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The Probability of Intersection of Independent Random Walks in Four Dimensions*

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Abstract. Let S_1 and S_2 be independent simple random walks of length n in Z^4 starting at 0 and x_0 respectively. If $|x_0|^2 \approx n$, it is shown that the probability that the paths intersect is of order $(\log n)^{-1}$. If $x_0 = 0$, it is shown that the probability of no intersection of the paths decays no faster than $(\log n)^{-1}$ and no slower than $(\log n)^{-1/2}$. It is conjectured that $(\log n)^{-1/2}$ is the actual decay rate.

1. Introduction

Let $S_1(n,\omega)$ and $S_2(n,\omega)$ be independent simple random walks in Z^4 starting at 0 and x_0 respectively; that is, S_1 and S_2 are independent processes indexed by the nonnegative integers satisfying:

(i) $S_1(0,\omega) = 0$ a.s. (almost surely),

(ii) $S_2(0,\omega) = x_0$ a.s.,

(iii) for each $x \in Z^4$, $e \in Z^4$, $|e| = 1$,

$$P\{S_i(n+1,\omega) - S_i(n,\omega) = e \mid S_i(n,\omega) = x\} = 1/8 .$$

Let $\Pi_i(m,n,\omega)$ denote the random set

$$\Pi_i(m,n,\omega) = \{S_i(k,\omega) : m < k \leqq n\} .$$

In understanding the interaction of random particles, it is useful to understand the behavior of $\Pi_1(0,n) \cap \Pi_2(0,m)$. In [4], it was shown that with probability one,

$$\Pi_1(0,\infty) \cap \Pi_2(0,\infty) \neq \emptyset$$

(this is not true for simple random walk in Z^d, $d \geqq 5$). It is well known [1], however that if $W_1(s)$ and $W_2(t)$ are independent Wiener processes taking values in R^d and $\Gamma_i(0,s) = \{W_i(r) : 0 < r \leqq s\}$, then almost surely,

$$\Gamma_1(0,\infty) \cap \Gamma_2(0,\infty) = \emptyset ,$$

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if and only if $d \geqq 4$. In this way, $d=4$ is a critical dimension for intersections of random paths.

Let

$$p(n)=P\{\Pi_1(0,n)\cap\Pi_2(0,n)=\emptyset\}.$$

Although the result of [4] states that

$$\lim_{n\to\infty} p(n)=0, \tag{1.1}$$

no estimate of the rate of decay is given. Erdös and Taylor [2, 3] studied $p(n)$ and asserted theorems of the following type:

(i) if $|x_0|^2 \approx n$, then

$$p(n)\approx 1-(\log n)^{-1};$$

(ii) if $x_0=0$, then

$$p(n)\approx(\log n)^{-1}.$$

Note that assertion (i) agrees with our knowledge of Wiener paths; as a lattice spacing gets finer, the likelihood of intersections in a discrete approximation should go to 0. Note also that one could conclude (1.1) from (ii). The proofs of these assertions were incorrect, however. The basic idea of their arguments was to use a renewal argument based on the "first" intersection of two random paths. Since there are two time scales involved, however, this "first" intersection cannot be defined so that it has the properties of a usual stopping time.

In this paper, assertion (i) is proved and an analogue to assertion (ii) is given. We also give an argument, based on a plausible but unproven conjecture, that assertion (ii) is actually false.

In Sect. 2, we give an inequality in one direction for assertion (i). Here we make use of some of the ideas in [2, 3]. A key step is considering the random variable

$$I_n(\omega)=(\log n)^{-1}\sum_{i=0}^{n} G(S_1(i,\omega)),$$

where G is the standard Green's function for simple random walk, and showing that I_n approaches a constant random variable. This then allows a modified form of stopping time argument to work.

Let S_3 be another simple random walk independent of S_1 and S_2, and let Π_3 be defined accordingly. Assume all three walks start at the origin. Let

$$f(n)=P\{\Pi_1(0,n)\cap(\Pi_2(0,n)\cup\Pi_3(0,n))=\emptyset\}.$$

In Sect. 3 we show that $f(n)\approx(\log n)^{-1}$. From this we conclude that $p(n)$ (for two walks starting at the origin) decays no faster than $(\log n)^{-1}$ and no slower than $(\log n)^{-1/2}$.

A more precise result for assertion (i) is given in Sect. 4. From this we give an argument in Sect. 5 as to why we believe $p(n)$ decays slower than $(\log n)^{-r}$ for any $r>\frac{1}{2}$.

2. Estimate for Walks Starting at Different Points

In this section we prove the following two theorems:

Theorem 2.1. *Let S_1 and S_2 be independent simple random walks starting at the origin in Z^4. Then there exists a constant $c_1>0$ such that for every $n>0$ and $0<\alpha<\beta$,*

(a) $P\{\Pi_1(\alpha n, \beta n)\cap \Pi_2(0,\infty)\neq \emptyset\} \leqq c_1 \log(\beta/\alpha)(\log n)^{-1}$

(b) $P\{\Pi_1(0,n)\cap \Pi_2(\alpha n,\infty)\neq \emptyset\} \leqq c_1 \log(1+1/\alpha)(\log n)^{-1}$.

Theorem 2.2. *Let S_1 and S_2 be independent simple random walks starting at 0 and x_0 respectively in Z^4. Then there exists a constant $c_2>0$ such that for every $n>0$, if $\alpha=|x_0|^2/n$,*

$$P\{\Pi_1(0,n)\cap \Pi_2(0,\infty)\neq \emptyset\} \leqq c_2 \log(1+1/\alpha)(\log n)^{-1}.$$

Lemma 7 of [3] gives an estimate in the other direction for Theorem 2.2; more specifically, for fixed α, there is a constant $\tilde{c}>0$ such that

$$P\{\Pi_1(0,n)\cap \Pi_2(0,\infty)\neq \emptyset\} \geqq \tilde{c}(\log n)^{-1}.$$

Similar arguments can be used to give estimates in this direction for the probabilities in Theorem 2.1.

The main idea of the proofs will be to use a modified stopping time argument. However, we will be unable to choose an intersection of the two paths such that the walks after this intersection act like two independent simple random walks. We will show, instead, that the amount of conditioning imposed by our choice is not too large.

