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Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric σ-Model

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Abstract. A complete geometrical classification of supersymmetric σ-models is given. Extended supersymmetry requires covariantly constant complex structures, and Kahler and hyperkahler manifolds play a special role. As an application of the classification, it is shown that a particular class of these models is on-shell ultraviolet finite to all orders in perturbation theory.

Nonlinear σ-models are the quantum field theories of harmonic maps from space-time into a Riemannian manifold M. Recent work indicates that there is an intimate connection between supersymmetric versions of the models and differential geometry. There is a correlation between complex manifolds and extended supersymmetry, [1, 2] with the strong implication for models in two space-time dimensions that ultraviolet divergences are severely limited compared to expectations based on power counting [3]. In particular when the manifold M is Ricci-flat the associated supersymmetric σ-model is on-shell ultraviolet finite through 3-loop order [3–5]. For general manifolds there appears to be at most a 1-loop contribution [3] to the generalized β-function [6].

In this paper we present new results of two types.

I. A complete geometrical characterization of manifolds which permit extended supersymmetry is given. The possibilities are $N=2$ supersymmetry which occurs if and only if M is Hermitian and Kahler [7] and $N=4$ supersymmetry which requires that M is hyperkahler [8].

II. Strong restrictions on the on-shell ultraviolet counterterms are derived by combining general considerations of power counting and invariance with the complex manifold structure required by extended supersymmetry. In particular it is shown that the allowed ultraviolet counterterms of $N=4$ models must be zero modes of the Lichnerowicz Laplacian on M which are also algebraic functions of the curvature tensor. For a subclass of $N=4$ models, namely those for which M is a four-dimensional asymptotically locally Euclidean self-dual gravitational instanton [9], it is shown that there are no solutions to these requirements. There are therefore no on-shell ultraviolet counterterms to any order in perturbation theory.

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As far as we know these are the first quantum field theories which have been shown to be ultraviolet finite although power counting arguments indicate divergences to all orders.

Given an n-dimensional Riemannian manifold with metric $g_{ij}(\Phi^k)$ one can define a supersymmetric σ-model with $N=1$ supersymmetry [10]. The superfield action is

$$I[\Phi]=\frac{1}{4i}\int d^2x d^2\theta g_{ij}(\Phi^k)\bar{D}\Phi^i D\Phi^j . \tag{1}$$

Here $\Phi^i(x,\theta)$ is a real scalar superfield

$$\Phi^i(x,\theta)=\varphi^i(x)+\bar{\theta}\psi^i(x)+\tfrac{1}{2}\bar{\theta}\theta F^i(x), \tag{2}$$

where $\varphi^i(x)$ and $\psi^i(x)$ are real scalar and spinor fields and $F^i(x)$ is a real auxiliary field, and the spinor derivative $D_\alpha=\frac{\partial}{\partial\bar{\theta}_\alpha}-i(\bar{\theta}\gamma^\mu)_\alpha\partial_\mu$ involves the real Grassman variable, θ_α, $\alpha=1,2$. This leads to the following component action for the physical fields [10]

$$I[\varphi,\psi]=\tfrac{1}{2}\int d^2x\{g_{ij}(\varphi)\partial_\mu\varphi^i\partial_\mu\varphi^j+ig_{ij}(\varphi)\bar{\psi}^i\gamma^\mu D_\mu\psi^j+\tfrac{1}{6}R_{ik\ell j}(\bar{\psi}^i\psi^\ell)(\bar{\psi}^k\psi^j)\} \tag{3}$$

with covariant derivative $D_\mu\psi^i=\partial_\mu\psi^i+\Gamma^i_{jk}\partial_\mu\varphi^j\psi^k$. The action is invariant under the supersymmetry transformation

$$\begin{aligned}\delta\varphi^i&=\bar{\varepsilon}\psi^i\\ \delta\psi^i&=-i\partial\!\!\!/\varphi^i\varepsilon-\Gamma^i_{jk}(\bar{\varepsilon}\psi^j)\psi^k .\end{aligned} \tag{4}$$

The action is also invariant under reparametrizations of M, $\varphi'^i=\varphi'^i(\varphi)$, $\psi'^i=\frac{\partial\varphi'^i}{\partial\varphi^j}\psi^j$, and such diffeomorphisms commute with supersymmetry transformations [which requires that $\delta\varphi^j$ appears in the second term of $\delta\psi^i$ in (4)].

