Commun. math. Phys. 8, 89—131 (1968)

A Pattern Calculus for Tensor Operators in the Unitary Groups*

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Abstract. For a large class of tensor operators in $U(n)$, a surprisingly simple diagrammatic calculus of patterns is shown to exist; to each operator of this class a pattern may be assigned in terms of which the *complete algebraic formula for all matrix elements may be read off directly.* The class of operators includes all fundamental, elementary and extremal Wigner operators in all $U(n)$. Application of the pattern calculus toward the explicit determination of all tensor operators is discussed.

I. Introduction and Summary

The problem of extending the angular momentum techniques, developed principally by WIGNER and RACAH, to the n-dimensional unitary group, $U(n)$, has been a research goal of considerable interest in the past few years[1]. The ancillary problems (for example, a constructive definition of all unitary irreducible representations[2]) have been solved and the major problem, simple reducibility (or more accurately, the multiplicity problem), has been given a canonical setting [4] in terms of a general embedding proposition (embedding $U(n)$ in the totally symmetric irreps of $U(n^2)$) and a corresponding interpretation in terms of the 'boson factorization lemma'. These results — which nicely incorporate all our earlier work as special cases — lead to an explicit canonical definition of all $SU(3)$ tensor operators[3] (and correspondingly a Wigner-Racah system comprising the analogues for $SU(3)$ of the $(1j)$, $(3j)$ and $(6j)$ symbols of $SU(2)$).

* Supported in part by the U. S. Army Research Office (Durham), the National Science Foundation, and the U. S. Atomic Energy Commission.

[1] To attempt an adequate referencing here would be inappropriate; alternatively we refer to three recent texts — [1, 2, 3] — typifying widely different view-points, from which a suitable bibliography can be obtained.

[2] The term "irrep" is used for an irreducible representation; "rep" denotes any representation. All representations considered are assumed unitary.

[3] The extension to all $SU(n)$ has been obtained, but has not as yet been prepared for publication.

Let us note, for clarity, that 'canonical' is used here in the sense that there are *no free choices* involved in the resolution of the multiplicity, aside from that choice of *order* $(1, 2, \ldots n)$ explicitly used in the Weyl canonical state labelling.

These general results provide the basic structure and framework for the present investigation; for the convenience of the reader a resumé is given in Section II, in sufficient detail that the present paper is essentially self-contained.

The results of the present paper arose from a detailed investigation of the explicit tensor operators of $U(n)$; for this investigation we exploited corollary (4) to the factorization lemma [4] (the analog for tensor operators of the Cartan theorem for constructing all irreps from the fundamental irrep). We found that, for a large class of tensor operators, there existed a 'calculus of patterns' in terms of which a truly remarkable, even astonishing, simplification occurs: *To each operator of this class we may assign a pattern in terms of which the complete algebraic formula for all matrix elements may be read off directly.*

It is only a slight exaggeration to say that our result makes the preparation of algebraic and numerical tables for these operators quite superfluous!

This pattern calculus is a natural continuation of the synthesis typified by the Young frame and the Young tableaux for the symmetric groups, and the Weyl and Gelfand patterns for the unitary groups; but it is also something more, in that an operator pattern is not merely a mnemonic, but literally *is* the operator it denotes, in virtue of its specification of all matrix elements. This fact distinguishes the pattern calculus from the many mnemonic schemes currently popular for the $SU(2)$, and other groups.

The class of operators for which the pattern calculus applies, in this completely explicit form, comprises first of all the *fundamental* Wigner operators of $SU(n)$, denoted by[4] $\langle 1\ \dot{0}\rangle$ (Section III) — these are the analogues to the spin-1/2 operators of $SU(2)$; secondly, it applies to all *elementary operators*, $\langle \dot{1}_k, \dot{0}_{n-k}\rangle$, ($k$, n arbitrary) (Section IV); lastly, it applies to all *extremal* reduced operators (Section VI). (Each of these classes includes the preceding class, but it is convenient to make the distinctions indicated.)