We omit the proof of the next lemma which was alluded to in the proof of Theorem 10 in [2]. The proof requires a combination of a tedious calculation and an approximation of simple random walk by a normal random variable. For the remainder of this paper, we will use $c_3, c_4, \ldots$ to denote arbitrary positive constants.

Lemma 2.3. *Let $Z_n(\omega)=(\log n)^{-1}\sum_{j=1}^{n}(1+|S(j,\omega)|^2)^{-1}$, where S is a simple random walk in Z^4. Then*

(a) $\lim_{n\to\infty} E(Z_n)=c_3$ *exists*;

(b) $\operatorname{Var} Z_n = O[(\log n)^{-1}]$.

Now let S_1 and S_2 be independent simple random walks starting at the origin in Z^4, and for $x\in Z^4$ let $G(x)$ be the Green's function

$$G(x)=\sum_{j=0}^{\infty} P\{S(j)=x\},$$

and

$$I_n(\omega)=\sum_{j=1}^{n} G(S_1(j,\omega)).$$

Lemma 2.4.

$$\operatorname{Var}((\log n)^{-1} I_n)=O[(\log n)^{-1}].$$

Proof. It is known [5] that there exists a constant $c_4>0$ such that for $x\in Z^4$,

$$G(x)=c_4|x|^{-2}(1+O(|x|^{-2})).$$

The result then follows from Lemma 2.3 (b).

Proof of Theorem 2.1. We will prove (b); (a) is similar.

Let $J_n(\omega)$ be the number of intersections of the paths, i.e.,

$$J_n(\omega)=\#\{(j,k):S_1(j,\omega)=S_2(k,\omega),\ 0\leqq j\leqq n,\ \alpha n\leqq k\},$$

where $\#(\cdot)$ denotes cardinality, and let $E_n=E(J_n)$. If $p_k=P\{S_1(k)=0\}$,

$$\begin{aligned} E_n &= \sum_{j=0}^{n}\sum_{k>\alpha n} P\{S_1(j)=S_2(k)\} \\ &= \sum_{j=0}^{n}\sum_{k=0}^{\infty} P\{S_1([j+k+\alpha n])=0\} \\ &= \sum_{j=0}^{n}\sum_{k=0}^{\infty} p_{[j+k+\alpha n]} \\ &\sim \sum_{j=0}^{n}\sum_{k=0}^{\infty} c_5(j+k+\alpha n)^{-2} \\ &\sim c_5\log(1+1/\alpha). \end{aligned} \tag{2.1}$$

Here we have used the well-known asymptotic estimate $p_k\sim 2c_5k^{-2}$ for k even. A similar calculation will show that

$$\lim_{n\to\infty} E((\log n)^{-1}I_n)=c_5.$$

Consider $p(n/2)=P\{\Pi_1(0,n/2)\cap\Pi_2(\alpha n,\infty)\neq\emptyset\}$. For each j, let

$$I_n^j(\omega)=\sum_{l=1}^{n/2} G(S_1(j+l,\omega)-S_1(j,\omega)).$$

I_n^j measures how many intersections one should expect of a random walk starting at $S_1(j)$ with the first $n/2$ points of S_1 after $S_1(j)$. By Lemma 2.4 and Chebyshev's inequality, as n goes to ∞, $E(\log n/2)^{-1}I_n^j$ approaches c_5 and

$$P\{|(\log n/2)^{-1}I_n^j-c_5|>\tfrac{1}{2}c_5\}\leqq c_6(\log n/2)^{-1}.$$

Now fix n. Assume we have an intersection for an ω. For such an ω, consider the first intersection on the path S_2; more precisely, let

$$\tau_n(\omega)=\inf\{k:k\geqq\alpha n,\ S_1(j,\omega)=S_2(k,\omega) \text{ for some j with } 0\leqq j\leqq n/2\},$$

$$\sigma_n(\omega)=\inf\{j:S_1(j,\omega)=S_2(\tau_n(\omega),\omega)\}.$$

Note that $\{S_2(k)-S_2(\tau_n):k\geqq\tau_n\}$ is independent of $\{S_2(k):k\leqq\tau_n\}\cup\{S_1(j):j\leqq n\}$, although $\{S_1(j)-S_1(\sigma_n):j\geqq\sigma_n\}$ is not.

Call ω *good* if

$$(\log n/2)^{-1}I_n^{\sigma_n(\omega)}(\omega)\geqq\tfrac{1}{2}c_5,$$

and *bad* otherwise. On $\{\omega:\omega \text{ good}\}$ we have

$$P\{\omega \text{ good}\}\mathscr{E}(J_n|\omega \text{ good}) \leqq EJ_n = E_n,$$

where $\mathscr{E}$ denotes conditional expectation. But since S_2 proceeds independently after τ_n, and S_1 has at least $n/2$ steps after σ_n,

$$\mathscr{E}(J_n|\omega \text{ good}) \geqq \tfrac{1}{2}c_5 \log(n/2).$$

Therefore by (2.1),

$$P\{\omega \text{ good}\} \leqq 2\log(1+1/\alpha)(\log n/2)^{-1}.$$

Also,

$$\begin{aligned} P\{\omega \text{ bad}\} &\leqq \sum_{j=1}^{n} \sum_{k>\alpha n} P\{S_1(j)=S_2(k)\} P\{I_n^j < \tfrac{1}{2}c_5\} \\ &\leqq \sum_{j=1}^{n} \sum_{k>\alpha n} p_{j+k}(c_6(\log n/2)^{-1}) \\ &\leqq c_7 \log(1+1/\alpha)(\log n/2)^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} p(n/2) &= P\{\omega \text{ good}\} + P\{\omega \text{ bad}\} \\ &\leqq c_1 \log(1+1/\alpha)(\log n/2)^{-1}, \end{aligned}$$

where $c_1 = 2 + c_7$.

