partial_b = E_b^M(x, \bar{x})\partial/\partial x^M$ belongs also to the complex tangent plane (i.e. the field satisfies (2)). This means that in the decomposition

$$[\partial_a \partial_b] = \tau_{ab}^c \partial_c + \tau_{ab}^{\dot{c}} \bar{\partial}_{\dot{c}} + \tau_{ab}^r \partial_r, \tag{5}$$

the coefficients $\tau_{ab}^{\dot{c}}$ and τ_{ab}^r vanish. Thus one gets a constraint on the structure function for the induced CR-structure.

Next we investigate the part of the structure function related to the commutator

$$[\partial_a, \bar{\partial}_{\dot{b}}] = E_a^M \left(\frac{\partial}{\partial x^M} \bar{E}_{\dot{b}}^{\dot{M}} \right) \frac{\partial}{\partial \bar{x}^{\dot{M}}} - \bar{E}_{\dot{b}}^{\dot{M}} \left(\frac{\partial}{\partial \bar{x}^{\dot{M}}} E_a^M \right) \frac{\partial}{\partial x^M}. \tag{6}$$

---

5 The vector field $E^M(x)$ and the corresponding differential operator $E^M(x)\partial/\partial x^M$ are often identified

Writing $[\partial_a, \bar{\partial}_{\dot{b}}]$ in the form

$$[\partial_a, \bar{\partial}_{\dot{b}}] = \tau^c_{a\dot{b}} E^M_c \frac{\partial}{\partial x^M} + \tau^{\dot{c}}_{a\dot{b}} \bar{E}^{\dot{M}}_{\dot{c}} \frac{\partial}{\partial \bar{x}^{\dot{M}}} + \tau^r_{a\dot{b}}\left(E^M_r \frac{\partial}{\partial x^M} + \bar{E}^{\dot{M}}_r \frac{\partial}{\partial \bar{x}^{\dot{M}}} \right), \tag{7}$$

and assuming that the relation (4) holds, it is easy to find the correspondence between the coefficients $\tau^r_{a\dot{b}}$ and the Levi form. Actually, applying the operator $\bar{\partial}_b$ to the equality (2) one gets

$$\bar{E}^{\dot{M}}_{\dot{b}} E^M_a \frac{\partial^2 f^k}{\partial x^M \partial \bar{x}^{\dot{M}}} = - \bar{E}^{\dot{M}}_{\dot{b}} \left(\frac{\partial}{\partial \bar{x}^{\dot{M}}} E^M_a \right) \frac{\partial f}{\partial x^M}.$$

On the other hand, it follows from (6), (7) that

$$\tau^c_{a\dot{b}} E^M_c \frac{\partial}{\partial x^M} + \tau^r_{a\dot{b}} E^M_r \frac{\partial}{\partial x^M} = - \bar{E}^{\dot{M}}_{\dot{b}} \left(\frac{\partial}{\partial \bar{x}^{\dot{M}}} E^M_a \right) \frac{\partial}{\partial x^M}. \tag{8}$$

Applying the operator equality (8) to the function f^k and taking (2), (4) into account we obtain

$$i\tau^k_{a\dot{b}} = \bar{E}^{\dot{M}}_{\dot{b}} E^M_a \frac{\partial^2 f^k}{\partial x^M \partial \bar{x}^{\dot{M}}}. \tag{9}$$

The right hand side is identical to the Levi form (see Sect. 3).

The parts of the function τ which were not considered here do not contribute to the structure function of the CR-structure. In other words, we may assume that the structure function takes its values in the space $\mathscr{C} \subset \mathscr{T}$, determined by the vanishing of all the components of the function τ, except $\tau^{\dot{c}}_{ab}$, τ^r_{ab}, $\tau^r_{a\dot{b}}$, and their complex conjugates. Thus for an arbitrary structure with the considered group (i.e. the CR-structure) one may assume that the structure function takes its values in the described space $\mathscr{C}$. If the CR-structure is generated by the induction from the trivial $\mathrm{GL}(m, \mathbb{C})$-structure, then the constraints $\tau^{\dot{c}}_{ab} = 0$ and $\tau^r_{ab} = 0$ are imposed.

It is seen that in the example in view the structure function of the induced structure is not arbitrary. Now we will show the constraints specific for the structure function of the induced structure in the general case.

Let Ω be a (p, q)-dimensional surface in the space $\mathbb{R}^{p',q'}$, provided with the trivial G-structure. Suppose, as in the previous discussion, that the surface is regular, i.e. that the tangent plane in any point of the surface Ω is obtained by means of a transformation, belonging to the group G, from the (p, q)-dimensional subspace $E \subset \mathbb{R}^{p',q'}$ spanned by the first vectors of the basis in $\mathbb{R}^{p',q'}$. An admissible frame in the surface Ω may be continued up to an admissible frame in the trivial G-structure. The admissible frames in the trivial G-structure are $E^M_A = g^M_A$, where g^M_A are elements of a matrix $g \in G$. Respectively, an admissible frame for the surface Ω is $E^M_R = g^M_R$(here and in the following the indices R, S run through the first p values in the even sector, and first q values in the odd sector). Of course, it is supposed that the vectors E^M_R are tangent to the surface.