The classes just enumerated comprise the largest set of tensor operators which the pattern technique directly provides the complete answer. However, as a calculus of patterns — in which patterns operate

[4] We use the notation that the dotted numerals, $\dot{0}$ and $\dot{1}$, signify the numeral is repeated as often as necessary. The more explicit notation $\dot{1}_k$ denotes that the 1 appears k-times.

on each other — the method generalizes such that it then applies to the explicit construction of the general tensor operator. This further development of the pattern calculus is incomplete at present. In Section VII we discuss this development for the simplest cases; the results obtained (in particular, Eqs. (58) and (60)) show that there exists a close connection between the pattern calculus and the boson polynomials of $U(n) \times U(n)$.

II. Resume of Previous Work and Notational Conventions

It is the purpose of the following section to discuss the background material on which our results are based in sufficient detail that the present paper be self-contained. It might be useful to emphasize again that the pattern calculus — which it is our aim to develop in succeeding sections — has such a surprisingly simple structure that one might almost be content merely to know that an adequate justification exists.

The essential results, which we review, have been developed by Baird, Giovannini and the present authors in several papers, [4—11]. There are three principal topics: (a) the concepts of canonical Wigner operators and reduced Wigner operators[5] for the group $U(n)$; (b) the boson calculus[6] applied to $U(n)$ state vectors [6, 10]; and (c) the factorization lemma [4] which reveals the composite structure of boson operators in terms of Wigner operators.

Let us summarize now the basic results — and especially the notational conventions — for the (unitary) irreps of the n-dimensional unitary group, $U(n)$. Every irrep of $U(n)$ is characterized by a partition $[m] \equiv [m_{in}] \equiv [m_{1n}, m_{2n}, \ldots, m_{nn}]$ of n non-negative integers obeying the relation $m_{in} \geqq m_{i+1,n}$; conversely every such partition denotes a unique irrep. To accord with our motivation from quantum mechanics ("complete set of commuting observables") the integers $\{m_{in}\}$ are to be defined as the eigenvalues of a set of n commuting invariant operators [4, 5] denoted by $\{I_k\}$, $k = 1, 2, \ldots n$. (The invariant operator I_k is of (algebraic) degree k and the unique definition of the $\{m_{in}\}$ requires sign conventions on the root extractions. Alternatively, the $\{m_{in}\}$ may be defined more directly as quantum numbers of the vector of highest weight.) To characterize uniquely the states (that is, the orthonormal

[5] These ideas stem from the 1951 Princeton Lectures of G. Racah [12] and the famous unpublished manuscript of E. P. Wigner [13]. The essential contribution of [8] is the analysis of the tensor operator decomposition, in which the undefined quantum numbers used by Racah are shown to be operator patterns.

[6] The boson calculus was first applied to $U(n)$ by P. Jordan [14], and has been extensively developed for $SU(2)$ by J. Schwinger and V. Bargmann. The literature on this subject is very large; further references may be found in the bibliography cited in [1, 2, 3] and in the reprint volume cited in [13].

vectors — dim $[m]$ in number, where dim $[m]$ is the Weyl dimension formula for $U(n)$ — belonging to an irrep $[m]$ one uses the *Weyl branching law for* $U(n)$. This law — the foundation (and model, as well) for all the work to follow — asserts that under restriction of $U(n)$ to $U(n-1)$ the irrep $[m_{in}]$ of $U(n)$ splits into the irreps $[m_{i,n-1}]$ of $U(n-1)$, where the non-negative integers $\{m_{i,n-1}\}$ obey the *betweenness conditions*: $m_{i,n} \geqq m_{i,n-1} \geqq m_{i+1,n}$ and each irrep $[m_{i,n-1}]$ obeying the betweenness conditions occurs *once and only once*. It follows that every vector of the irrep $[m_{in}]$ of $U(n)$ may be characterized uniquely by the quantum numbers $\{m_{i,j}\}$ of the chain of subgroups $U_n \to U_{n-1} \to U_{n-2} \cdots \to U_1$, the final subgroup U_1 having, of course, only one-dimensional irreps.