Proof of Theorem 2.2. The proof proceeds as in Theorem 2.1 (b) once we have an estimate for $E_n = E(J_n)$, where

$$J_n(\omega) = \#\{(j,k): 0 \leqq j \leqq n,\ 0 \leqq k \leqq \infty,\ S_1(j,\omega) = S_2(k,\omega)\}.$$

If we let $P(x,l) = P\{S_1(l) = x\}$ for $x \in Z^4$, note that

$$P\{S_1(j) = S_2(k)\} = P\{S_1(j+k) = x_0\} = P(x_0, j+k).$$

From (2.3) and (2.5) of [3] we get that there is a positive constant c_8 such that

$$P(x_0, j+k) \leqq \begin{cases} c_8(j+k)^{-2} (j+k) \geqq |x_0|^2 \\ c_8(j+k)^{-2} \exp\{-|x_0|^2/2(j+k)\} (j+k) < |x_0|^2. \end{cases}$$

Therefore,

$$\begin{aligned} E_n &= \sum_{j=0}^{n} \sum_{k=0}^{\infty} P\{S_1(j) = S_2(k)\} \\ &\leqq \sum_{j=0}^{\alpha n} \sum_{k=0}^{\alpha n - j} c_8(j+k)^{-2} \exp\{-|x_0|^2/2(j+k)\} + \sum_{j=0}^{\infty} \sum_{|k+j| \geqq \alpha n} c_8(j+k)^{-2} \\ &\leqq c_9 + c_{10}(1 + \log(1+1/\alpha)) \\ &\leqq c_{11} \log(1+1/\alpha). \end{aligned}$$

3. Estimate for Walks Starting at the Same Point

Let Λ_n be the set of (one-sided) simple random walk paths of length n, i.e. the set of all $\xi=[\xi(0), \ldots, \xi(n)]$ with $\xi(0)=0$ and $|\xi(i)-\xi(i-1)|=1$. Let Λ_n^A be the set of all $\xi\in\Lambda_n$ with $\xi(i)\neq 0$ for $i>0$.

Let Φ_n be the set of two-sided simple random walks of length $2n$, i.e. the set of all $\xi=[\xi(-n), \ldots, \xi(0), \ldots, \xi(n)]$ with $\xi(0)=0$ and $|\xi(i)-\xi(i-1)|=1$. Again. let Φ_n^A be the set of all $\xi\in\Phi_n$ with $\xi(i)\neq 0$ for $i>0$ (note there is no restriction for $i<0$). If $\xi\in\Phi_n$, we let ξ^+ (respectively ξ^-) be the element of Λ_n given by $[\xi(0), \ldots, \xi(n)]$ $([\xi(0), \ldots, \xi(-n)])$.

Let P denote the standard probability measure on Λ_n (Φ_n), i.e. $P(\xi)=8^{-n}(P(\xi)=8^{-2n})$. Let g_n be the probability that a simple random walk in four dimensions does not return to the origin in n steps, and $g=\lim_{n\to\infty} g_n$. Note that $g_n\geqq g>0$.

Let $S_1(n,\omega)$, $n\in Z$, be a *two-sided* simple random walk in Z^4, i.e. a process indexed by all the integers which gives the measure P on Φ_n. Let $S_2(n,\omega)$, $n\in Z_+$, denote a (one-sided) simple random walk independent of S_1. We will assume that $S_1(0)=0$ with probability one. Our first goal of this section is to give bounds for

$$f(n)=P\{S_1(i,\omega)\neq S_2(j,\omega): -n\leqq i\leqq n,\ 0<j\leqq n\}.$$

For S_1 and S_2 we define the last intersection of the two paths up through time n, where "last" is taken on the S_2 time scale. More precisely, we define $(\sigma_n(\omega), \tau_n(\omega))$ by

$$\tau_n(\omega)=\sup\{k: k\leqq n,\ S_2(k,\omega)\in\{S_1(i,\omega): -n\leqq i\leqq n\}\},$$

$$\sigma_n(\omega)=\sup\{j: j\leqq n,\ S_1(j,\omega)=S_2(\tau_n(\omega),\omega)\}.$$

Then,

(a) $S_1(\sigma_n(\omega),\omega)=S_2(\tau_n(\omega),\omega)$,

(b) $\{S_2(j,\omega):\tau_n(\omega)<j\leqq n\}\cap\{S_1(i,\omega): -n\leqq i\leqq n\}=\emptyset$,

(c) $S_1(i,\omega)\neq S_1(\sigma_n(\omega),\omega)$ for $\sigma_n(\omega)<i\leqq n$.

By Theorem 2.1, we have

$$\begin{aligned} P\{|\sigma_n|\geqq n(\log n)^{-1} &\quad\text{or}\quad \tau_n\geqq n(\log n)^{-1}\}\\ &\leqq P\{\{S_1(i): n(\log n)^{-1}\leqq i\leqq n\}\cap\{S_2(j): 0\leqq j\leqq n\}\neq\emptyset\}\\ &\quad+P\{\{S_1(i): -n\leqq i\leqq n\}\cap\{S_2(j): n(\log n)^{-1}\leqq j\leqq n\}\neq\emptyset\}\\ &\leqq c_{12}(\log\log n)(\log n)^{-1}. \end{aligned}\tag{3.1}$$

Define new random walks T_1^n and T_2^n by

$$T_1^n(i,\omega)=S_1(\sigma_n(\omega)+i,\omega)-S_1(\sigma_n(\omega)),\qquad -\infty<i<\infty,$$

$$T_2^n(j,\omega)=S_2(\tau_n(\omega)+j,\omega)-S_2(\tau_n(\omega)),\qquad 0\leqq j<\infty.$$

Note that $T_1^n(i,\omega) \neq 0$ for $0 \leqq i \leqq n - \sigma_n(\omega)$. Using (3.1) and a simple estimate on the return of random walk, one gets (with perhaps a slight change of the constant),

$$P\{[T_1^n(-n), \ldots, T_1^n(n)] \in \Phi_n^A\} \geqq 1 - c_{12}(\log\log n)(\log n)^{-1}.$$

Also, by definition,

$$\{T_2^n(j): 0 < j \leqq n - \tau_n\} \cap \{T_1^n(i): -n - \sigma_n \leqq i \leqq n - \sigma_n\} = \emptyset.$$

By Theorem 2.1,

$$\begin{aligned} &P\{\{S_2(j): n \leqq j \leqq n + n(\log n)^{-1}\} \cap \{S_1(i): -\infty < i < \infty\} = \emptyset\} \\ &\leqq c_{13}(\log n)^{-1}. \end{aligned}$$