Let us consider the field $E^M_R(x)$ of the admissible frames in the surface Ω and continue it to the field $E^M_A(x)$ of the admissible frames for the trivial G-structure. (The indices A, B, C run through the whole set of p' even and q' odd values.) The structure function for the trivial G-structure vanishes, so

$$[\partial_A, \partial_B\} = \tau_{AB}^C \partial_C,$$

where τ_{AB}^C is an element of $\mathscr{A}$, i.e.

$$\tau_{AB}^C = l_{AB}^C \pm l_{BA}^C,$$

where l_{AB}^C belongs to the Lie algebra $\mathscr{G}$ of the group G at any fixed B.

On the other hand the commutator $[\partial_R, \partial_S\}$ of two vector tangent fields to the surface Ω is again a tangent vector field. Hence $\tau_{AA'}^C = 0$, if A and A' are in the first group of the indices while C is in the last group of the indices (we mean that $\tau_{AA'}^C = 0$ if $A = R$, $A' = R'$ and C is above p in the even case and above q in the odd case). Thus

$$[\partial_R, \partial_{R'}\} = \tau_{RR'}^S \partial_S,$$

and it is possible to find $l_{AR'}^C$ in such a way that for a fixed R' the matrix $l_{AR'}^C$ belongs to the Lie algebra $\mathscr{G}$,

$$\tau_{RR'}^S = l_{RR'}^S \pm l_{R'R}^S \text{ and } l_{RR'}^D \pm l_{RR'}^D = 0.$$

(Here D is in the last group of the indices.)

The obtained information on the function $\tau_{RR'}^S$ leads to constraints on the structure function of the induced G'-structure in the surface Ω.

A natural question arises: is it possible to state that the G'-structure with a structure function satisfying the above constraints is realized in a surface embedded into the space with the trivial G-structure? In general, the answer is negative. It results, in particular, from the fact that besides the constraints on the structure function considered above (the first structure function), one may obtain also similar constraints for the socalled second structure function (these constraints generalize the Gauss–Codazzi theorem of the Riemannian geometry). However, a condition may be indicated under which the constraints for the first structure function are sufficient already for the possibility to embed the G'-structure into the trivial G-structure. (Namely, one should impose the condition that a certain subspace of the total matrix space would be involutive.) The aspects in view are considered in some detail in a work by A. Rosly and the author [5]. In the case of the minimal $N = 1$ supergravity it is possible to avoid application of the general results of [5], by exploiting a special method (see Sect. 6).

6. Supergravity and G-Structures

As it was mentioned above, in the supergravity the role of fields is played by (4, 4)-dimensional surfaces Ω in the complex space $\mathbb{C}^{4,2}$, while the action functional is invariant under the group L of analytical transformations conserving the super-volume. The group L may be considered as the automorphism group of the trivial $\mathrm{SL}(4, 2, \mathbb{C})$-structure in $\mathbb{C}^{4,2}$. According to the general methods of Section 5, this $\mathrm{SL}(4, 2, \mathbb{C})$-structure induces the SCR-structure[6] in any surface Ω. By SCR we

6 Omitting the requirement of the super-volume conservation, one has a structure known in mathematics as the Cauchy–Riemann structure, denoted also the CR-structure. This fact suggests the notation for the structure in view, the SCR-structure, i.e. the special Cauchy–Riemann structure

denote here the group containing the transformations like the following one

$$\xi'^a = A^a_b \xi^b,$$
$$v'^\alpha = A^\alpha_b \xi^b + A^\alpha_\beta v^\beta, \tag{1}$$

where $A^a_b = \overline{(A^a_b)}$, $\det(A^a_b) = \det(A^\alpha_\beta)$. Here ξ^a, $a = 1, 2, 3, 4$ are real even coordinates and $v^\alpha(\bar{v}^{\dot\alpha})$, $\alpha = 1, 2$ are complex odd coordinates in the (4, 4)-dimensional real space. Writing down an arbitrary linear transformation in this superspace as follows:

$$\xi'^a = \xi^b A^a_b + v^\beta A^a_\beta + \bar{v}^{\dot\beta} \bar{A}^a_{\dot\beta},$$
$$v'^\alpha = \xi^b A^\alpha_b + v^\beta A^\alpha_\beta + \bar{v}^{\dot\beta} A^\alpha_{\dot\beta},$$

one may fix the SCR group by means of the conditions

$$A^a_\beta = 0, \quad A^\alpha_{\dot\beta} = 0, \tag{2}$$
$$\det(A^a_b) = \det(A^\alpha_\beta). \tag{3}$$

For elements of the Lie algebra corresponding to the SCR group, these conditions are substituted by

$$l^a_\beta = 0, \quad l^\alpha_{\dot\beta} = 0 \tag{4}$$
$$l^a_a = l^\alpha_\alpha. \tag{5}$$

We assume that the surface Ω is generic. Therefore, in particular, the equations[7] $dz^A \partial_A f^k = 0$, specifying the tangent complex plane, are linearly independent at any point of the surface, so the tangent complex plane $C_z(\Omega)$ is (0, 2)-dimensional.