We now wish to study the possibility of additional fermionic invariances of the action (3). The most general Ansatz for transformation rules which is consistent with dimensional arguments and Lorentz and parity invariance is

$$\begin{aligned}\delta\varphi^i&=f^i{}_j\bar{\varepsilon}\psi^j\\ \delta\psi^i&=-ih^i{}_j\partial\!\!\!/\varphi^j\varepsilon-S^i_{jk}(\bar{\varepsilon}\psi^j)\psi^k\\ &\quad-V^i_{jk}(\bar{\varepsilon}\gamma^\mu\psi^j)\gamma_\mu\psi^k-P^i_{jk}(\bar{\varepsilon}\gamma_5\psi^j)\gamma_5\psi^k ,\end{aligned} \tag{5}$$

where f, h, S, V, and P are allowed to be functions of the dimensionless field $\varphi^i(x)$. Commutation with diffeomorphisms implies that f, h, V, and P are tensors. We require that the action (3) be stationary under the variations (5). Absence of the linear term in ψ of δI requires the conditions $g_{ik}f^k{}_j=g_{jk}h^k{}_i$ and $\nabla_k f^i{}_j=0$. Thus $f^i{}_j$ (and $h^i{}_j$) are covariantly constant, which implies via the Ricci identity

$$f^i{}_m R^m{}_{jk\ell}-R^i{}_{mk\ell}f^m{}_j=0 . \tag{7}$$

Cancellation of the $O(\psi^3)$ terms in δI requires that the tensors $V^i{}_{jk}$ and $P^i{}_{jk}$ vanish and that

$$S^i_{jk} = \Gamma^i_{k\ell} f^\ell{}_j . \tag{8}$$

Cancellation of the $O(\psi^5)$ terms then follows. The relation $f^i{}_j h^j{}_h = \delta^i{}_k$ is used in the $O(\psi^3)$ and $O(\psi^5)$ calculations and will be derived below.

Thus invariance of the action requires, essentially, that $f^i{}_j$ is a covariantly constant tensor. Such a tensor commutes with the holonomy group [12] of M, a fact which implies a great deal of further information about the manifold. For the moment, however, we prefer to derive this information from a more physical standpoint, and therefore assume that the fermionic invariances sought obey the supersymmetry algebra

$$\{Q^a, \bar{Q}^b\} = 2\delta^{ab} \not{P} . \tag{9}$$

When applied to (5) this implies the condition $f^i{}_j h^j{}_k = \delta^i{}_k$, which can be combined with the previous relation between f and h to give

$$g_{ij} f^i{}_k f^j{}_\ell = g_{k\ell} . \tag{10}$$

Thus the net result is that a supersymmetry requires a covariantly constant tensor $f^i{}_j(\varphi)$ satisfying (10), and that

$$\begin{aligned} \delta\varphi^i &= f^i{}_j \bar{\varepsilon} \psi^j \\ \delta(f^i{}_j \psi^j) &= [-i\not{\partial}\varphi^i + \tfrac{1}{2}\Gamma^i_{jk} f^j{}_\ell f^k{}_m (\bar{\psi}^\ell \psi^m)]\varepsilon . \end{aligned} \tag{11}$$

(where Fierz rearrangement has been used).

Note that the transformation (11) can be obtained by the simple substitution $\psi^i \to f^i{}_j \psi^j$ in the superfield (2) (working on the auxiliary shell) or equivalently by the same replacement in (4). Thus the action $I[\varphi, f\psi]$ is guaranteed to be invariant. Therefore a quick proof of invariance of $I[\varphi, \psi]$ under (11) can be obtained by demonstrating that $I[\varphi, \psi] = I[\varphi, f\psi]$, which follows easily using (7) and (10).