Therefore, again by (3.1),

$$\begin{aligned} &P\{T_1^n(i) = T_2^n(j) \quad \text{for some} \quad -n \leqq i \leqq n,\ 0 < j \leqq n\} \\ &\leqq c_{14}(\log\log n)(\log n)^{-1}. \end{aligned}$$

We will define two random variables on Φ_n. Let $S(k,\omega)$ be another (one-sided) simple random walk starting at the origin and independent of S_1 and S_2. Let

$$F_n(\xi) = P\{S(k) \neq \xi(i) \quad \text{for} \quad 0 < k \leqq n \quad \text{and} \quad -n \leqq i \leqq n\}$$

$$\begin{aligned} G_n(\xi) &= \sum_{i=-n}^{n} G(\xi(i)) \\ &= \sum_{i=-n}^{n} \sum_{k=0}^{\infty} P\{S(k) = \xi(i)\}. \end{aligned}$$

G_n is the analogue of I_n from Sect. 2. Lemma 2.4 holds for G_n; more specifically,

$$\lim_{n \to \infty} (\log n)^{-1} EG_n = 2c_5, \tag{3.2}$$

$$\operatorname{Var}((\log n)^{-1} G_n) = O[(\log n)^{-1}]. \tag{3.3}$$

Here, of course, the expectations are taken with respect to P.

If $\xi \in \Phi_n$ and $k < n$, we will also use ξ to denote the element of Φ_k, $[\xi(-k), \ldots, \xi(k)]$; we will do similarly for $\xi \in \Lambda_n$. Let $m_n = [n + n(\log n)^{-1}]$. For any $\xi \in \Phi_{m_n}^A$, let

$$\tilde{P}_n(\xi) = P\{T_1^n(i) = \xi(i),\ -n \leqq i \leqq n\},$$

and for fixed (i_0, j_0) with $\max\{|i_0|, j_0\} \leqq n(\log n)^{-1}$,

$$\tilde{P}_n(\xi, i_0, j_0) = P\{T_1^n(i) = \xi(i),\ -n \leqq i \leqq n,\ \sigma_n = i_0,\ \tau_n = j_0\}.$$

Then, by definition,

$$\begin{aligned} \tilde{P}_n(\xi, i_0, j_0) = &P\{S_1(i + i_0) - S_1(i_0) = \xi(i),\ -n - i_0 \leqq i \leqq n - i_0\} \\ &\cdot P\{S_2(j_0) = \xi(i_0)\} \\ &\cdot P\{S_2(j + j_0) - S_2(j_0) \neq \xi(i),\ 0 < j \leqq n - j_0,\ -n - i_0 \leqq i \leqq n - i_0\}. \end{aligned}$$

Summing over all i_0, j_0 with $\max\{|i_0|, j_0\} \leqq n(\log n)^{-1}$,

$$\begin{aligned}
&P(\xi)\bar{G}_n(\xi)P\{S_2(j) \neq \xi(i): 0<j\leqq n,\ -m_n \leqq i \leqq m_n\} \\
&\quad \leqq P\{T_1^n(i) = \xi(i),\ -n \leqq i \leqq n,\ \max\{|\sigma_n|, \tau_n\} \leqq n(\log n)^{-1}\} \\
&\quad \leqq P(\xi)\bar{G}_n(\xi)P\{S_2(j) \neq \xi(i): 0<j\leqq n - n(\log n)^{-1},\ -n \leqq i \leqq n\}, \qquad (3.4)
\end{aligned}$$

where

$$\bar{G}_n(\xi) = \sum_{|i| < n(\log n)^{-1}} \sum_{j < n(\log n)^{-1}} P\{S(j) = \xi(i)\}.$$

Obviously $\bar{G}_n \leqq G_n$ and a straightforward calculation shows

$$E(G_n - \bar{G}_n) \leqq c_{15}(\log\log n). \qquad (3.5)$$

Let $Es(k_1, k_2, \xi) = P\{S_2(j) \neq \xi(i),\ 0<j\leqq k_2,\ -k_1 \leqq i \leqq k_1\}$. Then summing (3.4) over all $\xi \in \Phi_{m_n}^A$ and using (3.1) and (3.5) produces

$$\overline{\lim_{n\to\infty}}\ E[G_n(\xi)\, Es(m_n, n, \xi)\, 1_n^A(\xi)] \leqq 1 \leqq \underline{\lim_{n\to\infty}}\ E[G_n(\xi)\, Es(n, n - n(\log n)^{-1}, \xi)\, 1_n^A(\xi)],$$

where 1_n^A is the indicator function of the set Φ_n^A.

Using Theorem 2.1 we may conclude

$$\lim_{n\to\infty} E(F_n G_n 1_n^A) = 1.$$

By (3.2) and (3.3) and Chebyshev's inequality, there is a $c_{16} > 0$ such that for every $\varepsilon > 0$, if $A_{n,\varepsilon} = \{\omega : |(\log n)^{-1} G_n(\omega) - 2c_5| > \varepsilon\}$,

$$P(A_{n,\varepsilon}) \leqq c_{16}(\varepsilon^2 \log n)^{-1}. \qquad (3.6)$$

Therefore,

$$\begin{aligned}
E(F_n 1_n^A) &= \int_{A_{n,\varepsilon}} F_n 1_n^A + \int_{(A_{n,\varepsilon})^c} F_n 1_n^A \\
&\leqq c_{16}(\varepsilon^2 \log n)^{-1} + E(F_n G_n 1_n^A) \min\{G_n : \omega \in (A_{n,\varepsilon})^c\} \\
&\leqq c_{16}(\varepsilon^2 \log n)^{-1} + ((2c_5 - \varepsilon)\log n)^{-1}.
\end{aligned}$$

Similarly we may show an inequality in the other direction. We therefore have established

$$c_{20}(\log n)^{-1} \leqq E(F_n 1_n^A) \leqq c_{21}(\log n)^{-1}. \qquad (3.7)$$