Let us describe the SCR-structure in Ω directly. We consider as admissible frame field in Ω a field containing two odd vector fields $\partial_\alpha = E^M_\alpha \partial/\partial z^M$, $\alpha = 1, 2$, forming a complex basis in the tangent complex plane $C_z(\Omega)$, two complex conjugated fields $\bar\partial_{\dot\alpha} = \bar{E}^{\dot M}_{\dot\alpha} \partial/\partial \bar{z}^M$, four tangent real vectors $\partial_a = E^M_a \partial/\partial z^M + \bar{E}^{\dot M}_a \partial/\partial \bar{z}^{\dot M}$. A single condition to be imposed for the even vectors ∂_a is that the Berezinian of the complex matrix $E^M_A = (E^M_a, E^M_\alpha)$ be unity. (The requirement that ∂_α and ∂_a are tangent to Ω are written as $\partial_\alpha f^k = 0$, $\partial_a f^k = 0$.) It is easily verified that the described tangent frame to the surface Ω is determined up to a transformation belonging to the SCR group; thus our definition of the admissible frame does fix the SCR-structure in Ω. The definition in view is invariant, evidently, with respect to the transformation belonging to the group L. (If the surface Ω' is obtained from Ω by means of an analytical transformation, the tangent complex plane is transformed again into a tangent complex plane. Using this remark and the super-volume conservation for any element of L, one gets the proof that any admissible frame in Ω is transformed into an admissible frame in Ω'.)

An example of the construction is the induced SCR-structure in a surface in $\mathbb{C}^{4,2}$, corresponding to the plane geometry. The surface is given by the equation

$$\operatorname{Im} x^a = \theta^\alpha (\sigma^a)_{\alpha\dot\beta} \bar\theta^{\dot\beta}. \tag{6}$$

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7 As in Sect. 3, the complex coordinates in $\mathbb{C}^{4,2}$ are denoted by $z^A = (x^a, \theta^\alpha)$; the surface is given by four real equations $f^k(z, \bar{z}) = 0$

(As in Sect. 3, σ's here are the Pauli matrices.) The induced SCR-structure in the surface may be given by the frame field containing the vector fields

$$\partial_\alpha = \partial/\partial\theta^\alpha + 2i\sigma^c_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial/\partial x^c,$$
$$\partial_a = \partial/\partial x^a + \partial/\partial\bar{x}^{\dot{a}}.$$

Note that for the SCR-structure in the surface (6) the described operators ∂_α coincide with the so-called spinor covariant derivatives.

A subgroup of the supergroup L, containing the elements transforming the surface (6) into itself, is isomorphic to the super Poincaré group. Hence, in particular, the SCR-structure in (6) is invariant with respect to the super Poincaré group. (This fact is intimately related to the commutativity of the spinor derivatives and of the super-translations.)

Now we turn to a consideration of the structure function of an arbitrary SCR-structure, with no preliminary assumption that the structure is obtained by induction. The quantities

$$\tau^{\dot{\gamma}}_{\alpha\beta},\ \tau^c_{\alpha\beta},\ \tau^c_{\alpha\dot{\beta}},\ \tau^a_{a\dot{\beta}} - \tau^\alpha_{\alpha\dot{\beta}} \tag{7}$$

may be considered as components of the structure function for the SCR-structure; in other words, these quantities depend only on a value of the frame field at the point, but not on the derivatives.

As it was seen in Sect. 4, in order to verify this statement it is sufficient to see that the expressions (7) vanish in a space $\mathscr{A}$ composed of elements represented as $\tau^A_{BC} = l^A_{BC} \pm l^A_{BC}$, where for any C the element l^A_{BC} belongs to the Lie algebra of the SCR-group. For example, in this space the combination $\tau^a_{a\dot{\beta}} - \tau^\alpha_{\alpha\dot{\beta}}$ may be written as

$$\begin{aligned}\tau^a_{a\dot{\beta}} - \tau^\alpha_{\alpha\dot{\beta}} &= l^a_{a\dot{\beta}} - l^a_{\dot{\beta}a} - l^\alpha_{\alpha\dot{\beta}} - l^\alpha_{\dot{\beta}\alpha} = \\ &= l^a_{a\dot{\beta}} - l^\alpha_{\alpha\dot{\beta}} = 0.\end{aligned}$$

(The relations (4), and then (5), were exploited.)