We now assume that there are several supersymmetries with covariantly constant tensors $f^{(a)i}{}_j$. Then, (9) implies

$$f^{(a)} f^{(b)-1} + f^{(b)} f^{(a)-1} = 2\delta^{ab} \tag{12}$$

in matrix notation. We can assume that one transformation is the original (4), i.e. $f^{(0)i}{}_j = \delta^i{}_j$. Then with $b=0$ and $a \neq 0$, we get from (12) that $f^{(a)} = -f^{(a)-1}$ which implies

$$f^{(a)i}{}_k f^{(a)k}{}_j = -\delta^i{}_j \tag{13a}$$

$$f^{(a)}{}_{ij} = -f^{(a)}{}_{ji} , \tag{13b}$$

where (10) is used. Then (12) implies a Clifford algebra structure

$$f^{(a)} f^{(b)} + f^{(b)} f^{(a)} = -2\delta^{ab} \tag{14}$$

for all supersymmetries beyond (4).

We now show that the previous results are extremely natural from the viewpoint of complex differential geometry [11]. A tensor $f^i{}_j$ on a given manifold satisfying (13a) is an almost complex structure. This means that multiplication by the imaginary unit i can be smoothly defined on the tangent space of M (namely "i" $V = fV$). If (10) is satisfied by the Riemannian metric, g, then an almost Hermetian structure is defined on M. Finally if $f^i{}_j$ is covariantly constant with respect to the Riemannian metric, then M is a complex Kahler manifold. This means that one can cover M with complex coordinate charts with holomorphic transition functions. In a complex coordinate system the line element takes the form

$$ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \tag{15}$$

and covariant constancy of f implies the Kahler condition

$$\frac{\partial}{\partial z^\gamma} g_{\alpha\bar{\beta}} = \frac{\partial}{\partial z^\alpha} g_{\gamma\bar{\beta}} \,. \tag{16}$$

Thus a supersymmetric σ-model on a Riemannian manifold M admits a second supersymmetry if and only if M is Kahler.

Further supersymmetries require additional parallel complex structures satisfying (14). Notice that if $f^{(1)}$ and $f^{(2)}$ exist and satisfying (14), then the product $f^{(3)i}{}_j = f^{(1)i}{}_k f^{(2)k}{}_j$ automatically generates a third supersymmetry. Then one has a quaternionic structure in the tangent space. Given a quaternion $q = (q_0, q_1, q_2, q_3)$, then the product of q with a vector V is defined by

$$(qV)^i = (q_0 \delta^i{}_j + q_1 f^{(1)i}{}_j + q_2 f^{(2)i}{}_j + q_3 f^{(3)i}{}_j) V^j \,. \tag{17}$$

A manifold possessing this quaterionic structure is called a hyperkahler manifold [8], and we have now shown that it implies and is required by $N=4$ supersymmetry.

Given a vector V in the tangent space at a point φ^i of M and any closed curve γ, then parallel transport of V around γ defines a new vector V'. The linear transformation $A^i{}_j(\gamma)$ which changes V into V' is an element of the holonomy group Ψ_φ. The full group is formed by repeating this operation for arbitrary V and all closed curves γ. If M is connected Ψ_φ is the same group at each point, and if parallel transport is performed with Riemannian connection, as it is here, then Ψ is a subgroup of $O(n)$. Tensors transform under the holonomy group as the appropriate tensor product of vectors, and one can show [12] that a covariantly constant (1, 1) tensor commutes with all elements of Ψ, i.e.

$$A^i{}_k(\gamma) f^k{}_j - f^i{}_k A^k{}_j(\gamma) = 0 \,. \tag{18}$$

Since the Riemann tensor can be viewed as the infinitesimal generator of Ψ, the infinitesimal form of (18) is nothing but (7).