An elegant notational convention to denote the unique states of the irrep $[m_{in}]$ is the *Gelfand pattern*, a triangular array, denoted by (m), of $\frac{n(n+1)}{2}$ non-negative integers $\{m_{ij}\}$ arranged as:

$$\begin{array}{ccccccccc} m_{1n} & & m_{2n} & & \cdots & & m_{n-1,n} & & m_{n,n} \\ & m_{1,n-1} & & m_{2,n-1} & & \cdots & & m_{n-1,n-1} & \\ & & & & \cdots & & & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{array}$$

It is clear that the betweenness condition of the Weyl branching law becomes a sort of geometrical constraint, very much in the tradition of the Young pattern for the symmetric group. State vectors belonging to the Gelfand pattern (m) are denoted by $|(m)\rangle$; to designate more clearly the irrep to which these state vectors belong, the Gelfand pattern will be written as: $\begin{pmatrix}[m]_n \\ (m)_{n-1}\end{pmatrix}$ where $[m]_n$ is the partition denoting an irrep in $U(n)$.

It is helpful to discuss in more detail certain special Gelfand patterns. The state vector of highest weight has the labels $m_{ij} = m_{in}$ for all i, j; this Gelfand pattern is termed *maximal* and denoted $\begin{pmatrix}[m]_n \\ \max\end{pmatrix}$. The Gelfand pattern having the labels $m_{ij} = m_{i+1,j+1}$ all i, j is termed *minimal* and denoted $\begin{pmatrix}[m]_n \\ \min\end{pmatrix}$. These are but two special cases of the set of $n!$ patterns termed *extremal* — patterns in which all m_{ij} take on values given by the top row, m_{in}, of the Gelfand pattern. Such extremal patterns are used, and discussed further, in Section VI.

We may summarize the above results by saying that the Weyl branching law constitutes 'a canonical resolution of the multiplicity for the state labelling problem'; since this resolution is the prototype for the analogous operator multiplicity problem, it is useful to discuss the Weyl resolution a bit further. The essential observation is that the Weyl labelling really involves a free choice, namely the *order* in which specific variables are treated under the restrictions. Because of the underlying Weyl group for $U(n)$, all such orderings are abstractly equivalent; thus one should more properly say that the labelling denoted by the Gelfand pattern is canonical only in the sense that it is unique to within equivalence.

Let us turn now to the closely related multiplicity problem for tensor operators in $U(n)$. Following the classic work of JORDAN, RACAH, WIGNER, . . ., one defines a tensor operator in this way: under a unitary transformation of the state vectors, $|(m)\rangle \to |(m)\rangle' \equiv U|(m)\rangle$. The generic operator $\mathcal{O}$ transforms as:

$$\mathcal{O} \to \mathcal{O}' \equiv U \mathcal{O} U^{-1},$$

such that all brackets (probability amplitudes) $\langle(m_{\text{final}})|\ \mathcal{O}|(m_{\text{initial}})\rangle$ remain invariant. A tensor operator is then a set of operators $\{\mathcal{O}_i\}$ such that under transformation the set transforms linearly into itself, i.e.,

$$\{\mathcal{O}_i\} \to \{\mathcal{O}_i\}' = \{U \mathcal{O}_i U^{-1}\} = \{\mathcal{D}_{i'i} \mathcal{O}_{i'}\}.$$

In this way irreducible tensor operators are assigned Gelfand patterns, $\mathcal{O}_{(M)}$, denoting that such operators constitute the carrier space of an irrep of $U(n)$. (To distinguish operators from state vectors, capital letters are used for operator Gelfand patterns.)

The tensor operator labelling afforded by the Gelfand pattern (M) does not furnish a unique designation; this is the familiar multiplicity problem. The essential contribution of [8] was to show that the tensor operator multiplicity problem could be put in one-to-one correspondence with the state labelling problem. It follows that a unique tensor operator notation comprises *two* Gelfand patterns, which may be denoted most economically by the pattern:

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{matrix} \right\rangle \tag{1}$$

This notation for $U(n)$ Wigner operators (*unit* tensor operators) incorporates in the notation itself the fact that the tensor operator multiplicity problem can be put in one-to-one correspondence with the state labelling problem; this operator notation consists of two patterns, a lower (triangular) pattern and an upper (inverted triangular) pattern sharing a common row. In the notation (1), the *lower pattern*

$$(M)_n \equiv \begin{pmatrix} [M]_n \\ (M)_{n-1} \end{pmatrix} \equiv \begin{pmatrix} M_{1n} & & M_{2n} & & \cdots & & M_{nn} \\ & M_{1,n-1} & & M_{2,n-1} & \cdots & M_{n-1,n-1} & \\ & & \cdot & & & \cdot & \\ & & & M_{12} & & M_{22} & \\ & & & & M_{11} & & \end{pmatrix} \tag{2}$$

