Commun. Math. Phys. 83, 171–191 (1982)

Communications in
Mathematical
Physics
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The Global Existence of Yang–Mills–Higgs Fields in 4-Dimensional Minkowski Space

I. Local Existence and Smoothness Properties*

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Abstract. In this paper and its sequel we shall prove the local and then the global existence of solutions of the classical Yang–Mills–Higgs equations in the temporal gauge. This paper proves local existence uniqueness and smoothness properties and improves, by essentially one order of differentiability, previous local existence results. Our results apply to any compact gauge group and to any invariant Higgs self-coupling which is positive and of no higher than quartic degree.

I. Introduction

This is the first paper in a series of two in which we shall prove the local and then the global existence of solutions to the Yang–Mills–Higgs equations in 4-dimensional Minkowski space. In this paper we establish local existence, uniqueness and smoothness properties of Yang–Mills–Higgs fields in the temporal gauge. In the sequel, we shall extend this result to global existence by showing that an appropriate norm of the solutions cannot blow up in a finite time. Our results apply to any compact gauge group and to any invariant, positive Higgs self-coupling of no higher than quartic degree.

Our work on Yang–Mills theory was motivated by an interest in the cosmic censorship conjecture in general relativity. This conjecture states (roughly) that singularities which develop from regular initial data are always hidden inside black holes. Some heuristic arguments given elsewhere by the authors [1, 2] strongly suggest that the cosmic censorship conjecture is equivalent to a certain global existence conjecture about the Einstein equations. One hopes to prove the global existence conjecture and thereby to establish the validity of cosmic censorship.

* Supported in part by the National Science Foundation (Grant No. PHY79-16482 at Yale)

0010-3616/82/0083/0171/$04.20

Yang–Mills theories provide some simpler models on which to develop the needed analytical techniques.

The local existence results of this paper are derived using Segal's general existence theory for semi-linear evolution equations [3]. Segal himself has already given a local existence argument for (pure SU(2)) Yang–Mills fields in the temporal gauge [4, 5]. However, our approach (which is technically somewhat more complicated) improves the local result by (roughly) one order of differentiability. This in turn simplifies the global existence argument to follow.

Segal proved local existence for the evolution equations by taking [5] $(A, E, B) \in H_s \times H_s \times H_s$ for $s \geqq 2$. Here A is the vector potential, E is the electric and B the magnetic field associated to A, and H_s represents the Sobolev space of square integrable functions with square integrable derivatives up to order s. However, the "constraint" equation which relates B to A,

$$B = \nabla \times A + \tfrac{1}{2}[A \times, A],$$

shows that $\nabla \times A \in H_s$ and thus that the transverse part of A lies in H_{s+1}. In addition the initial value equation,

$$\nabla \cdot E = [E \cdot, A],$$

shows that the longitudinal part of E lies in H_{s+1} and thus, since $\frac{dA}{dt} = E$, that the longitudinal part of A remains in H_{s+1} provided its initial value lies in this space. Thus the actual solutions (A, E, B) resulting from Segal's approach lie in $H_{s+1} \times H_s \times H_s$ for $s \geqq 2$.

In our approach we take (A, E, B) to lie in $H_{s+1} \times H_s \times H_s$ *ab initio* and prove local existence for all $s \geqq 1$. To achieve this improvement, however, we must modify the evolution equations in such a way that the modified equations (i) reduce to the original equations when the constraints are satisfied, (ii) preserve the constraint equations, and (iii) have the needed Lipshitz and smoothness properties for local existence in the stated spaces.

The reason that the unmodified equations fail to give an immediate local existence result in $H_{s+1} \times H_s \times H_s$ is discussed more fully in Sect. II below. To globally extend Segal's result by the energy methods of paper II one would have to show that the $H_3 \times H_2 \times H_2$ norm does not blow up in a finite time. To globally extend our result, however, we shall only need to show that the $H_2 \times H_1 \times H_1$ norm does not blow up.

The choice of ordinary Sobolev spaces as function spaces for our analysis was made primarily for simplicity. However, the functions in these spaces decay faster at infinity than $1/r$ and thus cannot accommodate the description of a non-zero magnetic charge. The treatment of spontaneously broken symmetry (even with zero magnetic charge) is also excluded from consideration since in that case the Higgs field would have to decay to a non-zero constant at infinity. For technical reasons (involving the elliptic theory done in the appendix) it is not sufficient to simply subtract a background field with the "bad" asymptotic behaviour and work with equations for the subtracted fields. Nonetheless, we believe there to be no fundamental difficulty in extending our results to the treatment of spontaneously broken theories and magnetic charges. The needed analytical tool would seem to be the weighted Sobolev spaces of Nirenberg and Walker [6], Cantor [7] and McOwen [8]. These have recently been applied to solve certain existence problems in general relativity by Christodoulou [9] and Choquet–Bruhat [10].

Since writing this paper we received a preprint by Ginibre and Velo [11] which significantly elaborates and extends Segal's Yang–Mills work. In particular they prove global existence for Yang–Mill–Higgs fields in 2 + 1 dimensions by extending a technique developed by one of us (VM) [12] to prove global existence for the Maxwell–Klein–Gordon equations.

II. Local Existence and Smoothness

In this section we shall establish local existence, uniqueness and smoothness properties of solutions of the Yang–Mills–Higgs (YMH) equations in the temporal gauge by applying Segal's [3] general theory for semi-linear evolution equations. We shall show that any initial data $(A, \dot{A}, \phi, \dot{\phi})$ lying in the Sobolev space $(H_{s+1} \times H_s \times H_{s+1} \times H_s) \equiv (H_{s+1} \times H_s)^2$ for $s \geqq 1$ generates a unique solution to the integral equation associated to the YMH evolution equations on some region $R^3 \times (t_a, t_b)$ of (Minkowski) spacetime. We shall further show that if the initial data is restricted to lie in $(H_{2+k} \times H_{1+k})^2$ for $k \geqq 2$ then it generates a solution for which the potentials (A, ϕ) are C^k functions on spacetime which satisfy the YMH equations in the classical sense.

To simplify the presentation slightly, we shall specify initial data at $t = 0$ and consider only evolution into the future so that the local existence intervals will always have the form $[0, T)$ for some $T > 0$. Since the YMH equations are time translational and time reversal invariant, it's clear that this specialization entails no loss of generality.

Our conclusions will apply to any compact gauge group and any quartic Higgs self-coupling with positive energy.

A. Notation and Basic Equations

Let G be an arbitrary, compact Lie group and let $\{\theta_a\}$ be a real matrix representation of the Lie algebra $\mathscr{g}$ of G so that

$$[\theta_a, \theta_b] = f^{abc}\theta_c \tag{2.1}$$

for some constants f^{abc}. We may regard the Yang–Mills potential ${}^{(4)}A$ as a $\mathscr{g}$-valued one-form field over Minkowski space and write

$$^{(4)}A = A_\mu^{(a)}\theta_a dx^\mu = A_\mu dx^\mu \tag{2.2}$$

The curvature of ${}^{(4)}A$ is a $\mathscr{g}$-valued two-form field ${}^{(4)}F$ defined by

$$^{(4)}F = (F_{\mu\nu}^{(a)}\theta_a)dx^\mu \wedge dx^\nu = F_{\mu\nu}dx^\mu \wedge dx^\nu, \tag{2.3}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2.4}$$

The fields induced by ${}^{(4)}A$ and ${}^{(4)}F$ on a flat, $t = \text{const}$ spacelike surface in Minkowski space will be denoted by

$$A = A_i^{(a)}\theta_a dx^i = A_i dx^i,$$
$$F = F_{ij}^{(a)}\theta_a dx^i \wedge dx^j = F_{ij} dx^i \wedge dx^j, \tag{2.5}$$

where i, j range over 1, 2, 3 and $x^0 = t$ is the time coordinate.