It is remarkable that all the components of the structure function are expressed in terms of the components (7). (In other words, any linear function in the space $\mathscr{T}$, vanishing in $\mathscr{A}$, may be represented as a linear combination of the expressions (7).)

The considered SCR-structure in the surface $\Omega \subset \mathbb{C}^{4,2}$ is induced by the trivial $\mathrm{SL}(4, 2, \mathbb{C})$-structure in $\mathbb{C}^{4,2}$, so that constraints on the structure function do arise. Applying the reasoning of Sect. 5, we get

$$\tau^{\dot{\gamma}}_{\alpha\beta} = 0, \quad \tau^c_{\alpha\beta} = 0, \quad \tau^a_{a\dot{\beta}} - \tau^\alpha_{\alpha\dot{\beta}} = 0. \tag{8}$$

The first two constraints are evident: their meaning is that the (anti) commutator of two vector fields, belonging to the tangent complex plane lies in the same plane.

Thus only the components $\tau^c_{\alpha\dot{\beta}}$ of the structure function are nontrivial for the induced SCR-structure. The considerations of Sect. 5 show that this part of the structure function may be identified with the Levi form. A more accurate statement is as follows. As it was mentioned already, the SCR group acts in the coset space $\mathscr{T}/\mathscr{A}$, where the structure function takes its values. It stems from the assertion just formulated that the structure function for the induced SCR-structure in the general case takes its values in a single orbit of the group action. This fact may be verified directly, or one may use the following arguments. It is known (see [12], [9]) that a vicinity of a point in a generic surface may be reduced to the form

$$\operatorname{Im} x^a = \theta^\alpha \sigma^a_{\alpha\beta} \bar{\theta}^\beta + \ldots \tag{9}$$

(higher-order terms are omitted) by means of a transformation belonging to the group L. The omitted terms do not effect the structure function at the point $x=0$, $\theta=0$, so the structure function of the surface (9) coincides at this point with the structure function of the surface (6), corresponding to the flat geometry. Thus we see that the structure function for an arbitrary generic surface at any point is obtained from the structure of the surface (6) at the point $x=0$, $\theta=0$, applying a transformation belonging to the SCR group. This fact means that in the general case the values of the structure function all belong to a single orbit.

The SCR-structure function over the surface (6) may be easily calculated with the admissible frame field in the surface, as constructed above. The relation $\{\partial_\alpha, \bar{\partial}_{\dot\beta}\} = 2i\sigma^a_{\alpha\dot\beta}\partial_a$ shows that $\tau^a_{\alpha\dot\beta} = 2i\sigma^a_{\alpha\dot\beta}$ for the surface in view. The statement that for a generic surface the structure function takes its values in a single orbit is equivalent to the following fact. For any point z it is possible to represent $\tau^a_{\alpha\dot\beta}(z)$ as $2iA^a_b\sigma^b_{\gamma\dot\delta}C^\gamma_\alpha\bar{C}^{\dot\delta}_{\dot\beta}$, where A^a_b is a real matrix, and C^α_β is a complex matrix, and they satisfy the relation $A^\alpha_\beta C^\beta_\gamma = \delta^\alpha_\gamma$, $\det(A^a_b) = \det(A^\alpha_\beta) = \det(C^\alpha_\beta)^{-1} \neq 0$.

The expression for the supergravity action functional over a (4, 4) surface Ω in $\mathbb{C}^{4,2}$ may be written in the form

$$S = C\int |\Gamma|^{-1/3} dV, \tag{10}$$

where $\Gamma = \det(\Gamma^a_b)$, and the matrix $\Gamma^a_b = \tau^a_{\alpha\dot\beta}\sigma^{\alpha\dot\beta}_b$. The symbol dV means the volume element for the surface, corresponding to the frame field fixing the induced SCR-structure in Ω. (By definition, the tangent frame at a point $z\in\Omega$ is a basis for the tangent subspace at this point, so it specifies the volume element in the tangent subspace.)

Note first of all, that the expression (10) is independent of the choice of an admissible frame field. In fact, the transformation law for $\tau^a_{\alpha\dot\beta}$ under local transformations belonging to the SCR group is

$$\tilde{\tau}^a_{\alpha\dot\beta} = A^a_b \tau^b_{\gamma\dot\delta} C^\gamma_\delta \bar{C}^{\dot\delta}_{\dot\gamma},$$

where

$$A^\alpha_\beta C^\beta_\gamma = \delta^\alpha_\gamma, \ \det(A^a_b) = \det(A^\alpha_\beta) = \det(C^\alpha_\beta)^{-1} \neq 0.$$

Hence $|\tilde{\Gamma}| = |\Gamma|\,|\det(A^a_b)|^{-3}$. Having in mind the transformation law for the volume element, when a new admissible frame is introduced,

$$d\tilde{V} = |\det(A^a_b)|\cdot|\det(A^\alpha_\beta)|^{-2} dV = |\det(A^a_b)|^{-1}\, dV,$$

we see that the integrand in (10) does not depend on the choice of the admissible frame.