The action of the holonomy group in the tangent space of M is said to be reducible (irreducible) according to whether there is (there is not) a non-trivial invariant subspace. Full reducibility follows since Ψ is necessarily compact [12]. The manifold is accordingly called reducible (irreducible). In the reducible case one can show that local coordinate charts $(\varphi^{i_1}, \varphi^{i_2})$ can be chosen such that the line element takes the form

$$ds^2 = g_{i_1j_1}(\varphi^{k_1})d\varphi^{i_1}d\varphi^{j_1} + g_{i_2j_2}(\varphi^{k_2})d\varphi^{i_2}d\varphi^{j_2}. \tag{19}$$

Physically this corresponds to a σ-model with two sets of fields which do not interact locally. Thus it is natural to restrict ourselves to irreducible manifolds.

We come to the question of the maximal number of conserved spinor charges Q^a in a supersymmetric σ-model. If M is irreducible then one can apply Schur's lemma for real representations of the holonomy group [13], which implies that the matrices which commute with all elements of Ψ form a division algebra over the reals. The only possibilities are the real, complex, and quaternion fields. Thus a supersymmetric σ-model on an irreducible manifold has at most 4 conserved spinor charges. It is intriguing that this limit coincides with the heuristic physical argument based on dimensional reduction from 6 dimensional space-time which is the largest dimension where there is a supersymmetric multiplet containing only scalar and spinor fields. However, this physical argument is imprecise, because more than 4 fermionic invariances can be defined if M is reducible. For example in a product manifold of two hyperkahler factors one can define 16 fermionic invariances by combining complex structures in each factor. However at most 4 of these transformations can be considered supersymmetries in the technical sense that their anticommutator gives a uniform translation of all fields. One should also note that a second supersymmetry beyond (4) requires a Kahler manifold independent of reducibility, and there is a theorem [14] that a reducible Kahler manifold always splits into a product of Kahler factors.

It is clear that such relations as $fh=1$ and the Clifford algebra condition appeared because we assumed that the algebra of fermionic invariances was that of supersymmetry. That assumption can be removed if M is irreducible. We show this now using only differential geometry and the conditions on f and h obtained from cancellation of the $O(\psi)$ term in δI. Since $f^i{}_j$ must be covariantly constant, it commutes with the holonomy group which acts irreducibly. Schur's lemma implies that $f^i{}_j$ (and $h^i{}_j$) are a representation of division algebra. If the division algebra is the reals, then necessarily $f^i{}_j = c\delta^i{}_j = h^i{}_j$, and there is only one supersymmetry up to normalization.

If the division algebra is the complex numbers then there is a basis with covariantly constant units $\delta^i{}_j$ and $I^i{}_j$ with $I^i{}_k I^k{}_j = -\delta^i{}_j$. To show that the metric g_{ij} is Hermitian with respect to the almost complex structure $I^i{}_j$, we consider the eigenvalue problem

$$(I^k{}_i g_{k\ell} I^\ell{}_j - \lambda g_{ij})v^i = 0. \tag{20}$$

Since all tensors are invariant under the action of the holonomy group, given any single eigenvector v^k and an element $A^j{}_k(\gamma)$ of Ψ, then $A^j{}_k(\gamma)v^k$ is also an eigenvector with the same eigenvalue. Since the holonomy group acts irreducibly, vectors of the form $A(\gamma)v$ span the tangent space of M. Hence there is only one possible eigenvalue, necessarily positive and

$$I^k{}_i g_{k\ell} I^\ell{}_j = \lambda g_{ij}. \tag{21}$$

Covariant constancy of I and g implies that $\lambda(\phi)$ is actually constant, and the condition $I^2=-1$ implies that $\lambda=1$. This is the desired Hermiticity property and implies antisymmetry, $I_{ij}=-I_{ji}$. Thus $I^i{}_j(\phi)$ defines an almost complex and Hermitian structure. Covariant constancy implies that M is a Kahler manifold. The real dimension of M is even, i.e. $2n$, and the holonomy group is contained [14] in $U(n)$.