The Higgs field ϕ is a vector valued function on Minkowski space with its values in the real vector space associated to the representation $\{\theta_a\}$. The covariant derivative of ϕ has the components

$$D_\mu \phi = \partial_\mu \phi + A_\mu \phi. \tag{2.6}$$

If $\mathscr{U}$ is a smooth G-valued function over spacetime it generates gauge transformations of $({}^{(4)}A, \phi)$ according to

$$\phi' = \mathscr{U}\phi, A'_\mu = \mathscr{U} A_\mu \mathscr{U}^{-1} + \mathscr{U}\partial_\mu \mathscr{U}^{-1} \tag{2.7}$$

from which follow the transformations

$$F'_{\mu\nu} = \mathscr{U} F_{\mu\nu} \mathscr{U}^{-1}$$
$$(D_\mu \phi)' = \mathscr{U}(D_\mu \phi) \tag{2.8}$$

The (gauge invariant) Lagrangian for the YMH equations is

$$\mathscr{L} = \mathrm{Tr}\{-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\} - \tfrac{1}{2}(D_\mu\phi)\cdot(D^\mu\phi) - P(\phi), \tag{2.9}$$

where $P(\phi)$ is an invariant polynomial in ϕ (i.e., $P(\mathscr{U}\phi) = P(\phi)\forall\phi$ and $\forall\mathscr{U}\in G$) of no higher than quartic degree and where Tr represents a trace over the representation of $\mathscr{g}$ and "." represents an invariant contraction in the vector space associated to the chosen representation. One can always choose the basis $\{\theta_a\}$ for $\mathscr{g}$ such that the θ_a are real antisymmetric matrices, the f^{abc} are completely antisymmetric and the trace is[1]

$$\mathrm{Tr}\{\theta_a\theta_b\} = \delta_{ab} \tag{2.10}$$

In this case the contraction "." is given by

$$\phi\cdot\phi = \phi_\kappa\phi_\kappa, (D_\mu\phi)\cdot(D^\mu\phi) = (D_\mu\phi)_\kappa(D^\mu\phi)_\kappa \tag{2.11}$$

etc., where κ ranges over $1, \ldots, d =$ the dimension of the representation.

The Hamiltonian associated to the Lagrangian (2.9) is given by

$$H = \int_{R_3} d^3x\{\mathrm{Tr}[\tfrac{1}{2}E_iE_i + \tfrac{1}{4}F_{ij}F_{ij}] + \tfrac{1}{2}\pi\cdot\pi + \tfrac{1}{2}(D_i\phi)\cdot(D_i\phi) + P(\phi) + \mathrm{Tr}[A_0\mathscr{C}]\}, \tag{2.12}$$

where

$$E_i = E_i^{(a)}\theta_a = \partial_t A_i - \partial_i A_0 + [A_0, A_i] = F_{0i}$$
$$\pi = \partial_t\phi + A_0\phi = D_0\phi \tag{2.13}$$

and

$$\mathscr{C} = \mathscr{C}^{(a)}\theta_a = -\partial_j E_j + [E_j, A_j] - (\pi\cdot\theta_a\phi)\theta_a. \tag{2.14}$$

1 For convenience we have defined the trace operation Tr to be the negative of the usual matrix trace

The initial value constraint is $\mathscr{C}=0$ and E_i and π are the momenta conjugate to A_i and ϕ respectively.

The Hamilton equations, specialized to the temporal ($A_0=0$) gauge become

$$\frac{\partial}{\partial t}\begin{bmatrix} A_i \\ E_i \\ \phi \\ \pi \end{bmatrix} = \begin{bmatrix} E_i \\ \Delta A_i - \partial_i \partial_j A_j \\ \pi \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} 0 \\ \{-\partial_j[A_i, A_j] - [A_j, F_{ij}] - ((D_i\phi)\cdot\theta_a\phi)\theta_a\} \\ 0 \\ \left\{\partial_i(A_i\phi) + A_i\partial_i\phi + A_iA_i\phi - \dfrac{\partial \mathrm{P}}{\partial\phi}\right\} \end{bmatrix}, \tag{2.15}$$

where we have split the linear from the non-linear terms on the right-hand side. For reasons that we shall explain below, it is desirable to replace these evolution equations by a different set which reduces to the above when the constraint is satisfied. The needed modification is obtained by first splitting $E=E_i dx^i$ into its unique transverse (divergence free) and longitudinal (curl free) parts (see the appendix for the definition of this splitting on H_s vector fields and the proof of its smoothness),

$$E_i = E_i^T + E_i^L, \quad \partial_i E_i^T = 0, \quad \varepsilon^{ijk}\partial_j E_k^L = 0, \tag{2.16}$$

and then by replacing E^L with an expression which equals E^L when the constraint is satisfied. Thus we let

$$E_i^L \to E_i^C \equiv \partial_i\left\{-\frac{1}{4\pi r}*([E_j, A_j] - (\pi\cdot\theta_a\phi)\theta_a)\right\} \tag{2.17}$$

where $-\dfrac{1}{4\pi r}*(\rho)$ represents convolution of ρ with the fundamental solution of Poisson's equation, i.e.,

$$-\frac{1}{4\pi r}*(\rho) = -\frac{1}{4\pi}\int_{R^3} dx'\left(\frac{\rho(x')}{|x-x'|}\right). \tag{2.18}$$

In the appendix we prove for any $(A, E, \phi, \pi)\in(H_{s+1}\times H_s)^2$, and with $s\geqq 1$, that E^C is a well-defined curl-free vector field lying in H_{s+1} and satisfying

$$\partial_i E_i^C = [E_j, A_j] - (\pi\cdot\theta_a\phi)\theta_a. \tag{2.19}$$

We also show that if in addition the constraint is satisfied, i.e., if

$$\partial_i E_i = \partial_i E_i^L = [E_j, A_j] - (\pi\cdot\theta_a\phi)\theta_a, \tag{2.20}$$

then $E_i^C = E_i^L$. Note that E^C (and thus E^L when the constraint is satisfied) is in general smoother than E^T (i.e., $E^C\in H_{s+1}$ whereas $E^T\in H_s$). This will be quite important in the following analysis.

With the above modification we may write the evolution equations as

$$\frac{du}{dt} = \mathscr{A}u + J(u), \tag{2.21}$$

where

$$u=\begin{bmatrix}A_i\\E_i\\\phi\\\pi\end{bmatrix},\quad \mathscr{A}u=\begin{bmatrix}E_i^T\\\Delta A_i-\partial_i\partial_jA_j\\\pi\\\Delta\phi\end{bmatrix}\tag{2.22}$$

and

$$J(u)=\begin{bmatrix}\partial_i\left[-\frac{1}{4\pi r}*\left([E_j,A_j]-(\pi\cdot\theta_a\phi)\theta_a\right)\right]\\\{-\partial_j[A_i,A_j]-[A_j,F_{ij}]-((D_i\phi)\cdot\theta_a\phi)\theta_a\}\\0\\\left\{\partial_i(A_i\phi)+A_i\partial_i\phi+A_iA_i\phi-\frac{\partial P}{\partial\phi}\right\}\end{bmatrix}.\tag{2.23}$$

By splitting A and E into transverse and longitudinal parts we may express the linearized equations

$$\frac{du}{dt}=\mathscr{A}u\tag{2.24}$$

in the form

$$\frac{d}{dt}\begin{pmatrix}A^T\\E^T\end{pmatrix}=\begin{pmatrix}0&I\\\Delta&0\end{pmatrix}\begin{pmatrix}A^T\\E^T\end{pmatrix},\ \frac{d}{dt}\begin{pmatrix}\phi\\\pi\end{pmatrix}=\begin{pmatrix}0&I\\\Delta&0\end{pmatrix}\begin{pmatrix}\phi\\\pi\end{pmatrix},\ \frac{d}{dt}\begin{pmatrix}A^L\\E^L\end{pmatrix}=0,\tag{2.25}$$

which shows at once that the pairs (A^T, E^T) and (ϕ, π) each satisfy the linear wave equation and that (A^L, E^L) is a constant of the (linearized) motion. It follows that the linearized equations are globally defined on any of the Sobolev spaces

$$(A,E,\phi,\pi)\in(H_{s+1}\times H_s)^2,\quad s\geqq 0.\tag{2.26}$$

Had we attempted to work with the unmodified equations (2.15), the longitudinal projection of the linearized equations would have been

$$\frac{d}{dt}\begin{pmatrix}A^L\\E^L\end{pmatrix}=\begin{pmatrix}0&I\\0&0\end{pmatrix}\begin{pmatrix}A^L\\E^L\end{pmatrix},\tag{2.27}$$

which has the general solution

$$E^L(t)=E^L(0),\quad A^L(t)=A^L(0)+E^L(0)t.\tag{2.28}$$

Thus $A^L(t)$ is no smoother than $E^L(t)=E^L(0)$. This is the difficulty in attempting to treat the linearized (unmodified) equations as defining a group on the space $H_{s+1}\times H_s$. In pure Maxwell theory this procedure causes no difficulty because the Maxwell constraint is simply $E^L=0$ which allows $A^L(t)$ to persist in H_{s+1}. In Yang–Mills theory one might hope to solve the constraint for E^L as a functional of (A, E^T) and show that this solution forces E^L to lie in H_{s+1}. Unfortunately, however, one cannot globally solve the constraint for E^L since the functional derivative of the constraint with respect to E^L is not an everywhere surjective map (this fact was first pointed out by Gribov [13]). Nonetheless, it is true that the constraint forces E^L to lie in H_{s+1}. Our device of replacing E^L by E^C takes advantage of this "smoothing" property of the constraint without in fact attempting to solve the constraint.