Theorem 3.1. *Let* $f(n) = P\{S_1(i,\omega) \neq S_2(j,\omega),\ -n \leqq i \leqq n,\ 0<j\leqq n\}$. *Then*

$$c_{20}(\log n)^{-1} \leqq f(n) \leqq c_{21} g^{-1}(\log n)^{-1}.$$

Proof. Clearly $f(n) \geqq E(F_n 1_n^A)$, which gives the first inequality. To prove the second inequality it suffices to show

$$E(F_n) \leqq \mathscr{E}(F_n | \xi \in \Phi_n^A).$$

Intuitively, this states that a random walk which returns to the origin is more likely to be hit than one which does not. We omit the details; the key idea is to note that if $\eta = [\eta(-n), \ldots, \eta(0), \ldots, \eta(n), \ldots]$ is any half-infinite random walk path and

$$\Delta(\eta)=\inf\{k:k\geqq 0,\ \eta(k+i)\neq 0 \quad \text{for} \quad 1\leqq i\leqq n\},$$

then $E(F_n)=E(F_n(\eta))$, but $\mathscr{E}(F_n|\xi\in\Phi_n^A)=E(F_n(\eta_1))$, where

$$\eta_1=[\eta(-n),\ldots,\eta(0),\ \eta(\Delta+1),\ldots,\eta(\Delta+n)].$$

Using an argument similar to the one referred to above, we can show that if $\Phi_n^a=\{\xi\in\Phi_n^A:\xi^-\in\Lambda_n^A\}$, and 1_n^a is the indicator function of Φ_n^a,

$$gc_{20}(\log n)^{-1}\leqq E(F_n 1_n^a)\leqq c_{21}(\log n)^{-1}. \tag{3.8}$$

We use the notation $(n_1,n_2)\leqq(m_1,m_2)$ if $n_1\leqq m_1$ and $n_2\leqq m_2$; also, we use $(n_1,n_2)<(m_1,m_2)$ if $(n_1,n_2)\leqq(m_1,m_2)$ and $(n_1,n_2)\neq(m_1,m_2)$. For ξ_1, ξ_2, $\xi_3\in\Lambda_n$ and $\xi\in\Phi_n$, let

$$b_n(\xi_1,\xi_2)=\begin{cases}1 & \text{if} \quad \xi_1(i)\neq\xi_2(j) \quad \text{for} \quad (0,0)<(i,j)\leqq(n,n)\\ 0 & \text{otherwise,}\end{cases}$$

$$r_n(\xi_1,\xi_2,\xi_3)=b_n(\xi_1,\xi_3)b_n(\xi_2,\xi_3),$$
$$r_n(\xi,\xi_3)=r_n(\xi^+,\xi^-,\xi_3),$$
$$e_n(\xi_1)=\mathscr{E}(b(\xi_1,\xi_2)|\xi_1)=\mathscr{E}(b_2(\xi_2,\xi_1)|\xi_1).$$

Note that

$$E(F_n 1_n^a)=E(r_n),$$
$$(e_n(\xi_3))^2=\mathscr{E}(r_n(\xi,\xi_3)|\xi_3).$$

Theorem 3.2. *If* $p(n)=P\{S_1(i)\neq S_2(j),\ (0,0)<(i,j)\leqq(n,n)\}$, *then*

$$p(n)\leqq g^{-2}c_{21}{}^{1/2}(\log n)^{1/2}.$$

Proof. Again, it can be shown that $p(n)\leqq q(n)$ where

$$q(n)=P\{S_1(i)\neq S_2(j),\ (0,0)<(i,j)\leqq(n,n)|S_1(i)\neq 0,\ S_2(j)\neq 0,\ 0<i,j\leqq n\}.$$

But by (3.8),

$$\begin{aligned}(q(n))^2&=(g_n^{-2}E(e_n))^2\\ &\leqq g_n^{-4}E(e_n^2)\\ &\leqq g^{-4}E(r_n)\\ &=g^{-4}E(F_n 1_n^a)\\ &\leqq g^{-4}c_{21}(\log n)^{-1}.\end{aligned}$$

Remark. It is clear that Theorem 3.1 implies that

$$p(n)\geqq c_{20}(\log n)^{-1}.$$

Erdös and Taylor [3] contains a rigorous derivation of this bound.

Note that for $\xi\in\Phi_n^a$,

$$F_n(\xi)\leqq\min\{e_n(\xi^+),e_n(\xi^-)\},$$

and hence for $\alpha>0$,

$$P\{F_n 1_n^a\geqq\alpha\}\leqq P\{e_n\geqq\alpha\}^2. \tag{3.9}$$

Lemma 3.3. *Let $c>0$. Then*

(i) $\lim_{n\to\infty} \sup \int_D F_n 1_n^a \log n = 0,$

(ii) $\lim_{n\to\infty} \sup \int_D F_n 1_n^A \log n = 0,$

(iii) $\lim_{n\to\infty} \sup \int_D F_n \log n = 0,$

where in each case the supremum is taken over all sets D with $P(D) \leqq c(\log n)^{-1}$.

Proof. We only prove (i); the other two follow from "avoiding the origin" arguments as used in the proof of Theorem 3.1. Assume there exists a $\beta>0$, and a sequence n_i going to infinity, and D_i with $P(D_i) \leqq c(\log n_i)^{-1}$ such that

$$\int_{D_i} F_{n_i} 1_{n_i}^a \log n_i \geqq \beta .$$

Then a simple argument gives

$$P\{F_{n_i} 1_{n_i}^a \geqq \beta/2c\} \geqq \beta(2\log n)^{-1}.$$

By (3.9), if $B_i = \{\xi_3 : e_{n_i}(\xi_3) \geqq \beta/2c\}$,

$$P(B_i) \geqq \beta^{1/2}(2\log n)^{-1/2}.$$

But this implies

$$\begin{aligned}\int_{B_i} r(\xi, \xi_3) &= \int_{B_i} (e_n(\xi_3))^2 \\ &\geqq (\beta/2c)^2 \beta^{1/2}(2\log n)^{-1/2},\end{aligned}$$

and this contradicts (3.8).