is an n-rowed *Gelfand pattern*, in which, for later convenience, we make explicit the shared irrep labels $[M]_n \equiv [M_{1n} M_{2n} \ldots M_{nn}]$ of $U(n)$. The significance of the complete lower pattern $(M)_n$ is that it designates fully the transformation properties in $U(n)$ of the designated tensor operator. The upper pattern in the notation (1) is introduced in consequence of the one-to-one mapping of the multiplicity of the tensor operators onto states. The pattern

$$(\Gamma)_n \equiv \begin{pmatrix} (\Gamma)_{n-1} \\ [M]_n \end{pmatrix} \equiv \begin{pmatrix} & & & \Gamma_{11} & & & \\ & & \Gamma_{12} & & \Gamma_{22} & & \\ & \cdot & & & & \cdot & \\ & \Gamma_{1n-1} & & & & \Gamma_{n-1,n-1} & \\ M_{1,n} & & M_{2,n} & \ldots & M_{n-1,n} & & M_{nn} \end{pmatrix} \tag{3}$$

is of the same form as a Gelfand pattern in so far as the nonnegative integers Γ_{ij} satisfy the same "betweenness conditions"

$$\Gamma_{i,j+1} \geqq \Gamma_{i,j} \geqq \Gamma_{i+1,j+1} \tag{4}$$

as do the integers M_{ij} of the Gelfand pattern (2). However, the significance of the two patterns is very different. A (state labelling) Gelfand pattern such as in Eq. (2) is an expression of the Weyl branching law, and accordingly has direct group theoretical significance. The significance of the upper pattern is as follows: It designates the fact that the Wigner operator (1) carries an arbitrary vector belonging to irrep $[m]_n = [m_{1n} m_{2n} \ldots m_{nn}]$ of $U(n)$ into a vector belonging to irrep $[m']_n$ of $U(n)$, where

$$m'_{in} = m_{in} + \Delta_{in}(\Gamma) \tag{5}$$

$$\Delta_{in}(\Gamma) \equiv \sum_{j=1}^{i} \Gamma_{ji} - \sum_{j=1}^{i-1} \Gamma_{j,i-1}, \quad (i = 1, 2, \ldots, n), \tag{6}$$

$$[\Delta(\Gamma)]_n \equiv [\Delta_{1n}(\Gamma), \Delta_{2n}(\Gamma), \ldots, \Delta_{nn}(\Gamma)]. \tag{7}$$

(The use of square brackets [...] for Δ is intended to help identify the Δ_{in} as the *changes* induced on partition labels.)

Because the significance of the upper pattern for tensor operators is rather different from that of a Gelfand pattern (which is defined completely group theoretically for $U(n)$) it is more appropriate to call these upper patterns *operator patterns* to afford a distinction.

It will be recognized from the definitions, Eqs. 5—7, that an operator pattern determines n additive quantum numbers, the Δ_{in}. These quantum numbers play a role precisely analogous to that of the n additive quantum numbers defined by operators H_i $(i = 1, \ldots n)$ of the group $U(n)$. (The operators H_i — the generalized charges in quantum applications of $U(n)$ — are the generators of the maximal abelian (Cartan) subalgebra.) The eigenvalues of the H_i define the weight of a given vector $|(m)\rangle$ of $U(n)$; it is very familiar that the weights do not uniquely identify the vector (the state labelling problem), and that it takes a complete Gelfand pattern to specify a unique vector. Thus it is quite understandable that the quantum numbers Δ_{in} fail to identify a unique tensor operator, and that it takes in fact a complete *operator pattern* to designate a unique tensor operator (assuming, of course, that the lower (Gelfand) pattern is already fixed).

It can be demonstrated for $U(2)$ that operator patterns have indeed a group theoretic significance (this is the content of [11]); this explicit example shows at the same time that it is neither an easy nor an obvious task to give a precise meaning to the 'group of upper pattern space' in general. For the purposes of the present paper it is fortunate that such considerations are completely inessential (since all operator patterns which occur below are uniquely specified by the Δ_{in})[7].