B. Local Existence and Uniqueness

As discussed above, the linear operator $\mathscr{A}$ generates a one-parameter group on any of the Sobolev spaces $(H_{s+1} \times H_s)^2$ for $s \geqq 0$. For $s \geqq 1$ we may establish local existence and uniqueness of solutions to the integral equation

$$u(t) = \exp(\mathscr{A}(t-t_0))u(t_0) + \int_{t_0}^{t} ds \exp(\mathscr{A}(t-s))J(u(s)) \tag{2.29}$$

associated to (2.21) by showing that J is a continuous map from $\mathscr{H} = (H_{s+1} \times H_s)^2$ to itself which satisfies the Lipshitz condition

$$\|J(u) - J(v)\| \leqq C(\|u\|, \|v\|)\|u - v\| \tag{2.30}$$

for all $u, v \in \mathscr{H}$. Here $\| \ \|$ is the $(H_{s+1} \times H_s)^2$ norm and $C(,)$ is some monotonically increasing, everywhere finite function of the norms indicated. It will follow from Segal's general theory [3] that any initial data $u(t_0) = u_0 \in \mathscr{H}$ will determine a unique solution $u(t)$ to (2.29) on some interval (t_a, t_b) containing t_0 and that either $(t_a, t_b) = (-\infty, +\infty)$ or $\|u(t)\| \to \infty$ as $t \to t_a$ or t_b.

The proof that J is a continuous, Lipshitz map from $\mathscr{H}$ to $\mathscr{H}$ is, except for the non-local term E^C, straightforward and was in fact given by Segal for the case $\mathscr{H} = H_2 \times H_1$ (with $G = SU(2)$ and Higgs fields not included). A proof in the general case (including $s > 2$) is facilitated by using the Sobolev estimate

$$\begin{aligned} \|Df\|_{L^4} &\leqq K \|D^2 f\|_{L^2}^{7/8} \|f\|_{L^2}^{1/8} \\ &\leqq K' \|f\|_{H_2} \end{aligned} \tag{2.31}$$

and the Schauder ring property of H_s maps (i.e., the property that H_s fields over R^n form a ring under pointwise multiplication of components if $s > n/2$ [14]). The proof that E^C, defined by Eq. (2.17), is a continuous map from $\mathscr{H}$ to H_{s+1} is given in the appendix where it is also shown that E^C has the Lipshitz property needed for (2.30). Combining these results one finds that

$$\|J(u) - J(v)\| \leqq (C_1 + C_2(\|u\| + \|v\|) + C_3(\|u\| + \|v\|)^2)\|u - v\|, \tag{2.32}$$

where C_1, C_2 and C_3 are positive constants and u and v are arbitrary elements of $\mathscr{H}$.

To show that solutions $u(t)$ of the integral equation (2.29) actually satisfy the original differential equation (2.21), we need to establish some smoothness of the non-linear operator J and to restrict the initial data u_0 to lie in the domain of $\mathscr{A}$.

J is in fact a C^∞ map from $\mathscr{H}$ to itself. One can prove this by directly computing the Frechet derivatives of J (all derivatives higher than the third vanish identically). For the local terms in J (i.e., those not involving E^C) one needs only the Sobolev estimate (2.31) and the Schauder ring property of H_s maps to prove the indicated smoothness. The proof that E^C is a smooth map from $\mathscr{H}$ to H_{s+1} is given in the appendix.

Since J is a C^∞ map (C^1 would be sufficient here) it follows from Segal's general theory that any initial data in the domain of $\mathscr{A} \equiv D_{\mathscr{A}}$ determines a solution of (2.29) which remains in $D_{\mathscr{A}}$ throughout its interval of existence and which thus satisfies the differential equation

$$\frac{du}{dt}(t) = \mathscr{A}u(t) + J(u(t)), \tag{2.33}$$

with $\frac{du}{dt}(t)$ a continuous curve in $\mathscr{H}$.

The domain of $\mathscr{A}$ consists of all (A, E, ϕ, π) satisfying

$$\begin{gathered} A^T \in H_{s+2}, A^L \in H_{s+1}, E^T \in H_{s+1}, E^L \in H_s \\ \phi \in H_{s+2}, \pi \in H_{s+1} \end{gathered} \tag{2.34}$$

We shall now show that solutions $u(t) \in D_{\mathscr{A}}$ satisfy the constraint $\mathscr{C}(t) = 0$ throughout their interval of existence provided they satisfy this constraint initially.

Computing $\frac{d\mathscr{C}}{dt}$ for any solution $u(t) \in D_{\mathscr{A}}$ we get

$$\frac{d\mathscr{C}}{dt} = [\Delta E_i^L, E_i], \tag{2.35}$$

where

$$\begin{aligned} \Delta E_i^L &\equiv E_i^L - E_i^C \\ &= E_i^L - \partial_i \left\{ -\frac{1}{4\pi r} * ([E_j, A_j] - (\pi \cdot \theta_a \phi)\theta_a) \right\}. \end{aligned} \tag{2.36}$$

Thus, using Hölder's inequality,

$$\begin{aligned} \frac{d}{dt} \int_{R^3} \mathscr{C}^{(a)} \mathscr{C}^{(a)} &= +2 \int_{R^3} f^{abc} \mathscr{C}^{(c)} (\Delta E^L)_j^{(a)} E_j^{(b)} \\ &\leqq 2 \sum_{a,b,j} \| f^{abc} \mathscr{C}^{(c)} \|_{L^2} \| E_j^{(b)} \|_{L^3} \| (\Delta E^L)_j^{(a)} \|_{L^6}. \end{aligned} \tag{2.37}$$

From the Sobolev estimate (valid for H_1 functions over R^3)

$$\| f \|_{L^6} \leqq K \| Df \|_{L^2}, \tag{2.38}$$

we get that

$$\begin{aligned} \| (\Delta E^L)_j \|_{L^6} &\leqq K \left(\int_{R^3} \sum_k (\partial_k (\Delta E^L)_j)^2 \right)^{1/2} \\ &\leqq K \left(\int_{R^3} \sum_{k,j} (\partial_k (\Delta E^L)_j)^2 \right)^{1/2} \\ &\leqq K \left(\int_{R^3} \left(\sum_k \partial_k (\Delta E^L)_k \right)^2 \right)^{1/2} \\ &= K \left(\int_{R^3} \mathscr{C}^{(b)} \mathscr{C}^{(b)} \right)^{1/2} = K \| \mathscr{C} \|_{L^2}. \end{aligned} \tag{2.39}$$

where, in the intermediate steps, we have integrated by parts and used the fact (see the appendix) that ΔE^L is a gradient obeying

$$\partial_i \Delta E_i^L = -\mathscr{C}. \tag{2.40}$$

It follows from (2.37) and (2.39) that

$$\frac{d}{dt}(\|\mathscr{C}(t)\|_{L^2}^2) \leqq K'\|E(t)\|_{H_1}(\|\mathscr{C}(t)\|_{L^2})^2, \tag{2.41}$$

(since $\|E(t)\|_{L^3} \leqq C\|E(t)\|_{H_1}$) and thus, using Gronwall's inequality, that

$$\|\mathscr{C}(t)\|_{L^2}^2 \leqq \|\mathscr{C}(0)\|_{L^2}^2 \exp\left(\int_0^t ds K'\|E(s)\|_{H_1}\right). \tag{2.42}$$

Thus $\|\mathscr{C}(t)\|_{L^2}$ and hence $\mathscr{C}(t)$ vanishes throughout the interval of existence of $u(t)$ provided $\mathscr{C}$ vanishes initially. Furthermore since $\mathscr{C}(t)=0$ implies that $E^L(t)=E^C(t)$ (see the appendix) it follows that solutions to the modified equations (2.21–2.23) which satisfy $\mathscr{C}(t)=0$ are in fact solutions of the original unmodified equations (2.14 and 2.15).