In order to prove the expression (10) one may use the relation between $\tau^c_{\alpha\dot\beta}$ and the Levi form, given in Sect. 5. This relation enables one to identify (10) with the expression (3.8) of Sect. 3 for the supergravity action functional. It is simpler, however, to ascertain that the expression (10) is invariant under the group L, and to observe that the structure function is expressed in terms of the first and second derivatives of the functions present in the surface equations. As in Sect. 3, this enables one to use the uniqueness of the action satisfying the indicated requirements, as it was proven in [3]. (The invariance of the expression (1) with respect to the group L is a consequence of the mentioned L-invariance of the induced SCR-structure.)

It is remarkable that the expression (10) is determined completely in terms of the internal geometry of the surface Ω (to be more precise, by the SCR-structure in Ω). This remark leads us to a new formulation of the minimal supergravity. In this formulation the major object is the frame field in the (4,4)-dimensional region, determined up to transformations belonging to the SCR group (i.e. the SCR-structure in the region). It is assumed that the SCR-structure function satisfies the conditions (8). This approach leads directly to the action functional given by (10).

Let us consider a relation between the present formulation and that by Ogievetsky and Sokatchev. Let a (4, 4)-dimensional surface Ω in $\mathbb{C}^{4,2}$ be given by a parametric equation (i.e. by means of a mapping of some (4, 4)-dimensional region into $\mathbb{C}^{4,2}$). Then the SCR-structure is defined naturally in this region. Thus a field in the Ogievetsky–Sokatchev formalism corresponds to a field in the present approach; evidently the action functionals in both cases do coincide.

To complete the proof of the equivalence between the Ogievetsky-Sokatchev formalism and the present formalism, it is sufficient to show that any SCR-structure in a (4, 4)-dimensional domain satisfying the constraints (8) on the structure function may be obtained by means of the induction from the trivial SL (4, 2, $\mathbb{C}$)-structure in $\mathbb{C}^{4,2}$. We shall show that this statement is true indeed (at least, locally). It is appropriate to base the proof upon the following remark. Let $V(z)$ be an analytical function defined in $\mathbb{C}^{4,2}$, or in a domain of $\mathbb{C}^{4,2}$. We establish a SL(4, 2, $\mathbb{C}$)-structure in the space $\mathbb{C}^{4,2}$, assuming that the frame $E_A^M \partial/\partial z^M$ is admissible, if the vectors E_A^M form a complex basis in the space $\mathbb{C}^{4,2}$ and satisfy the condition $\mathrm{Ber}(E_A^M) = V(z)$. (As always, we assume that the conjugated vectors $\bar{E}_A^M$ belong to the basis together with E_A^M.) The trivial SL (4, 2, $\mathbb{C}$)-structure considered before corresponds to the case $V(z) \equiv 1$. Note that, at least locally, the new SL (4, 2, $\mathbb{C}$) structure is equivalent to the trivial one. (This stems from the fact that in a vicinity of any point z_0 such an analytical transformation $\Phi(z)$ can be found that $V(z) = \mathrm{Ber}(\partial\Phi^A/\partial z^B)$.) Thus if the geometry in the (4, 4)-dimensional surface is induced by a new SL (4, 2, $\mathbb{C}$) structure, then (at least, locally) the isomorphic geometry is induced in another surface by the trivial SL (4, 2, $\mathbb{C}$)-structure[8].

Let us consider the SCR-structure in a (4, 4)-dimensional domain, satisfying the constraints (8) for the structure function. We shall assume that the admissible frames for this structure contain complex odd vector fields $\nabla_\alpha = e_\alpha^N \partial/\partial\zeta^N$, their conjugated

8 It is noteworthy that the presented consideration enables one to modify somewhat the Orievetsky–Sokatchev formulation. Actually, one may assume that the basic objects are the (4, 4)-dimensional surface in $\mathbb{C}^{4,2}$ and an analytical function $V(z)$ in $\mathbb{C}^{4,2}$. Then one can suppose that the action is invariant under arbitrary (not only the super-volume conserving) analytical transformations; and at an analytical transformation $\Phi(z)$ the function $V(z)$ is multiplied by Ber $(\partial\Phi^A/\partial z^B)$. The reduction of the modified formulation to that by Ogievetsky and Sokatchev is performed by imposing the gauge condition $V(z) \equiv 1$. The modified formulation was used in a number of papers [6], [11]; the reduction of the supergravity field to the normal form was investigated within this framework in [11]

partners $\bar{\nabla}_{\dot{\alpha}}$, and real even vector fields $\nabla_a = e_a^N \partial/\partial\zeta^N$ (the symbol ζ^N means real coordinates in the domain). A particular consequence of (8) is

$$\{\nabla_\alpha, \nabla_\beta\} = \tau^\gamma_{\alpha\beta}\nabla_\gamma, \tag{11}$$

$$\{\bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}}\} = \tau^{\dot{\gamma}}_{\dot{\alpha}\dot{\beta}}\bar{\nabla}_{\dot{\gamma}}. \tag{12}$$