The next essential concept is that of decomposing a $U(n)$ Wigner operator into a reduced Wigner operator and a $U(n-1)$ Wigner operator [8]:

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{matrix} \right\rangle = \sum_{(\gamma)_{n-2}} \begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix} \left\langle \begin{matrix} (\gamma)_{n-2} \\ [M]_{n-1} \\ (M)_{n-2} \end{matrix} \right\rangle . \tag{8}$$

Iterating this result — for $U(n-1)$, $U(n-2), \ldots U(2)$ — establishes the canonical decomposition of a $U(n)$ tensor operator.

In Eq. (8) the symbol

$$\begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix} \tag{9}$$

denotes a *reduced Wigner operator.* Note that the operator pattern

$$(\gamma)_{n-1} \equiv \begin{pmatrix} [M]_{n-1} \\ (\gamma)_{n-2} \end{pmatrix}$$

is inverted as an upper pattern in the $U(n-1)$ Wigner operator, but stands in uninverted form as the lower pattern of the reduced Wigner operator. *Both upper and lower patterns in a reduced Wigner operator are "operator patterns".*

(The *order* of the operators in Eq. (8) is not important. The result is not different, though it might seem to depend on whether the reduced Wigner operator acts before or after the $U(n-1)$ Wigner operators shifts the labels

$$[m]_{n-1} = [m_{1\,n-1} m_{2\,n-1} \ldots m_{n-1\,n-1}]$$

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[7] It may be of interest, nevertheless, to summarize the present status of this problem. We can show that there exists a generalization of the Weyl branching law to tensor operators in $U(n)$, such that the betweenness condition holds for operator patterns, with the labelling determined uniquely by the dimensions of the operators under splitting. The labelling so determined is unique once a Weyl labelling for the state vectors has been chosen. We conjecture that a group theoretic significance for $U(n)$ operator patterns probably exists.

of a vector belonging to irrep $[m]_{n-1}$. The point is that *both* the reduced operator and the $U(n-1)$ operator are defined to "see" the same state vector in $U(n-1)$ and — since they share the same $U(n-1)$ operator pattern — to cause the same shift.)

Now consider the matrix elements of Eq. (8) between the initial canonical Gelfand basis vector

$$|(m)_n\rangle = \left|\begin{pmatrix} [m]_n \\ (m)_{n-1} \end{pmatrix}\right\rangle = \left|\begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (m)_{n-2} \end{pmatrix}\right\rangle \tag{10}$$

and the final basis vector

$$|(m')_n\rangle = \left|\begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n \\ [m]_{n-1} + [\Delta(\gamma)]_{n-1} \\ (m')_{n-2} \end{pmatrix}\right\rangle, \tag{11}$$

where $[\Delta(\Gamma)]_n$ is the shift associated with operator pattern $(\Gamma)_n$, and $[\Delta(\gamma)]_{n-1}$ is the shift associated with a definite, but arbitrary $(\gamma)_{n-1}$ which appears as a lower operator pattern in the reduced Wigner operator in Eq. (8). The final labels $(m')_{n-2}$ are left arbitrary. Then, from Eq. (8), we obtain

$$\left\langle\begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n \\ [m]_{n-1} + [\Delta(\gamma)]_{n-1} \\ (m')_{n-2} \end{pmatrix}\right|\left\langle\begin{matrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{matrix}\right\rangle\left|\begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (m)_{n-2} \end{pmatrix}\right\rangle$$
$$= \sum_{(\gamma')_{n-2}} \left\langle\begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n \\ [m]_{n-1} + [\Delta(\gamma)]_{n-1} \\ (m')_{n-2} \end{pmatrix}\right|\begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma')_{n-1} \end{bmatrix}\left|\begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (m)_{n-2} \end{pmatrix}\right\rangle \tag{12}$$
$$\left\langle\begin{pmatrix} [m]_{n-1} + [\Delta(\gamma)]_{n-1} \\ (m')_{n-2} \end{pmatrix}\right|\left\langle\begin{matrix} (\gamma')_{n-2} \\ [M]_{n-1} \\ (M)_{n-2} \end{matrix}\right\rangle\left|\begin{pmatrix} [m]_{n-1} \\ (m)_{n-2} \end{pmatrix}\right\rangle,$$

where now the sum $(\gamma')_{n-2}$ is over all operator patterns $\begin{pmatrix} [M]_{n-1} \\ (\gamma')_{n-2} \end{pmatrix}$ which have shifts $[\Delta(\gamma)]_{n-1}$; that is, the sum is over all $(\gamma')_{n-2}$ such that the operators

$$\left\langle\begin{matrix} (\gamma')_{n-2} \\ [M]_{n-1} \\ (M)_{n-2} \end{matrix}\right\rangle$$

effect the same change $[\Delta(\gamma)]_{n-1}$ in the $U(n-1)$ labels $[m]_{n-1}$.