We have thus proven:

Theorem 1. *If $u_0=(A,E,\phi,\pi)$ is any initial data lying in $\mathscr{H}=(H_{s+1}\times H_s)^2$ for $s\geqq 1$ then there exists a unique solution $u(t)$ to the integral equation (2.29), with $u(0)=u_0$, defined on some interval $[0,T)$. $u(t)$ is a continuous curve in $\mathscr{H}$ and either $T=+\infty$ or $\|u(t)\|_{\mathscr{H}} \underset{t\to T}{\longrightarrow} \infty$.*

Furthermore if $u_0\in D_{\mathscr{A}}$ and satisfies the constraint $\mathscr{C}(u_0)=0$ then the solution curve $u(t)$ remains in $D_{\mathscr{A}}$, has a first derivative $\frac{du}{dt}(t)$ which is continuous in $\mathscr{H}$, and satisfies the differential equation (2.15), and the constraint $\mathscr{C}(u(t))=0$ throughout its interval of existence $[0,T)$. Again, either $T=\infty$ or $\|u(t)\|_{\mathscr{H}} \underset{t\to T}{\longrightarrow} \infty$.

C. Smoothness of Solutions

The solutions discussed above have $u(t)\in D_{\mathscr{A}}$ and $\frac{du}{dt}(t)\in\mathscr{H}=(H_{s+1}\times H_s)^2$ throughout their intervals of existence. Because of the constraint however, they are actually somewhat smoother than this. As shown in the appendix (see Lemmas A5 and A6) the constraint equation

$$E_i^L=\partial_i\left\{-\frac{1}{4\pi r}*([E_j,A_j]-(\pi\cdot\theta_a\phi)\theta_a)\right\} \tag{2.43}$$

forces E^L to satisfy

(i) $$E^L\in H_2\,\forall(A,E,\phi,\pi)\in(H_2\times H_1)^2 \tag{2.44}$$

and

(ii) $$E^L\in H_{s+2}\,\forall(A,E,\phi,\pi)\in(H_{s+1}\times H_{s+1})^2 \tag{2.45}$$

for $s\geqq 1$. From (i) and (ii) it follows that any $u\in D_{\mathscr{A}}$ has $E^L\in H_{s+1}$ if the constraint is satisfied. But this means that $(A,E,\phi,\pi)\in(H_{s+1}\times H_{s+1})^2$ (with ϕ and A^T actually in H_{s+2}) and thus from (ii) that $E^L\in H_{s+2}$. The constraint thus forces E^L to have two more orders of differentiability than inclusion in $D_{\mathscr{A}}$ requires.

The longitudinal projection of the equations of motion gives

$$\frac{dA^L}{dt}(t) = E^L(t) \in H_{s+2}, \tag{2.46}$$

which shows that if we further restrict the initial data so that $A^L(0) \in H_{s+2}$ then $A^L(t)$ will persist in this space throughout the interval of existence. We thus find that if the initial data is restricted to lie in the natural "subdomain" $\tilde{D}_{\mathscr{A}} \subset D_{\mathscr{A}}$ defined by

$$\tilde{D}_{\mathscr{A}} = \{(A, E, \phi, \pi) \in (H_{s+2} \times H_{s+1})^2\}, \tag{2.47}$$

then $u(t)$ will persist in this space throughout its interval of existence and in fact have E^L lying in H_{s+2} by virtue of the constraint.

We have thus proven:

Corollary 2. *If $u_0 = (A, E, \phi, \pi)$ is any initial data lying in $\tilde{D}_{\mathscr{A}} = (H_{s+2} \times H_{s+1})^2$ for $s \geqq 1$ which satisfies the constraint $\mathscr{C}(u_0) = 0$ then there is a unique, once continuously differentiable solution $u(t)$ of the differential equation (2.33) and the constraint $\mathscr{C}(u(t)) = 0$ having $u(0) = u_0$. This solution has $u(t) \in \tilde{D}_{\mathscr{A}}$ and $\frac{du}{dt}(t) \in \mathscr{H} = (H_{s+1} \times H_s)^2$ throughout its interval of existence $[0, T)$ and either $T = +\infty$ or $\lim_{t \to T} \| u(t) \|_{(H_{s+1} \times H_s)^2} = \infty$. Here continuously differentiable means that $u(t)$ and $\frac{du}{dt}(t)$ are continuous curves in $\mathscr{H}$.*

We can combine this result with Segal's general smoothness theorem [3] to show that if the initial data is restricted to lie in $(H_{s+1+k} \times H_{s+k})^2$ with k an integer $\geqq 1$, then the solutions obtained will in fact be k times continuously differentiable with

$$u(t) \in (H_{s+1+k} \times H_{s+k})^2, \frac{du}{dt}(t) \in (H_{s+k} \times H_{s+k-1})^2,$$

$$\ldots, \frac{d^{k-1}u}{dt^{k-1}}(t) \in (H_{s+2} \times H_{s+1})^2, \frac{d^k u}{dt^k}(t) \in (H_{s+1} \times H_s)^2 \tag{2.48}$$

throughout their intervals of existence.

First note that for any integer $j \geqq 1$ we have, for any $v \in (H_{s+1+j} \times H_{s+j})^2$, that

$$\mathscr{A} v \in (H_{s+j} \times H_{s+j-1})^2. \tag{2.49}$$

Furthermore, since the non-linear term $J(u)$ is a smooth map from $(H_{s+1} \times H_s)^2$ to itself for all $s \geqq 1$, we find, computing successive derivatives of the equation of motion,

$$\frac{du}{dt} = \mathscr{A} u + J(u), \tag{2.50}$$

$$\frac{d^2 u}{du^2} = \mathscr{A} \frac{du}{dt} + DJ(u) \cdot \frac{du}{dt},$$

$$\frac{d^3 u}{dt^3} = \mathscr{A} \frac{d^2 u}{dt^2} + DJ(u) \cdot \frac{d^2 u}{dt^2} + D^2 J(u) \left(\frac{du}{dt}, \frac{du}{dt} \right),$$

etc.,

that if $u(0)=u_0$ lies in $(H_{s+1+k}\times H_{s+k})^2$ then the successive derivatives $u_0'=\frac{du}{dt}(0)$, $u_0''=\frac{d^2u}{dt^2}(0)$, etc., computed formally from the above, satisfy

$$\begin{aligned}&u_0'\in(H_{s+k}\times H_{s+k-1})^2, u_0''\in(H_{s+k-1}\times H_{s+k-2})^2,\\ &\quad\ldots, u_0^{(k-1)}\in(H_{s+2}\times H_{s+1})^2, u_0^{(k)}\in(H_{s+1}\times H_s)^2.\end{aligned} \tag{2.51}$$

Thus, in particular the first $(k-1)$ derivatives $(u_0',\ldots,u_0^{(k-1)})$ lie in $(H_{s+2}\times H_{s+1})^2=\tilde{D}_{\mathscr{A}}\subset D_{\mathscr{A}}$.

It follows from Segal's general smoothness theorem that the solution $u(t)$ generated by u_0 will be k times continuously differentiable[2] with $u(t), \frac{du}{dt}(t),\ldots, \frac{d^{k-1}u}{dt^{k-1}}(t)$ all lying in $D_{\mathscr{A}}$ and $\frac{d^ku}{dt^k}(t)$ lying in $\mathscr{H}$ throughout its interval of existence. In fact, the solution will be somewhat smoother than this as the following argument shows.