Consider now the equation

$$\bar{\nabla}_{\dot{\alpha}}\varphi = 0, \tag{13}$$

where φ is a complex function. The relations (12) show that the Frobenius theorem is applicable to Eq. (13) (to be more accurate, a super-extension of the theorem [13]). In view of this theorem, (13) has, locally, four independent even and two independent odd solutions. Let us denote the solutions $x^a(\zeta)$ and $\theta^\alpha(\zeta)$, where ζ is a point in the domain. The functions x^a, θ^α determine an embedding $z^A(\zeta) = (x^a(\zeta), \theta^\alpha(\zeta))$ of the domain in view into $\mathbb{C}^{4,2}$; this embedding is fixed up to an analytical transformation. (If $\lambda^A(z)$ is an analytical transformation of the space $\mathbb{C}^{4,2}$, then if one has a solution $z^A = (x^a, \theta^\alpha)$ of Eq. (13), one can get another solution by setting $z'^A(\zeta) = \lambda^A(z(\zeta))$.)

Use the notations $\partial_\alpha, \bar{\partial}_{\dot{\alpha}}, \partial_a$ for the vector fields obtained at the mapping $\zeta \to z(\zeta)$ from the vector fields $\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}, \nabla_a$. It follows from $\bar{\nabla}_{\dot{\alpha}}\, z(\zeta) = 0$, and from the relations

$$\bar{\partial}_{\dot{\alpha}} F(z, \bar{z}) = \bar{\nabla}_{\dot{\alpha}} F(z(\zeta), \bar{z}(\zeta))$$

that $\bar{\partial}_{\dot{\alpha}} = E_{\dot{\alpha}}^{\dot{M}} \partial/\partial\bar{z}^{\dot{M}}$. The real vector fields ∂_a are written as $\partial_a = E_a^M \partial/\partial z^M + \bar{E}_a^{\dot{M}} \partial/\partial\bar{z}^{\dot{M}}$. The vector fields ∂_α, $\bar{\partial}_{\dot{\alpha}}$, ∂_a form the tangent frame for a surface Ω, obtained by embedding of the domain considered into $\mathbb{C}^{4,2}$. They determined the SCR-structure in the surface Ω. The fields $\partial_\alpha = E_\alpha^M \partial/\partial z^M$ belong, evidently, to the tangent complex subspace to the surface Ω. Hence, the frame $\partial_\alpha, \bar{\partial}_{\dot{\alpha}}, \partial_a$ is admissible for the CR-structure induced in Ω by the trivial GL$(4, 2, \mathbb{C})$-structure in $\mathbb{C}^{4,2}$. Let us define a function v in the surface Ω:

$$v(z) = \operatorname{Ber}(E_A^M),$$

where $E_A^M = (E_a^M, E_\alpha^M)$. Note that the function $v(z)$ is independent of the choice of the frame fields fixing the SCR-structure. It may be verified that the function $v(z)$ satisfies the Cauchy–Riemann equations; to be more precise, at any point of the surface the differential of the function $v(z)$ is an analytical linear function in the tangent complex subspace. This statement can be written as the relation $\bar{\partial}_{\dot{\alpha}} v = 0$. Verifying the Cauchy–Riemann condition for the function $v(z)$ we shall assume that at the point of interest the tangent subspace is given by the equation $\operatorname{Im} x^a = 0$, while the frame field in the vicinity of the point is

$$\partial_a = \partial/\partial x^a + \partial/\partial\bar{x}^{\dot{a}} + \varepsilon_a^M \partial/\partial z^M + \bar{\varepsilon}_a^{\dot{M}} \partial/\partial\bar{z}^{\dot{M}},$$

$$\partial_\alpha = \partial/\partial\theta^\alpha + \varepsilon_\alpha^M \partial/\partial z^M,$$

where $\varepsilon_a^M, \varepsilon_\alpha^M$ vanish at the point in view. (With no loss of generality, we can take this assumption, since the function $v(z)$ is independent of the choice of the frame, and the condition $\bar{\partial}_{\dot{\alpha}} v = 0$ is invariant with respect of transformations belonging to the group L.) At the imposed restrictions, it is easily verified in an infinitely small vicinity of the point in view one has $v(z) = 1 + \varepsilon_a^a - \varepsilon_\alpha^\alpha + \ldots$, so at the point itself

$$\bar{\partial}_{\dot{\beta}} v = \bar{\partial}_{\dot{\beta}} \varepsilon_a^a - \bar{\partial}_{\dot{\beta}} \varepsilon_\alpha^\alpha = -\tau_{a\dot{\beta}}^a + \tau_{\alpha\dot{\beta}}^\alpha = 0.$$

In view of the Cauchy–Riemann conditions, the function $v(z)$ may be continued up to a function $V(z)$ which is analytical in a neighbourhood of the surface Ω. (The relevant theorem is well known in the theory of functions of several complex variables.) It is easily seen that the SCR-structure in the surface Ω, given by the frame $\partial_\alpha, \bar{\partial}_\alpha, \partial_a$ is induced by the SL(4, 2, $\mathbb{C}$)-structure in a neighbourhood of the surface Ω, constructed by means of the function $V(z)$.