We now return to supersymmetry and note that $f^i{}_j$ and $h^i{}_j$ are superpositions of the units $\delta^i{}_j$ and $I^i{}_j$ with constant coefficients. The condition $f_{ij}=h_{ji}$ implies $f^i{}_j=c\delta^i{}_j+dI^i{}_j$ and $h^i{}_j=c\delta^i{}_j-dI^i{}_j$. This is sufficient to prove the vanishing of the $O(\psi^3)$ and $O(\psi^5)$ terms in δI. The units $\delta^i{}_j$ and $I^i{}_j$ can be used to define two fermionic invariances of the action which then obey (9). The most general invariance is then a superposition of these with constants c and d. Thus a second fermionic invariance appears if and only if M is Kahler, and that invariance is necessarily a supersymmetry. In particular this implies that there are no central charges in the supersymmetry algebra of σ-models defined on irreducible manifolds.

In the case that the division algebra is the quaternions, there is a basis with three imaginary units which satisfy the Clifford algebra property (14). By the previous discussion each imaginary unit defines a Hermitian Kahler structure on M, and the four units may be used to define four supersymmetry transformations which obey (9). Further, as discussed above, there is a quaternionic structure in the tangent space which is preserved by parallel transport. This means that M is a hyperkahler manifold. Its real dimension is $4n$, and the holonomy group is contained in $\mathrm{Sp}(n)$.

$\mathrm{Sp}(n)$ is embedded in $\mathrm{SU}(2n)$ and this is sufficient to show that hyperkahler manifolds are necessarily Ricci-flat [14]. The curvature tensor is the generator of the holonomy group and can be written in complex coordinates as $R^\alpha{}_{\beta\gamma\bar\delta}$, where the indices α,β are those of the holonomy group. In the case that the holonomy group is $\mathrm{SU}(2n)$ [or more generally $\mathrm{SU}(m)$ for an m-complex dimensional case], the trace $R^\alpha{}_{\alpha\gamma\bar\delta}=0$. Applying the cyclic identity for a Kahler manifold ($R_{\bar\lambda\mu\nu\bar\varrho}=R_{\bar\lambda\nu\mu\bar\varrho}$), one finds that $R_{\gamma\bar\delta}=0$. By previous work on the ultraviolet structure [3, 15], this implies that supersymmetric σ-models on hyperkahler manifolds are at least 3-loop finite.

We now apply these results to the study of the ultraviolet behavior of supersymmetric σ-models. One expects, ab initio, that as a consequence of power counting these models have the same generalized renormalizable structure found by Friedan [6] in the bosonic σ-model. The important class of ultraviolet counterterms are reparametrization invariant and take the superfield form

$$\Delta I[\Phi]=\frac{1}{4i}\int d^2x d^2\theta\, T_{ij}(\Phi)\bar D\Phi^i D\Phi^j\,, \tag{22}$$

where $T_{ij}(\Phi)$ is a second rank tensor algebraically constructed from the curvature tensor on M and its covariant derivatives. An ℓ-loop order tensor $T_{ij}^{(\ell)}$ must scale [15] as $T_{ij}^{(\ell)}\to\Lambda^{\ell-1}T_{ij}^{(\ell)}$ under the constant conformal transformation $g_{ij}\to\Lambda^{-1}g_{ij}$ of the metric. In the treatment of Friedan, the invariant counterterms require the introduction of a renormalized metric $g_{ij}^R(\mu)$ on M which varies with energy scale μ.

There is an additional class of "off-shell" counterterms which vanish when the classical equations of motion are imposed and correspond to divergences compensated by field redefinitions or reparametrizations of M. This gives rise to the appellation "on-shell" for the invariant counterterms (22). It is these which are of primary importance since they correspond to renormalization of the physical parameters of the theory. The background field method and superspace perturbation theory permit direct calculation [15] of the invariant counterterms in their geometrical form.