Since for any $k\geqq 1$ u_0 lies in $(H_{s+2}\times H_{s+1})^2\subset D_{\mathscr{A}}$ it follows from Segal's theorem that $u(t)\in D_{\mathscr{A}}$ throughout the interval of existence $[0,T)$. But the argument preceding Corollary (2) shows that in fact $u(t)\in\tilde{D}_{\mathscr{A}}=(H_{s+2}\times H_{s+1})^2$ on $[0,T)$. If $k\geqq 2$ then we have in addition that

$$\begin{aligned}&u_0\in(H_{s+3}\times H_{s+2})^2\subset D_{\mathscr{A}},\\ &u_0'\in(H_{s+2}\times H_{s+1})^2\subset D_{\mathscr{A}},\end{aligned} \tag{2.52}$$

and the general theory gives

$$u(t)\in D_{\mathscr{A}},\quad \frac{du}{dt}(t)\in D_{\mathscr{A}} \tag{2.53}$$

on $[0,T)$. But we already know that $u(t)\in(H_{s+2}\times H_{s+1})^2$ on $[0,T)$ and thus (since J is a smooth map) that $J(u(t))\in(H_{s+2}\times H_{s+1})^2$ on $[0,T)$. It follows that the linear term, $\mathscr{A}u(t)$, must separately lie in $D_{\mathscr{A}}$ and thus that $u(t)$ must lie in $D_{\mathscr{A}^2}$ on $[0,T)$. It follows that

$$A^T\in H_{s+3}, E^T\in H_{s+2}, \phi\in H_{s+3}, \pi\in H_{s+2}, A^L\in H_{s+2}, E^L\in H_{s+1} \tag{2.54}$$

on $[0,T)$. However the constraint forces $E^L(t)$ to lie in H_{s+3} and since $A^L(0)\in H_{s+3}$, the equation of motion $\frac{dA^L}{dt}=E^L$ shows that $A^L(t)\in H_{s+3}$ on $[0,T)$. Therefore $u(t)\in(H_{s+3}\times H_{s+2})^2$ and (from the equation of motion) $\frac{du}{dt}(t)\in(H_{s+2}\times H_{s+1})^2$ on $[0,T)$.

We can continue this argument if $k\geqq 3$. The general theory gives

$$u(t)\in D_{\mathscr{A}},\quad \frac{du}{dt}(t)\in D_{\mathscr{A}},\quad \frac{d^2u}{dt^2}(t)\in D_{\mathscr{A}} \tag{2.55}$$

2 C^k as a map from $[0,T)$ to $\mathscr{H}$. See however Corollary (4) below

on $[0, T)$. But the above argument has already shown that $u(t) \in (H_{s+3} \times H_{s+2})^2$ and $\frac{du}{dt}(t) \in (H_{s+2} \times H_{s+1})^2$ on $[0, T)$. It follows that $DJ(u(t)).\ \frac{du}{dt}(t) \in (H_{s+2} \times H_{s+1})^2 \subset D_{\mathscr{A}}$ and thus that the linear term $\mathscr{A}\frac{du}{dt}(t)$ in the $\frac{d^2u}{dt^2}(t)$ equation must lie in $D_{\mathscr{A}}$. However, since $\mathscr{A}J(u(t)) \in (H_{s+2} \times H_{s+1})^2 \subset D_{\mathscr{A}}$ it follows that $\mathscr{A}^2 u(t)$ must separately lie in $D_{\mathscr{A}}$ or that $u(t)$ must be in $D_{\mathscr{A}^3}$ on $[0, T)$. These conclusions imply that

$$A^T \in H_{s+4}, E^T \in H_{s+3}, \phi \in H_{s+4}, \pi \in H_{s+3}, A^L \in H_{s+3}, E^L \in H_{s+2}. \tag{2.56}$$

But the constraint forces $E^L(t)$ to lie in H_{s+4} and since $A^L(0) \in H_{s+4}$, the equation of motion gives $A^L(t) \in H_{s+4}$ on $[0, T)$. Therefore we get

$$\begin{aligned} &u(t) \in (H_{s+4} \times H_{s+3})^2, \frac{du}{dt}(t) \in (H_{s+3} \times H_{s+2})^2, \\ &\frac{d^2u}{dt}(t) \in (H_{s+2} \times H_{s+1})^2, \frac{d^3u}{dt^3} \in (H_{s+1} \times H_s)^2 \end{aligned} \tag{2.57}$$

throughout the interval of existence.

Clearly one can continue this pattern of argument to successively larger values of k, thereby deriving the smoothness theorem:

Theorem 3. *If $u_0 \in (H_{s+1+k} \times H_{s+k})^2$ for integers $s \geqq 1$ and $k \geqq 1$ and $\mathscr{C}(u_0) = 0$, then the solution $u(t)$ generated by u_0 is k times continuously differentiable as a curve in $\mathscr{H} = (H_{s+1} \times H_s)^2$ and has*

$$\begin{aligned} &u(t) \in (H_{s+1+k} \times H_{s+k})^2, \frac{du}{dt}(t) \in (H_{s+k} \times H_{s+k-1})^2, \ldots, \\ &\frac{d^{k-1}u}{dt^{k-1}}(t) \in (H_{s+2} \times H_{s+1})^2, \frac{d^k u}{dt^k}(t) \in (H_{s+1} \times H_s)^2 \end{aligned}$$

throughout the interval of existence $[0, T)$.

The continuity referred to in Theorem (3) is continuity in the topology of $\mathscr{H}$. However, we can easily show that $u(t)$ and its derivatives are actually continuous as curves in the spaces indicated in Theorem (3). To see this, note that a solution having

$$\begin{aligned} &u(t) \in (H_{s+1+k} \times H_{s+k})^2 \equiv \mathscr{H}^{(k)}, \\ &\frac{du}{dt}(t) \in (H_{s+k} \times H_{s+k-1})^2 \equiv \mathscr{H}^{(k-1)}, \\ &\vdots \\ &\frac{d^k u}{dt^k}(t) \in (H_{s+1} \times H_s)^2 \equiv \mathscr{H}^{(0)} = \mathscr{H}, \end{aligned} \tag{2.58}$$

can be viewed as a C^1 solution in $\mathscr{H}^{(k-1)}$, a C^2 solution in $\mathscr{H}^{(k-2)}, \ldots,$ or as a C^k

solution in $\mathscr{H}$. Applying Theorem (3) we see that $\frac{d^j u}{dt^j}(t)$ is continuous in $\mathscr{H}^{(k-j)}$ for $1 \leqq j \leqq k$. As for $u(t)$ itself, it satisfies the integral equation in $\mathscr{H}^{(k)}$ and so, by Segal's basic existence theorem, is a continuous curve in $\mathscr{H}^{(k)}$. We thus have:

Corollary 4. *If $u_0 \in \mathscr{H}^{(k)}$ satisfies the constraint $\mathscr{C}(u_o) = 0$, then the solution $u(t)$ generated by u_0 is a continuous curve in $\mathscr{H}^{(k)}$ with its derivative $\frac{d^j u}{dt^j}(t)$ continuous in $\mathscr{H}^{(k-j)}$ for $1 \leqq j \leqq k$.*

Finally we complete the local existence and smoothness argument by showing that if $k \geqq 2$ then solutions $u(t) \in (H_{s+1+k} \times H_{s+k})^2$ provide C^k potentials (A, ϕ) and C^{k-1} momenta (E, π) on spacetime which satisfy the Yang–Mills–Higgs equations in the classical sense.

We first prove several lemmas.

Lemma 5. *Let $f(t)$ be a continuous curve in $H_{2+j}(R^3)$ for some integer $j \geqq 0$ and t in some open interval I. Then f together with its spatial derivatives up to order j are continuous functions on $R^3 \times I$.*

Proof. That f is a C^j function in the spatial variables for each fixed t follows from the Sobolev embedding lemma for R^3. We may therefore write $f(t,x), \partial_i f(t,x)$, etc. for f and its derivatives up to order j. If $g(t,x)$ is any one of these functions, we get

$$\begin{aligned} |g(t+\varepsilon, x+\delta x) - g(t,x)| &\leqq |g(t+\varepsilon, x+\delta x) - g(t, x+\delta x)| + |g(t, x+\delta x) - g(t,x)| \\ &\leqq \|g(t+\varepsilon) - g(t)\|_{L^\infty} + |g(t, x+\delta x) - g(t,x)| \\ &\leqq C\|g(t+\varepsilon) - g(t)\|_{H_2} + |g(t, x+\delta x) - g(t,x)|. \end{aligned} \tag{2.59}$$

The first term can be made less than any $\delta/2 > 0$ by choosing ε sufficiently small since $g(t)$ is a continuous curve in H_2. The second term can be made less than $\delta/2$ by choosing δx sufficiently small since $g(t,x)$ is continuous in x at fixed t. ■

Lemma 6. *Suppose that $f(t)$ is a continuous curve in H_{2+j} for $t \in I$ and that the (strong) derivatives*

$$\frac{df}{dt}(t) = V^{(1)}, \frac{d^2 f}{dt^2}(t) = V^{(2)}, \ldots, \frac{d^j f}{dt^j}(t) = V^{(j)} \tag{2.60}$$

all exist as continuous curves in (respectively) $H_{j+1}, H_j, \ldots, H_2$. Then f is a C^j function on $R^3 \times I$ and, in particular, $\frac{\partial^j f}{\partial t^j} = V^{(j)}$.