This completes the proof of the statement that the SCR-structure with the described constraints on the structure function is (locally) isomorphic to the SCR-structure induced in a (4, 4)-dimensional surface by the trivial SL(4, 2, $\mathbb{C}$)-structure in $\mathbb{C}^{4,2}$. Simultaneously, the proof is completed of the fact that the Ogievetsky–Sokatchev formalism is equivalent to the approach to the supergravity based upon the SCR-structure.

7. Equivalence of Different Approaches

Note first of all that the formulation of the supergravity presented in the preceding section might be given without the concept of the G-structure. Actually, the basic object to be considered must be the frame field in the (4, 4)-dimensional region, consisting of complex odd vector fields $\partial_\alpha, \bar{\partial}_\alpha$ and real even vector fields ∂_a. The condition (6.8) is imposed for the frame field. The action functional is given by (6.10). The condition (6.8) and the action (6.10) are invariant with respect to local transformations belonging to the group SCR. In other words, they are invariant under substitutions of the frame field given by the formulae

$$\begin{aligned} &\partial'_a = A_a^b \partial_b + A_a^\beta \partial_\beta + \bar{A}_a^{\dot{\beta}} \bar{\partial}_{\dot{\beta}}, \\ &\partial'_\alpha = A_\alpha^\beta \partial_\beta, \\ &\det(A_a^b) = \det(A_\alpha^\beta). \end{aligned} \tag{1}$$

(It is because of this fact that one can investigate the SCR-structure instead of the frame field.)

New formulations of the supergravity can be obtained by introduction of gauge conditions. For instance, let the gauge condition for the frame field in view be

$$\tau_{\alpha\dot{\beta}}^c = 2i\sigma_{\alpha\dot{\beta}}^c. \tag{2}$$

The arguments presented in Sec. 6 result in the statement that in the generic case the gauge condition (2) may be satisfied by application of the local SCR transformations (1). With the gauge condition (2) the action is extremely simple,

$$S = C \int dV, \tag{3}$$

where dV, as before, is the volume element corresponding to the considered frame field. The symmetry group is contracted after the gauge condition (2) is imposed. In fact, the new symmetry group contains those transformations of the form (1) which do not break the gauge condition (2). It is easily seen that the transformation (1) satisfies this requirement provided that $\det(A_\beta^\alpha) = 1$ (i.e. the matrix A_β^α is an element of the group SL (2, $\mathbb{C}$)), while the matrix A^a_b corresponds to the Lorentz transformation related to the matrix A^α_β. In other words, the considered theory is invariant under local TL-transformations, where TL is the transformation group for the real vector ξ^a and the complex two-component spinor v^α, generated by the Lorentz transformations and by the transformations of the form $\xi^a \to \xi^a + A^a_\alpha v^\alpha + \bar{A}^a_\alpha \bar{v}^\alpha$, $v^\alpha \to v^\alpha$. Clearly, the basic object in this formulation is, in fact, the frame field determined up to a local TL-transformation. Thus the formulation is based on the TL-structure concept.

Note that the described transition from the SCR-structure to the TL-structure is just the reduction procedure familiar in the theory of G-structures [4]. (If the G-structure function takes its values in an orbit generated by the action of the group G in the space $\mathscr{T}/\mathscr{A}$, then the G-structure may be reduced to the H-structure, where H is an isotropy subgroup of G. For the case in question, we have $G = \mathrm{SCR}$, $H = \mathrm{TL}$.)

The formulations of the supergravity given here may be somewhat modified. Namely, it is reasonable to consider as the basic concepts not the SCR-structure or the TL-structure, but these structures supplied with connections. Constraints should be introduced for the torsion of the connections in the case of the SCR-structures

$$T^{\dot\gamma}_{\alpha\beta} = 0,\ T^{c}_{\alpha\beta} = 0,\ T^{a}_{\alpha\dot\beta} - T^{\alpha}_{\alpha\dot\beta} = 0. \tag{4}$$

The constrains in the case of the TL-structures are

$$T^{\dot\gamma}_{\alpha\beta} = 0,\ T^{c}_{\alpha\beta} = 0,\ T^{a}_{a\dot\beta} - T^{\alpha}_{\alpha\dot\beta} = 0, \tag{5}$$
$$T^{c}_{\alpha\dot\beta} = 2i\sigma^{c}_{\alpha\dot\beta}.$$

As before, the action is given by (6.10) in the case of the SCR-structures, and by (3) in the case of the TL-structures. Thus in both cases the action is independent of the choice of the connection. The equivalence between the modified formulations and the original formulations is a consequence of the interpretation, given in Sect. 4, according to which the structure function is a part of the torsion independent of the choice of the connection. Actually, this interpretation suggests that the torsion for any connection in the SCR-structure, satisfying the conditions (6.8) for the structure function, is subject to the constraints (4). On the other hand, if in the SCR-structure there is a connection subject to the constraints (4), the conditions (6·8) holds for the SCR-structure function. The same statement is true also for the TL-structures.