The geometrical correlation between extended supersymmetry and complex manifolds may now be used to deduce strong restrictions on the form of the allowed tensor counterterms in (22). First note that the effective unrenormalized metric on M is the sum $g_{ij}+T_{ij}$ of the metric of the classical action and the quantum corrections, and that one expects that the quantum corrections preserve the symmetries of the classical action. Hence if g_{ij} is a Kahler metric so that the classical action has $N=2$ supersymmetry, then $g_{ij}+T_{ij}$ must allow $N=2$ supersymmetry and must also be a Kahler metric. Hence in complex coordinates T_{ij} must satisfy $T_{\alpha\beta}=T_{\bar{\alpha}\bar{\beta}}=0$ and $\partial_\gamma T_{\alpha\bar{\beta}}=\partial_\alpha T_{\gamma\bar{\beta}}$. It was shown previously [3] that the vanishing curl condition severely restricts the tensors which occur in one and two-loop order, and that these restrictions apply to both $N=1$ and $N=2$ models because of a "universality property". Extensions of this approach to higher order should be facilitated because the Hermiticity condition is now known to be required.

When the classical metric g_{ij} is that of a hyperkahler manifold, the quantum corrections must preserve the four supersymmetries of the classical theory. Thus the effective metric $g_{ij}+T_{ij}$ must also be hyperkahler and therefore Ricci-flat. By the Palatini identity [16] this requires that T_{ij} is a zero mode of Lichnerowicz Laplacian of g_{ij}, defined by

$$\Delta_L T_{ij}=\nabla^k\nabla_k T_{ij}+[\nabla_i,\nabla_k]T^i{}_j+[\nabla_j,\nabla_k]T^k{}_i. \tag{23}$$

Thus the problem of finding ultraviolet counterterms of these models reduces to the problem of finding the zero modes of (23) which are also algebraic functions of curvature of definite positive conformal weight.

We have been able to solve the zero mode problem for a special class of manifolds, namely the explicitly known family [9] of four dimensional asymptotically locally Euclidean self-dual gravitational instantons. Self-duality of the curvature tensor implies that the holonomy group is SU(2) which acts irreducibly in the tangent space in the four dimensional real representation. Since SU(2) is isomorphic to Sp(1), there are necessarily 3 parallel complex structures [8, 9] which satisfy the Clifford algebra condition. Thus if M is a gravitational instanton metric, the associated supersymmetric σ-model in two space-time dimensions has $N=4$ supersymmetry.

The particular property of the gravitational instanton metrics used here is that the solutions are known with the most general set of independent parameters for given topological class. Thus the zero modes of Δ_L are in 1 : 1 correspondence with parameter variations of these solutions. Such parameter variations necessarily have conformal weight -1, whereas positive conformal weight is required for any ultraviolet counterterm in perturbation theory. Therefore there are no ultraviolet counterterms and the supersymmetric σ-models are on-shell ultraviolet finite to all orders in perturbation theory.

Some technical comments should be made concerning the assumption that the on-shell counterterms are invariant under symmetry transformations of the classical actions. Some exceptions have been noted [17] in the case of nonlinear transformations because the "quantum field" in background field calculations has complicated transformation properties. Further, one must always note the possibility of quantum level anomalies for classical invariances. There is no problem with reparametrization of M, because the normal coordinate expansion allows one to choose a "quantum field" with linear transformation law, and there is no problem with the Riemannian $N=1$ supersymmetry, because superfield calculations are manifestly supersymmetric. The extended supersymmetry transformations (11) are nonlinear because of the elimination of auxiliary fields in our treatment. However in coordinates on M adapted to the complex structure $f^i{}_j$ the extended supersymmetry can be formulated as linear homogeneous transformations with auxiliary fields which coincide with the real form of the transformation of [1]. Thus the counterterms (22) which are coordinate independent should be invariant under the extended supertransformations.

We believe that the ultraviolet finiteness properties found here should be valid for supersymmetric σ-models on any hyperkahler manifold and probably for any Ricci-flat metric. Extension of the proof requires classification of the zero modes of Δ_L which is not a simple problem.

Improved ultraviolet divergence structure is well known in supersymmetry, but these are the first theories whose ultraviolet finiteness is established to all orders. One may hope that these geometrical methods or perhaps more traditional methods may be applied in four dimensional field theories, where the $N=4$ supersymmetric Yang-Mills theory has been shown by calculation to be ultraviolet finite in 2-loop [18] and 3-loop [19] order.

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