Proof. We first show that the partial time derivatives exist. Note that

$$\begin{aligned} \left| \frac{f(t+\varepsilon, x) - f(t,x)}{\varepsilon} - V^{(1)}(t,x) \right| &\leqq \left\| \frac{f(t+\varepsilon) - f(t)}{\varepsilon} - V^{(1)}(t) \right\|_{L^\infty} \\ &\leqq C \left\| \frac{f(t+\varepsilon) - f(t)}{\varepsilon} - V^{(1)}(t) \right\|_{H_2} \underset{\varepsilon \to 0}{\longrightarrow} 0. \end{aligned} \tag{2.61}$$

Thus $\frac{\partial f}{\partial t} = V^{(1)}$. In a similar way we get for $i<j$

$$\left| \frac{V^{(i)}(t+\varepsilon, x) - V^{(i)}(t,x)}{\varepsilon} - V^{(i+1)}(t,x) \right| \leqq C \left\| \frac{V^{(i)}(t+\varepsilon) - V^{(i)}(t)}{\varepsilon} - V^{(i+1)}(t) \right\|_{H_2} \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{2.62}$$

and thus that $\frac{\partial V^{(i)}}{\partial t} = \frac{\partial^{i+1} f}{\partial t^{i+1}} = V^{(i+1)}$.

If follows from Lemma (5) that each $V^{(i)} = \frac{\partial^i f}{\partial t^i}$ is a C^{j-i} function in the spatial variables on $R^3 \times I$. A standard calculus argument shows that continuity of $\partial_t g, \partial_i g$ and $\partial_i \partial_t g$ implies $\partial_i \partial_t g = \partial_t \partial_i g$. Proceeding inductively one sees that the order of differentiation is immaterial and thus that f is a C^j function on the spacetime region $R^3 \times I$. ∎

Recalling Theorem (3) and Corollary (4) we see that any initial data $u_0 \in (H_{s+1+k} \times H_{s+k})^2$ for $s \geqq 1, k \geqq 1$ generates a solution $u(t)$ having

$$\begin{aligned} &(A(t), \phi(t)) \in (H_{s+1+k})^2, \left(\frac{dA}{dt}(t), \frac{d\phi}{dt}(t) \right) \in (H_{s+k})^2, \\ &\quad \ldots, \left(\frac{d^k A}{dt^k}(t), \frac{d^k \phi}{dt^k}(t) \right) \in (H_{s+1})^2, \\ &(E(t), \pi(t)) \in (H_{s+k})^2, \left(\frac{dE}{dt}(t), \frac{d\pi}{dt}(t) \right) \in (H_{s+k-1})^2, \\ &\quad \ldots, \left(\frac{d^{k-1} E}{dt^{k-1}}, \frac{d^{k-1} \pi}{dt^{k-1}} \right) \in (H_{s+1})^2, \end{aligned} \tag{2.63}$$

each as continuous curves in the indicated spaces. Since $s \geqq 1$, Lemma (6) applies and show that (A, ϕ) are C^k functions and that (E, π) are C^{k-1} functions on their domains of definition in spacetime. It follows that if $k \geqq 2$ the solutions thus obtained satisfy the Yang–Mills–Higgs equations in the classical sense. We have thus proven

Corollary 7. *If $u_0 \in (H_{s+1+k} \times H_{s+k})^2$ for $s \geqq 1, k \geqq 2$ is initial data satisfying the constraint $\mathscr{C}(u_0) = 0$, then the solution $u(t)$ generated by u_0 has C^k potentials (A, ϕ) and C^{k-1} momenta (E, π) which satisfy the Yang–Mills–Higgs equations in the classical sense throughout their domain of definition.*

Appendix: Properties of E^C

For any $(A, E, \phi, \pi) \in (H_{s+1} \times H_s)^2$ with $s \geqq 1$ we let

$$\rho = \frac{1}{4\pi} \{ [E_j, A_j] - (\pi \cdot \theta_a \phi) \theta_a \}, \tag{A.1}$$

and attempt to define

$$E_i^C = -\partial_i\left(\frac{1}{r}*\rho\right) = -\partial_i \int_{R^3} dx' \frac{\rho(x')}{|x-x'|}. \tag{A.2}$$

Lemma A1. *The convolution* $\frac{1}{r}*\rho$ *is a well defined tempered distribution lying in* L^s *for all* $s>3$.

Proof. Using the fact that $\frac{1}{r}$ lies in the weak L^p space $L_w^3(R^3)$ it follows from the generalized Young inequality (see ref. (15), Sect. (IX.4)) that $\frac{1}{r}*\rho \in L^s(R^3)$ provided $\rho \in L^p(R^3)$ and $1<p,s<\infty$, where $\frac{1}{p}=\frac{1}{s}+\frac{2}{3}$. In that case

$$\left\|\frac{1}{r}*\rho\right\|_{L^s} \leqq C\|\rho\|_{L^p}\left\|\frac{1}{r}\right\|_{3,w} \leqq C'\|\rho\|_{L^p}. \tag{A.3}$$

However, since (A, E, ϕ, π) all lie (at least) in H_1 they all lie in $L^{p'}$ for $2 \leqq p' \leqq 6$. It follows that $\rho \in L^p$ for $1 \leqq p \leqq 3$. Hence $\frac{1}{r}*\rho \in L^s$ for $s>3$.

The continuity property required of distributions follows from the estimate

$$\begin{aligned}\left\langle f, \frac{1}{r}*\rho\right\rangle = \int_{R^3} dx\left(f\left(\frac{1}{r}*\rho\right)\right) &\leqq \|f\|_{L^{s'}}\left\|\frac{1}{r}*\rho\right\|_{L^s} \\ &\leqq \|f\|_{L^\infty}^{1-1/s'}\|f\|_{L^1}^{1/s'}\left\|\frac{1}{r}*\rho\right\|_{L^s} \\ &\leqq C\|f\|_{L^\infty}^{1-1/s'}(\|(1+x^2)^2|f|\|_{L^\infty})^{1/s'}\left\|\frac{1}{r}*\rho\right\|_{L^s}\end{aligned} \tag{A.4}$$

where f is any C^∞ function of rapid decrease and where $\frac{1}{s}+\frac{1}{s'}=1$ with $s>3$. The last step follows from

$$\begin{aligned}\|f\|_{L^1} &= \int_{R^3} dx((1+x^2)^{-2}((1+x^2)^2|f|)) \\ &\leqq C\|(1+x^2)^2|f|\|_{L^\infty}. \quad \blacksquare\end{aligned} \tag{A.5}$$

We define E^C to be the (distributional) gradient of $\left(-\frac{1}{r}*\rho\right)$. An alternative representation of E^C is given by

$$E_i^C = \left(\frac{\hat{r}}{r^2}*\rho\right)_i = \int_{R^3} dx'\left(\frac{(x^i-x^{i'})}{|x-x'|^3}\rho(x')\right). \tag{A.6}$$

To prove this let V_i be any vector field in $\mathscr{S}(R^3)=$ the Schwartz space of C^∞

functions of rapid decrease. Then we have

$$\begin{aligned}\langle V_i, E_i^C\rangle &= \left\langle \partial_i V_i, \frac{1}{r} * \rho \right\rangle \\ &= \left\langle \frac{1}{r} * \partial_i V_i, \rho \right\rangle, \end{aligned} \tag{A.7}$$

where we have appealed to Fubini's theorem to exchange the order of integration. Integrating by parts one can show that

$$\int_{R^3} dx \left[\frac{1}{|x - x'|} \partial_i V_i(x) \right] = \int_{R^3} dx \frac{(x^i - x^{i'})}{|x - x'|^3} V_i(x). \tag{A.8}$$

Using the facts that $\left|\frac{\hat{r}}{r^2}\right| \in L_w^{3/2}(R^3)$ and that $\rho \in L_p$ for all $1 \leqq p \leqq 3$, we find from the generalized Young inequality that

$$\left|\frac{\hat{r}}{r^2}\right| * |\rho| \in L^s \text{ for all } \tfrac{3}{2} < s < 3. \tag{A.9}$$

This fact together with Fubini's theorem allows us again to exchange the order of integration in (A.7) to obtain

$$\left\langle V_i, \partial_i\left(-\frac{1}{r} * \rho \right) \right\rangle = \left\langle V_i, \frac{\hat{r}}{r^2} * \rho \right\rangle, \tag{A.10}$$

which establishes

Lemma A2. $E_i^C = \left(\frac{\hat{r}}{r^2} * \rho\right)_i = -\partial_i\left(\frac{1}{r} * \rho\right)$.