We can now improve on (3.7):

Theorem 3.4.

$$\lim_{n\to\infty} E((\log n) F_n 1_n^A) = (2c_5)^{-1}.$$

Proof. Let $\varepsilon>0$. Using the notation in the derivation of (3.7),

$$E(F_n 1_n^A \log n) = \int_{A_{n,\varepsilon}} F_n 1_n^A \log n + \int_{(A_{n,\varepsilon})^c} F_n 1_n^A \log n .$$

By Lemma 3.3 (i), the first term on the right hand side goes to zero as n goes to infinity. Hence, by (3.6),

$$\begin{aligned}(2c_5+\varepsilon)^{-1} &\leqq \underline{\lim}_{n\to\infty} E(F_n 1_n^A \log n) \\ &\leqq \overline{\lim}_{n\to\infty} E(F_n 1_n^A \log n) \leqq (2c_5-\varepsilon)^{-1}.\end{aligned}$$

In the next section we will need to use the fact that conditioning a path so that it has no intersections does not affect the asymptotic behavior of the path. Let $C[-1, 1]$ denote the set of continuous functions from $[-1, 1]$ to R^4. Consider the following measures $\mathscr{P}_1^n, \ldots, \mathscr{P}_4^n$ on $C[-1,1]$:

$\mathscr{P}_1^n$: the measure generated by

$$X_1^n(t,\omega) = 2n^{-1/2}(S_1([nt]) + (nt-[nt])(S_1([nt]+1) - S_1([nt])));$$

$\mathscr{P}_2^n$: the measure generated by

$$X_2^n(t,\omega) = 2n^{-1/2}(T_1^n([nt]) + (nt-[nt])(T_1^n([nt]+1) - T_1^n(nt])));$$

$\mathscr{P}_3^n$: the measure generated by $X_1^n(t,\omega)$, but with the measure $(G_n F_n 1_n^A)(E(G_n F_n 1_n^A))^{-1} P$ on ω;

$\mathscr{P}_4^n$: the measure generated by $X_1^n(t,\omega)$, but with the measure $(F_n 1_n^A)(E(F_n 1_n^A))^{-1} P$ on ω.

Theorem 3.5. *Let $\mathscr{W}$ be the standard Wiener measure on $C[-1,1]$. Then for $i=1,\ldots,4$,*

$$\mathscr{P}_i^n \to \mathscr{W},$$

where the convergence is weak convergence of probability measures.

We omit the proof of this theorem since the methods are standard. The case $i=1$ is the standard invariance principle for simple random walk. The case $i=2$ follows from this because, by (3.1), for "almost all" ω, T_1^n is an infinitesimal shift of S_1. T_1^n gives measure approximately equal to $(F_n G_n 1_n^A)(E(F_n G_n 1_n^A))^{-1} P$ to paths of S_1 and this in turn is approximately equal to the measure $(F_n 1_n^A)(E(F_n 1_n^A))^{-1} P$; therefore, we get the cases $i=3, 4$.

We can also include S_2 in this analysis. Let $Y_i^n: [0,1] \times \Omega \to R^4$, $i=1,2,3$, be defined by

$$Y_1^n(t,\omega) = X_1^n(t,\omega),$$
$$Y_2^n(t,\omega) = X_1^n(-t,\omega),$$
$$Y_3^n(t,\omega) = 2n^{-1/2}(S_2([nt]) + (nt-[nt])(S_2([nt]+1) - S_2([nt]))).$$

Let

$$A_n = \{\omega : S_2(j,\omega) \notin \{S_1(i,\omega) : -n \leqq i \leqq n\},\ 0<j\leqq n \text{ and } S_1(i,\omega) \neq 0,\ 0<i\leqq n\}.$$

Then we can prove:

Theorem 3.6. *Let $Z^n(t) = \mathscr{E}((Y_1^n(t), Y_2^n(t), Y_3^n(t)|A_n)$. Then Z^n approaches the standard twelve dimensional Wiener process (i.e., three independent four dimensional Wiener processes) in distribution.*

We have done all our work with three walks (one two-sided and one one-sided) of length n. It is clear that we could have worked with walks of length $\beta_1 n$, $\beta_2 n$, $\beta_3 n$, for fixed positive β_1, β_2, β_3, and the theorems would still hold.

In the next section we will be interested in random walks S_1 and S_2 conditioned so that

(i) $\{S_1(i,\omega) : -\beta_1 n \leqq i \leqq \beta_2 n\} \cap \{S_2(j,\omega) : 0<j\leqq \beta_3 n\} = \emptyset$;
(ii) $S_1(i,\omega) \neq 0$ for $0<i\leqq \beta_2 n$;
(iii) $S_1([\beta_4 n],\omega) = S_2([\beta_5 n],\omega)$;

where $\beta_2<\beta_4$ and $\beta_3<\beta_5$. We wish to calculate the probability of such walks. By Theorem 3.4, the probability of walks satisfying (i) and (ii) is approximately $(2c_5 \log n)^{-1}$. We also know that the probability of (iii) is about $2c_5((\beta_4+\beta_5)n)^{-2}$, assuming $[\beta_4 n]+[\beta_5 n]$ is even. If the two events were independent we would know the probability; in fact, they are almost independent. We only sketch the argument here. Let A_n denote the set of ω which satisfy (i) and (ii).

By Theorem 3.6, applied to walks of lengths $[\beta_1 n]$, $[\beta_2 n]$, $[\beta_3 n]$, the distribution of

$$\mathscr{E}(2n^{-1/2}(S_1([\beta_2 n])-S_2([\beta_3 n]))|A_n) \tag{3.10}$$

approaches the normal distribution with covariance $(\beta_2+\beta_3)I$.

Assume $[\beta_4 n]+[\beta_5 n]$ is even. Then

$$\begin{aligned} &P\{S_1([\beta_4 n])-S_2([\beta_5 n])=0\} \\ &\quad=\sum_{x\in Z^4} P\{S_1([\beta_2 n])-S_2([\beta_3 n])=x\} \\ &\quad\quad\cdot P\{S_1([\beta_4 n]-[\beta_2 n])-S_2([\beta_5 n]-[\beta_3 n])=-x\}. \end{aligned}$$

If we substitute a random variable with distribution as in (3.10) for the first probability in the above sum, we do not change the result very much. Then standard asymptotic estimates for simple random walk can be used to show that the two sums are asymptotically equal.