Now we are in the position to prove the equivalence between the formalisms described above for the supergravity (in particular, the Ogievetsky–Sokatchev formalism) and the Wess–Zumino formalism. To this end, we shall start from the formulation based on the TL-structures and connections in these structures. We put an additional gauge condition

$$R^{D}_{\alpha\dot\beta C} = 0. \tag{6}$$

(Recall that having the connection $(\omega^A_B)_M$ one can determine the covariant derivative $\mathscr{D}_A$ acting on any vector X as follows

$$\mathscr{D}_A X^C = \partial_A X^C + X^B \omega^C_{BA}.$$

Here, as always, $\partial_A = E^M_A\, \partial_M$, $\omega^C_{BA} = E^M_A (\omega^C_B)_M$. The relation

$$[\mathscr{D}_A, \mathscr{D}_B]\, X^D = T^C_{AB} \mathscr{D}_C X^D + X^C R^D_{ABC}$$

determines the torsion tensor T^C_{AB} and the curvature tensor R^D_{ABC}. Note that using (5), the relation (6) may be rewritten as

$$\{\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\beta}\} = 2i\sigma^C_{\alpha\dot\beta}\mathscr{D}_C. \tag{7}$$

In order to prove that the condition (7) is admissible, we note a relation which holds for arbitrary connection in the TL-structure satisfying the conditions (5). The relation is

$$\begin{aligned}\{\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\beta}\} = 2i\sigma^c_{\alpha\dot\beta}\mathscr{D}_c + \tau^\gamma_{\alpha\dot\beta}\mathscr{D}_\gamma \\ + \tau^{\dot\gamma}_{\alpha\dot\beta}\bar{\mathscr{D}}_{\dot\gamma} + \hat{R}_{\alpha\dot\beta}.\end{aligned} \tag{8}$$

Instead of the TL-structure in view, we shall consider a new TL-structure and an associated connection, assuming that the covariant derivatives $\mathscr{D}_\alpha$ are not changed, and the covariant derivatives $\mathscr{D}_a$ are replaced by

$$\tilde{\mathscr{D}}_a = -\frac{i}{4}\sigma^{\alpha\dot\beta}_a\{\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\beta}\}. \tag{9}$$

(The definition of the vector covariant derivative in terms of the spinor derivatives was used in [2].) It is noteworthy that having determined the operators $\tilde{\mathscr{D}}_a$, one also gets a definition of the new vector fields $\tilde{\partial}_a$. It follows from (8) that

$$\tilde{\partial}_a = \partial_a - \frac{i}{4}\sigma^{\alpha\dot\beta}_a(\tau^\gamma_{\alpha\dot\beta}\partial_\gamma + \tau^{\dot\gamma}_{\alpha\dot\beta}\bar{\partial}_{\dot\gamma}).$$

The meaning of this relation is that the frame field $\partial_\alpha, \bar{\partial}_{\dot\alpha}, \tilde{\partial}_a$ is obtained from the field $\partial_\alpha, \bar{\partial}_{\dot\alpha}, \partial_a$ by means of a TL-transformation. The resulting conclusion is that the replacement of the covariant derivatives $\mathscr{D}_\alpha$, $\bar{\mathscr{D}}_{\dot\alpha}, \mathscr{D}_a$ by the covariant derivatives $\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\alpha}, \tilde{\mathscr{D}}_a$ enables one to satisfy the gauge conditions (6) while the action functional remains intact.

If the transformation (1) conserves not only the gauge condition (5), but also the gauge condition (6), it is the local Lorentz transformation. Therefore after the condition (6) is introduced we have a G-structure corresponding to the Lorentz group. (To be more precise, the group SL$(2, \mathbb{C})$ is considered as the transformation group for the superspace $\mathbb{R}^{4,4}$, and the coordinates in this space are considered as a real four-dimensional vector and a complex two-component spinor.) However, in general, the connections considered take their values not in the Lie algebra of the Lorentz group, but in the Lie algebra of the group TL. To establish the equivalence of the present approach to the Wess–Zumino formalism, one should introduce an additional gauge condition for the connections $\omega^b_{\beta A} = 0$. It is remarkable that in view of the constraint (9) it is sufficient to require that the connection coefficients in the covariant derivatives $\mathscr{D}_\alpha$ belong to the Lie algebra of the Lorentz group. (In other words, the condition $\omega^b_{\beta\alpha} = 0$ is sufficient.)

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