We may use this representation of E^C to prove

Lemma A3. *$E^C \in L^s$ for all $\frac{3}{2} < s < \infty$. In particular $E^C \in L^2$.*

Proof. From the generalized Young inequality

$$\left\| \frac{\hat{r}}{r^2} * \rho \right\|_{L^s} \leqq c \|\rho\|_{L^p} \left\| \frac{\hat{r}}{r^2} \right\|_{3/2,w} \leqq c' \|\rho\|_{L^p} \tag{A.11}$$

provided $1 < p, s < \infty$ and $\frac{1}{p} = \frac{1}{s} + \frac{1}{3}$. But we showed previously that $\rho \in L^p$ for all $1 \leqq p \leqq 3$. Therefore the conclusion follows. ■

Furthermore we have

Lemma A4. *E^C has vanishing curl and satisfies $\partial_i E_i^C = 4\pi\rho$.*

Proof. That E^C has vanishing (distributional) curl follows immediately from the fact that E^C is a gradient. To evaluate $\partial_i E_i^C$, we let f be an arbitrary element of

$\mathscr{S}(R^3)$ and compute

$$\begin{aligned}\langle f, \partial_i E_i^C \rangle &= -\langle \partial_i f, E_i^C \rangle \\ &= -\left\langle \partial_i f, \left(\frac{\hat{r}}{r^2} * \rho\right)_i \right\rangle \\ &= -\int_{R^3} dx' \left\{ \rho(x') \int_{R^3} dx \left(\frac{(x^i - x^{i'})}{|x - x'|^3} \partial_i f(x) \right) \right\}, \end{aligned} \tag{A.12}$$

where we have used Fubini's theorem to justify the last step. Integrating by parts one shows that

$$\int_{R^3} dx \frac{(x^i - x^{i'})}{|x - x'|^3} \partial_i f(x) = -4\pi f(x'), \tag{A.13}$$

and thus that

$$\begin{aligned}\langle f, \partial_i E_i^C \rangle &= \int_{R^3} dx' (\rho(x') 4\pi f(x')) \\ &= \langle f, 4\pi\rho \rangle. \end{aligned} \tag{A.14}$$

Thus, as a distribution E^C obeys $\partial_i E_i^C = 4\pi\rho$. ■

If V is any vector field in L^2 we can define its Fourier transform $\hat{V} = \mathscr{F}(V) \in L^2$, and decompose

$$\hat{V} = \hat{V}^T + \hat{V}^L, \tag{A.15}$$

where the transverse and longitudinal summands are given by

$$\begin{aligned}(\hat{V}^T)_j &= \left(\delta_{ij} - \frac{k_j k_i}{|k|^2} \right) \hat{V}_i, \\ (\hat{V}^L)_j &= \frac{k_j k_i}{|k|^2} \hat{V}_i. \end{aligned} \tag{A.16}$$

It is straightforward to show that

$$\|\hat{V}^T\|_{L^2} \leqq \|\hat{V}\|_{L^2}, \|\hat{V}^L\|_{L^2} \leqq \|\hat{V}\|_{L^2}, \tag{A.17}$$

and that

$$\hat{V}_j^T \hat{V}_j^L = 0, k_j \hat{V}_j^T = 0, \varepsilon^{ijk} k_j \hat{V}_k^L = 0. \tag{A.18}$$

It follows that the inverse transforms V^T and V^L satisfy

$$\begin{aligned}&\|V^T\|_{L^2} \leqq \|V\|_{L^2}, \|V^L\|_{L^2} \leqq \|V\|_{L^2}, \\ &\int_{R^3} V_j^T V_j^L = 0, \nabla \times V^L = 0, \nabla \cdot V^T = 0. \end{aligned} \tag{A.19}$$

In particular since $E^C \in L^2$ we can decompose E^C in this way. However since E^C has vanishing curl it follows that $k \times \hat{E}^C = 0$ which implies that $(\hat{E}^C)^T = 0$. Thus E^C is purely longitudinal and we may write

$$\hat{E}_j^C = \frac{k_j k_i}{|k|^2} \hat{E}_i^C = (\hat{E}^C)_j^L. \tag{A.20}$$

Furthermore since $\partial_i E_i = 4\pi\rho$ we get, taking the Fourier transform,

$$ik_i \hat{E}_i^C = 4\pi\hat{\rho}, \tag{A.21}$$

and thus that

$$\hat{E}_j^C = \frac{k_j}{|k|^2}(-i4\pi\hat{\rho}). \tag{A.22}$$

We can decompose the electric field E as above to get

$$E = E^T + E^L, \tag{A.23}$$

where

$$\partial_i E_i^T = 0, \nabla \times E^L = 0 \tag{A.24}$$

and

$$k_i \hat{E}_i = k_i \hat{E}_i^L. \tag{A.25}$$

Now suppose that (A, E, ϕ, π) satisfies the constraint equation

$$\partial_i E_i = \partial_i E_i^L = 4\pi\rho. \tag{A.26}$$

Taking the Fourier transform we get

$$ik_i \hat{E}_i^L = 4\pi\hat{\rho}, \tag{A.27}$$

and thus

$$\begin{aligned}\hat{E}_j^L = \frac{k_j k_i}{|k|^2}\hat{E}_i^L = \frac{k_j}{|k|^2}(-i4\pi\hat{\rho}) \\ = \hat{E}_j^C.\end{aligned} \tag{A.28}$$

Thus any solution of the constraints has $E^L = E^C$. Conversely if $E^L = E^C$, then the constraint is satisfied since (from Lemma (A4)) we have

$$\partial_i E_i = \partial_i E_i^L = \partial_i E_i^C = 4\pi\rho. \tag{A.29}$$

This gives:

Lemma A5. *The constraint equation* $\partial_i E_i = 4\pi\rho$ *is equivalent to* $E^C = E^L$.

We can now prove

Lemma A6. E^C *lies in* $H_2 \forall (A, E, \phi, \pi) \in (H_2 \times H_1)^2$ *and* E^C *lies in* $H_{s+2} \forall (A, E, \phi, \pi) \in (H_{s+1} \times H_{s+1})^2 \forall s \geqq 1$.