Through careful handling of constants in Theorem 3.6 and the above argument, we can also get uniformity in the covergence. Hence, we can prove the following:

Theorem 3.7. *Let* $0<\alpha_{11}<\alpha_{12}$, $0<\alpha_{21}<\alpha_{22}<\alpha_{41}<\alpha_{42}$, $0<\alpha_{31}<\alpha_{32}<\alpha_{51}<\alpha_{52}$. *Let*

$$\begin{aligned} A(\beta_1,\beta_2,\beta_3,n)&=\{\omega:\text{(i) } and \text{ (ii) } hold\}\,; \\ B(\beta_4,\beta_5,n)&=\{\omega:\text{(iii) } holds\}\,. \end{aligned}$$

Then

$$\lim_{n\to\infty}\sup\left|\frac{P(A(\beta_1,\beta_2,\beta_3,n)\cap B(\beta_4,\beta_5,n))}{P(A(\beta_1,\beta_2,\beta_3,n))P(B(\beta_4,\beta_5,n))}-1\right|=0,$$

where the supremum is taken over all $\beta_1,\ldots,\beta_5$ *satisfying*

(a) $\alpha_{j1}\leqq\beta_j\leqq\alpha_{j2}$

(b) $[\beta_4 n]+[\beta_5 n]$ *even.*

4. Exact Result for Walks Starting at Different Points

We now use the results of Sect. 3 to improve on Theorems 2.1 and 2.2. We return to the notation of Sect. 2.

Theorem 4.1. *Let S_1 and S_2 be independent simple random walks starting at the origin in Z^4. Then*

(a) *for* $0<\alpha<\beta<\infty$

$$\lim_{n\to\infty}(\log n)P\{\Pi_1(\alpha n, \beta n)\cap\Pi_2(0,\infty)\neq\emptyset\}=\tfrac{1}{2}\log(\beta/\alpha);$$

(b) *for* $0<\alpha<\infty$

$$\lim_{n\to\infty}(\log n)P\{\Pi_1(0,n)\cap\Pi_2(\alpha n,\infty)\neq\emptyset\}=\tfrac{1}{2}\log(1+1/\alpha).$$

Theorem 4.2. *Let S_1 and S_2^n be independent simple random walks in Z^4 starting at* 0 *and x_n respectively. Suppose $\alpha=\lim_{n\to\infty}|x_n|^2/n$ exists. Then*

$$\lim_{n\to\infty}(\log n)P\{\Pi_1(0,n)\cap\Pi_2^n(0,\infty)\neq\emptyset\}=\tfrac{1}{2}\log(1+1/\alpha).$$

We will only prove Theorem 4.1 (a); the other proofs are similar. We will also assume $\beta=1$. Before proceeding with the proof, we will give an intuitive argument. Fix α, $0<\alpha<1$, and let

$$B_n=\{\omega:\Pi_1(\alpha n, n,\omega)\cap\Pi_2(0,\infty,\omega)\neq\emptyset\}.$$

On B_n, let $(\sigma_n(\omega),\ \tau_n(\omega))$ be the first intersection of the paths defined by

$$\tau_n(\omega)=\inf\{j:S_2(j,\omega)=S_1(i,\omega)\text{ for some } i,\ \alpha n\leqq i\leqq n\};$$

$$\sigma_n(\omega)=\inf\{i:i\geqq\alpha n\text{ and } S_1(i,\omega)=S_2(\tau_n(\omega),\omega)\}.$$

As in the proof of Theorem 2.1 (b), let $J_n(\omega)$ denote the number of intersections of the two paths,

$$J_n(\omega)=\#\{(i,j):S_1(i,\omega)=S_2(j,\omega),\ \alpha n\leqq i\leqq n,\ 0\leqq j<\infty\}.$$

By an estimate similar to the one done before we can show

$$\lim_{n\to\infty}E(J_n)=c_5\log(1/\alpha).$$

It is clear that

$$E(J_n)=P(B_n)\mathscr{E}(J_n|B_n).$$

Therefore, if we could compute $\mathscr{E}(J_n|B_n)$ we would have the result. Consider the two paths at (σ_n,τ_n) as shown in this figure:

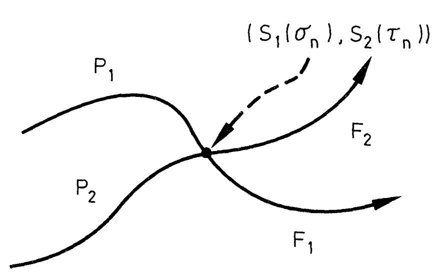

Fig. 1

In the figure, P_1 and F_1 denote the parts of Π_1 before and after σ_n respectively. The same is true for P_2 and F_2. By the definition of (σ_n, τ_n),

$$P_2 \cap P_1 = \emptyset, \qquad P_2 \cap F_1 = \emptyset.$$

Also, F_2 is independent of P_1. P_2, and F_1. Therefore, the intersections of the two paths can be broken down into two sets: those of F_2 with P_1 and those of F_2 with F_1. Ignoring the conditioning on P_1 and F_1 imposed by the selection of (σ_n, τ_n), we see that the expected number of intersections of the paths looks like *twice* the number of intersections of two paths starting at the origin, i.e. $2E(\tilde{J}_n)$, where

$$\tilde{J}_n(\omega) = \# \{(i,j) : S_1(i,\omega) = S_2(j,\omega),\ 0 \leqq i \leqq n,\ 0 \leqq j < \infty\}.$$

But an easy estimate gives

$$\lim_{n\to\infty} (\log n)^{-1} E(\tilde{J}_n) = c_5.$$

Therefore, assuming this argument could be made rigorous,

$$\mathscr{E}(J_n | B_n) \approx 2c_5 (\log n),$$

and hence

$$P(B_n) \approx \tfrac{1}{2} \log(1/\alpha).$$

Proof of Theorem 4.1(a). Let B_n, σ_n, τ_n, be as above and again fix α, $0<\alpha<1$. For $\omega \in B_n$, define random walks R_1^n and R_2^n by

$$R_1^n(i,\omega) = S_1(\sigma_n(\omega) - i, \omega) - S_1(\sigma_n(\omega), \omega),\ \sigma_n(\omega) - n \leqq i \leqq \sigma_n(\omega),$$
$$R_2^n(j,\omega) = S_2(\tau_n(\omega) - j, \omega) - S_2(\tau_n(\omega), \omega),\ 0 \leqq j \leqq \tau_n(\omega).$$