Equation (12) expresses the basic decomposition of $U(n)$ Wigner coefficients into reduced Wigner coefficients and $U(n-1)$ Wigner coefficients. To put this result in a more convenient, and compact form, recall that, by definition, the reduced Wigner operator, Eq. (9), is invariant under $SU(n-1)$ transformations. Thus the matrix element of the reduced Wigner operator in Eq. (12) is actually unchanged if we introduce maximal $U(n-2)$ labels. Next we note that for maximal $U(n-2)$ labels, this matrix element is fully specified by the two operator patterns in the reduced Wigner operator for an arbitrary but definite initial state. It follows that we may write this matrix element (denote it by #) in *operator form*:

$$\begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix} \left| \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle = \# \left| \begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n \\ [m]_{n-1} + [\Delta(\gamma)]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle, \qquad (9'\text{a})$$

where

$$\# \equiv \left\langle \begin{pmatrix} [m]_n + \Delta(\Gamma)]_n \\ [m]_{n-1} + [\Delta(\gamma)]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right| \begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix} \left| \begin{pmatrix} [M]_n \\ [M]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle, \qquad (9'\text{b})$$

and we have used the definition that Wigner operators between maximal states are unity if non-vanishing.

We may further extend the notation (more accurately perhaps this is an *abuse de notation*!) by restricting the admissible initial states to be maximal in $U(n-2)$. Then under the reduced Wigner operators this restriction propagates and we may consider *products* of reduced operators. The importance of this extension lies in the fact that it enables us to discuss operators in the space $U(n) : U(n-1)$; we will call these "projective operators", or, synonymously, "projections", having in mind to distinguish them from the different concept of projection operators. Projections will be denoted by the same notation as reduced Wigner operators.

(Strictly speaking projective operators should be distinguished from reduced Wigner operators, and using the same notation for both makes for certain paradoxical features. For example, a reduced Wigner operator is an $SU(n-1)$ *invariant;* thus matrix elements must have the *same* $SU(n-1)$ labels in both initial and final states. Yet the notation for both types of operator ascribes a change $[\Delta(\gamma)]_{n-1}$.

The paradox is easily resolved: A reduced Wigner operator is the $SU(n-1)$ invariant product of a $U(n)$ and a $U(n-1)$ Wigner operator; the changes in $U(n-1)$ labels induced by the $U(n)$ part are exactly compensated by changes induced by the $U(n-1)$ part; the matrix element of the $U(n)$ part is a matrix element of a projective operator; the $U(n-1)$ matrix element is unity, as befits a unit tensor operator.

Once these circumstances are clearly in mind, the use of the same symbol for both operators is no longer confusing.)

Equation (8) (and the resulting matrix element expression, Eq. (12)) is the first result basic to the present work.

The second significant result is the factorization lemma [4] for boson opreators. The use of boson variables as a convenient realization for the carrier space of $U(n)$ is very familiar. In order to realize all irreps of $U(n)$ it is necessary to assume n kinematically independent copies of an n-state boson variable; that is, one takes the variables a^i_j; $i, j = 1, 2, \ldots n$ with the commutators:

$$[\bar{a}^i_j, a^{i'}_{j'}] = \delta^{i'}_i \, \delta^{j'}_j \, ,$$

all other commutators defined to be zero. The generators E_{ij} of the group $U(n)$ are defined by the mapping:

$$E_{ij} \to \sum_{k=1}^{n} a_i^k \bar{a}_j^k .$$

It is clear, however, that these boson variables admit also of a second, isomorphic, $U(n)$ group generated by the operator mapping:

$$E^{ij} \to \sum_{k=1}^{n} a_k^i \bar{a}_k^j ,$$

and that, moreover, the two sets of operators $\{E_{ij}\}$ and $\{E^{ij}\}$ commute. Thus this boson realization involves the direct product group $U_n \times U_n$.