Proof. We already know that $E^C \in L^2$. Using the expression (A.22) for $\hat{E}^C$ we get

$$\int_{R^3} dx (\partial_i E_j^C)^2 = \int_{R^3} dk |k_i \hat{E}_j^C|^2 = (4\pi)^2 \int_{R^3} dk |\hat{\rho}|^2 = (4\pi)^2 \int_{R^3} dx (\rho)^2. \tag{A.30}$$

Similarly

$$\int_{R^3} dx (\partial_i \partial_j E_k^C)^2 = (4\pi)^2 \int_{R^3} dx (\partial_j \rho)^2. \tag{A.31}$$

From Lemma (A3) we have, taking $p = \frac{6}{5}$, that

$$\|E^C\|_{L^2} \leqq C' \|\rho\|_{L^{6/5}} \tag{A.32}$$

and, using Hölders' inequality, that

$$\int_{R^3} |\rho|^{6/5} dx \leqq K\left\{\left(\int_{R^3} |E|^2\right)^{3/5}\left(\int_{R^3} |A|^3\right)^{2/5} + \left(\int_{R^3} |\pi|^2\right)^{3/5}\left(\int_{R^3} |\phi|^3\right)^{2/5}\right\}, \tag{A.33}$$

and thus that

$$\begin{aligned}\|\rho\|_{L^{6/5}} &\leqq C(\|E\|_{L^2}\|A\|_{L^3} + \|\pi\|_{L^2}\|\phi\|_{L^3}) \\ &\leqq C'(\|E\|_{L^2}\|A\|_{H_1} + \|\pi\|_{L^2}\|\phi\|_{H_1}).\end{aligned} \tag{A.34}$$

In the last step we have used the Sobolev estimate

$$\|f\|_{L^p} < K'\|Df\|_{L^2}^{a}\|f\|_{L^q}^{1-a} \tag{A.35}$$

for $\frac{1}{p} = \frac{a}{6} + (1-a)\frac{1}{q}$ (with $0 \leqq a \leqq 1$) to show that $\|A\|_{L^3} \leqq K\|A\|_{H_1}$.
Using the similar estimate $\|f\|_{L^4} \leqq K\|f\|_{H_1}$ together with $\|f\|_{L^\infty} \leqq K\|f\|_{H_2}$ one shows that

$$\int_{R^3} dx(\partial_j\rho)^2 \leqq C\{\|A\|_{H_2}^2\|E\|_{H_1}^2 + \|\phi\|_{H_2}^2\|\pi\|_{H_1}^2\}. \tag{A.36}$$

Combining these results one gets

$$\|E^C\|_{H_2} \leqq K\{\|E\|_{H_1}\|A\|_{H_2} + \|\pi\|_{H_1}\|\phi\|_{H_2}\}, \tag{A.37}$$

which proves the first part of the lemma.

Now suppose that $(A, E, \phi, \pi) \in (H_{s+1} \times H_{s+1})^2$ for $s \geqq 1$. Then from the Schauder ring property of H_s maps we have $\rho \in H_{s+1}$. The H_{s+2} norm of E^C may be computed via

$$\begin{aligned}\|E^C\|_{H_{s+2}}^2 &= \int_{R^3} dk(1+k^2)^{s+2}\hat{E}^C\cdot\hat{E}^C \\ &\leqq \int_{R^3} dk\{[(1+k^2)^2 + Ck^2(1+k^2)^{s+1}](\hat{E}^C\cdot\hat{E}^C)\} \\ &\leqq \|E^C\|_{H_2}^2 + C'\int_{R^3} dk(1+k^2)^{s+1}|\hat{\rho}|^2 \\ &\leqq \|E^C\|_{H_2}^2 + C'\|\rho\|_{H_{s+1}}^2.\end{aligned} \tag{A.38}$$

But the Schauder ring property gives the estimate

$$\|\rho\|_{H_{s+1}} \leqq K\{\|E\|_{H_{s+1}}\|A\|_{H_{s+1}} + \|\pi\|_{H_{s+1}}\|\phi\|_{H_{s+1}}\}. \tag{A.39}$$

Thus for $s \geqq 1$ we get

$$\|E^C\|_{H_{s+2}} \leqq K'\{\|E\|_{H_{s+1}} \times \|A\|_{H_{s+1}} + \|\pi\|_{H_{s+1}} \times \|\phi\|_{H_{s+1}}\}, \tag{A.40}$$

which proves the second part of the lemma. ■

The result of Lemma (A6) is particularly useful in verifying the Lipshitz condition and the smoothness of E^C. To simplify the formulas slightly we shall consider only the terms in (A, E) in the expression for E^C since these are completely representative. We have

$$E^C(A',E') - E^C(A,E) = \frac{1}{4\pi}\frac{\hat{r}}{r^2} * \{[(E'-E)\cdot, A'] - [E\cdot, (A-A')]\}, \qquad (A.41)$$

and thus, recalling the estimate in the proof of Lemma (A6), that

$$\begin{aligned} \|E^C(A',E') - E^C(A,E)\|_{H_{s+1}} &\leqq K\{\|E'-E\|_{H_s}\|A'\|_{H_{s+1}} + \|E\|_{H_s}\|A-A'\|_{H_{s+1}}\} \\ &\leqq K(\|(A',E')\| + \|(A,E)\|)\|(A'-A, E'-E)\|, \qquad (A.42) \end{aligned}$$

where $\| \ \|$ designates the $H_{s+1} \times H_s$ norm for any $s \geqq 1$. This is the needed Lipshitz condition for E^C. The continuity of E^C as a map from $H_{s+1} \times H_s$ to H_{s+1} follows from the Lipshitz property.

Using the estimate (A.40) again it is straightforward to show that

$$\begin{aligned} &\left\| E^C(A+A', E+E') - E^C(A,E) - \frac{1}{4\pi}\frac{\hat{r}}{r^2} * ([E'\cdot, A] + [E\cdot, A']) \right\|_{H_{s+1}} \\ &= \left\| \frac{1}{4\pi}\frac{\hat{r}}{r^2} * ([E'\cdot, A']) \right\|_{H_{s+1}} \leqq K\|E'\|_{H_s}\|A'\|_{H_{s+1}} \\ &\leqq K(\|(A',E')\|_{H_{s+1}\times H_s})^2 \leqq \varepsilon\|(A',E')\|_{H_{s+1}\times H_s} \qquad (A.43) \end{aligned}$$

for all $\|(A',E')\|_{H_{s+1}\times H_s} \leqq \frac{\varepsilon}{K}$. Thus the first Fréchet derivative of E^C exists and is given by

$$DE^C(A,E)\cdot(A',E') = \frac{1}{4\pi}\frac{\hat{r}}{r^2} * ([E'\cdot, A] + [E\cdot, A']). \qquad (A.44)$$

Similarly we get

$$\begin{aligned} &\|DE^C(A,E)\cdot(A',E') - DE^C(A^0,E^0)\cdot(A',E')\|_{H_{s+1}} \\ &= \left\| \frac{1}{4\pi}\frac{\hat{r}}{r^2} * \{[E'\cdot, (A-A^0)] + [(E-E^0)\cdot, A']\} \right\|_{H_{s+1}} \\ &\leqq K\{\|E'\|_{H_s}\|A-A^0\|_{H_{s+1}} + \|E-E^0\|_{H_s}\|A'\|_{H_{s+1}}\}, \qquad (A.45) \end{aligned}$$

which shows that DE^C is continuous.

The second derivative is easily shows to be

$$D^2E^C(A,E)\cdot((A',E'),(A'',E'')) = \frac{1}{4\pi}\frac{\hat{r}}{r^2} * ([E'\cdot, A''] + [E''\cdot, A']), \qquad (A.46)$$

which is constant in (A, E) and thus obviously continuous. All higher derivatives vanish identically. Thus we have proven

Lemma A7. *E^C is a C^∞, Lipshitz map from $H_{s+1} \times H_s$ to H_{s+1} for all $s \geqq 1$.*

To justify the steps in (2.39) we note that

$$\partial_i \Delta E_i^L = \partial_i E_i^L - \partial_i E_i^C = \partial_i E_i - 4\pi\rho = -\mathscr{C}, \qquad (A.47)$$

which follows from Lemma (A4) and the properties of the decomposition (A.23).

These also give

$$\Delta\hat{E}_j^L = \hat{E}_j^L - \hat{E}_j^C = \frac{k_j k_i}{|k|^2}(\hat{E}_i^L - \hat{E}_i^C). \tag{A.48}$$

It follows that

$$\begin{aligned}\int_{R^3} dx \sum_{k,j} (\partial_k(\Delta E^L)_j)^2 &= \int_{R^3} dk \left| \frac{k_k k_j k_i}{|k|^2}(\hat{E}_i^L - \hat{E}_i^C) \right|^2 = \int_{R^3} dk |k_i(\hat{E}_i^L - \hat{E}_i^C)|^2 \\ &= \int_{R^3} dx(\partial_i \Delta E_i^L)^2 = \int_{R^3} \mathscr{C}\cdot\mathscr{C}\, dx,\end{aligned} \tag{A.49}$$

which completes the argument.

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Communicated by A. Jaffe

Received April 20, 1981; in revised form June 19, 1981