For any i_0, j_0, let

$$B_n(i_0, j_0) = \{\omega \in B_n : \sigma_n(\omega) = i_0,\ \tau_n(\omega) = j_0\}.$$

Then on $B_n(i_0, j_0)$, R_1^n and R_2^n are independent simple random walks starting at the origin (R_1^n is two-sided, R_2^n is one-sided) satisfying

(i) $R_2^n(j,\omega) \notin \{R_1^n(i,\omega) : i_0 - n \leqq i \leqq i_0 - \alpha n\}$, $0 < j \leqq j_0$;
(ii) $R_1^n(i,\omega) \neq 0$, $0 < i \leqq i_0 - \alpha n$;
(iii) $R_1^n(i_0, \omega) = R_2^n(j_0, \omega)$.

For each $\varepsilon > 0$, let

$$B_n^\varepsilon = \{\omega \in B_n : \alpha n(1+\varepsilon) \leqq \sigma_n(\omega) \leqq n(1-\varepsilon)\}.$$

Clearly

$$P(B_n^\varepsilon) \leqq P(B_n) \leqq P(B_n^\varepsilon) + P(C_n^\varepsilon),$$

where

$$C_n^\varepsilon = \{\omega \in B_n : (\Pi_1(\alpha n, \alpha n(1+\varepsilon)) \cup \Pi_1(n(1-\varepsilon), n)) \cap \Pi_2(0, \infty) \neq \emptyset\}.$$

By Theorem 2.1 (a) we can conclude,

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} (\log n) P(C_n^\varepsilon) = 0.$$

Therefore, if $P_\varepsilon = \lim_{n\to\infty} P(B_n^\varepsilon)(\log n)$ exists, we will have

$$\lim_{n\to\infty} (\log n) P(B_n) = \lim_{\varepsilon\to 0} P_\varepsilon .$$

Likewise if we set

$$D_n^\varepsilon = \{\omega \in C_n^\varepsilon : \varepsilon n \leqq \tau_n(\omega) \leqq \varepsilon^{-1} n\},$$

then

$$\lim_{n\to\infty} (\log n) P(B_n) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} (\log n) P(D_n^\varepsilon),$$

assuming the right hand limit exists.

Consider D_n^ε for a fixed $\varepsilon > 0$;

$$\lim_{n\to\infty} (\log n) P(D_n^\varepsilon) = \lim_{n\to\infty} (\log n) \sum_{i=\alpha n(1+\varepsilon)}^{n(1-\varepsilon)} \sum_{j=\varepsilon n}^{\varepsilon^{-1} n} P(B(i,j)).$$

By Theorem 3.7,

$$P(B(i,j)) \sim (c_5 (i+j)^{-2})(2c_5 \log n)^{-1},$$

where the asymptotic convergence is uniform for $\alpha n(1+\varepsilon) \leqq i \leqq n(1-\varepsilon)$, $\varepsilon n \leqq j \leqq \varepsilon^{-1} n$.

Therefore,

$$\lim_{n\to\infty} (\log n) \sum \sum B(i,j) = \lim_{n\to\infty} \sum \sum (\mathrm{i}+\mathrm{j})^{-2} (2\log n)^{-1}$$
$$= \frac{1}{2} \int_{\alpha(1+\varepsilon)}^{(1-\varepsilon)} \int_{\varepsilon}^{\varepsilon^{-1}} (x+y)^{-2} dx dy .$$

By (4.1) then,

$$\lim_{n\to\infty} (\log n) P(B_n) = \lim_{\varepsilon\to 0} \frac{1}{2} \int_{\alpha(1+\varepsilon)}^{(1-\varepsilon)} \int_{\varepsilon}^{\varepsilon^{-1}} (x+y)^{-2} dx dy$$
$$= \tfrac{1}{2} \log(1/\alpha) .$$

5. Conjecture for Walks Starting at the Same Point

Let S_1 and S_2 be independent random walks starting at the origin in Z^4, and again let

$$p(n) = P\{\Pi_1(0,n) \cap \Pi_2(0,n) = \emptyset\} .$$

For ease we will consider instead

$$q(n) = P\{\Pi_1(0,n) \cap \Pi_2(0,\infty) = \emptyset\} .$$

Clearly $q(n) \leqq p(n)$.

We will assume a plausible conjecture which we have been unable to prove, and from it conclude that for every $r > \frac{1}{2}$,

$$\lim_{n\to\infty} (\log n)^r q(n) = \infty , \tag{5.1}$$

and hence also for $p(n)$.

Let

$$T_n=\{\omega:\Pi_1(0,n,\omega)\cap\Pi_2(0,\infty,\omega)=\emptyset\}$$
$$V_n=\{\omega:\Pi_1(n,2n,\omega)\cap\Pi_2(0,\infty,\omega)=\emptyset\}.$$

Note that $q(n)=P(T_n)$.

Conjecture 5.1.

$$P(T_{2n})\geqq P(T_n)P(V_n).$$

This conjecture states that the probability that the second path intersects the first path at some time between n and $2n$ is not increased if we know that the second path has avoided all points of the first path up through time n. We actually believe that T_n and V_n are asymptotically independent events, and a similar argument as below will show it is sufficient for concluding (5.1) to assume asymptotic independence with a sufficiently small error term.

By Theorem 4.1 (a)

$$P(B_n)=1-\varrho_n(\log 2)(2\log n)^{-1},$$

where $\varrho_n\to 1$. Therefore, using Conjecture 5.1,

$$q(2n)\geqq q(n)(1-\varrho_n(\log 2)(2\log n)^{-1}). \tag{5.2}$$

If we let $f(k)=q(2^k)$, $\gamma_k=\varrho_{2^k}$, (5.2) becomes

$$f(k+1)\geqq f(k)(1-\gamma_k(2k)^{-1}). \tag{5.3}$$

It is then an easy exercise to conclude that (5.3) implies

$$\lim_{k\to\infty} k^r f(k)=\infty, \qquad r>\tfrac{1}{2},$$

and hence we get (5.1).

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