In fact, one sees at once that this boson realization $\{a_j^i\}$ really involves the group U_{n^2} and all totally symmetric irreps thereof. This defines a canonical imbedding of $U(n)$ in the sequence of groups $U_{n^2} \supset U_n \times \times U_n \supset U_n$, in which moreover the irrep labels of the two $U(n)$ groups in $U_n \times U_n$ *coincide* (we denote this by $U_n * U_n$). This structure is precisely the analog to that exhibited by the tensor operators of $U(n)$, and reference [4] discusses this canonical embedding in detail, proving the factorization lemma to which we now turn.

Let

$$\left| \begin{pmatrix} (M')_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} \right\rangle \tag{13}$$

denote a normalized basis vector in $U_n * U_n$. In this notation, the first $U(n)$ refers to the $U(n)$ group with generators E_{ij}, the second to the $U(n)$ group with generators E^{ij}. (These two $U(n)$ groups are isomorphic but distinct (and commuting); the placement of the indices is merely a reminder as to which group is which ("upper" vs. "lower") — there is no implication as to metric in this placement of indices. The star signifies that the Casimir invariants of the irreps of these two groups coincide.) Hence both

$$(M)_n = \begin{pmatrix} [M]_n \\ (M)_{n-1} \end{pmatrix} \quad \text{and} \quad (M')_n = \begin{pmatrix} (M')_{n-1} \\ [M]_n \end{pmatrix}$$

in Eq. (13) are Gelfand patterns, the second one being inverted. The basis vector (13) may also be written in the form

$$\left| \begin{pmatrix} (M')_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} \right\rangle = \mathcal{M}^{-1/2} \; B \begin{pmatrix} (M')_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} |0\rangle, \tag{14}$$

where (B denotes "boson")

$$B \begin{pmatrix} (M')_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} \tag{15}$$

is a polynomial in each of the boson creation operators a_j^i, the symbol $|0\rangle$ denotes the vacuum ket, and $\mathcal{M}$ is the measure of the highest weight tableau associated with $[M]_n$:

$$\begin{aligned} \mathcal{M} &\equiv \prod_{i=1}^{n} (M_{in} + n - i)!/ \prod_{i<j=1}^{n} (M_{in} - M_{jn} + j - i) , \\ &\equiv \mathcal{M}([M]_n) . \end{aligned} \tag{16}$$

The introduction of $\mathcal{M}^{-1/2}$ into Eq. (14) defines the manner in which the boson operators (15) are normalized: For example, if $(M')_n$ and $(M)_n$ are maximal, i.e.,

$$M'_{ij} = M_{in}, \quad M_{ij} = M_{in} \quad (\text{all } i, j) ,$$

then

$$B\begin{pmatrix} \max \\ [M]_n \\ \max \end{pmatrix} = \prod_{k=1}^{n} (a_{1\,2\,\ldots\,k}^{1\,2\,\ldots\,k})^{M_{kn} - M_{k+1,n}} , \tag{17}$$

where $a_{1\,2\,\ldots\,k}^{1\,2\,\ldots\,k}$ is the determinant formed from the k-bosons a_j^i $i, j \leqq k$.

The boson operator (15) is clearly a tensor operator in either its lower or upper Gelfand pattern with respect to transformations in the respective $U(n)$ subgroup of $U_n * U_n$. As such it must be bilinear in the canonical Wigner operators which are defined, respectively, on the two $U(n)$ groups. *The factorization lemma asserts that the precise form of this bilinear relation is*

$$B\begin{pmatrix} (M')_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} = \sum_{(\Gamma)_{n-1}} \mathcal{M}^{1/2} \left\langle \begin{matrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{matrix} \right\rangle_l \left\langle \begin{matrix} (\Gamma)_{n-1} \\ [M]_n \\ (M')_{n-1} \end{matrix} \right\rangle_u \mathcal{M}^{-1/2} , \tag{18}$$

where $\mathcal{M}$ is an invariant operator of $U_n * U_n$ which has eigenvalue equal to the measure $\mathcal{M}([m]_n)$ for an arbitrary vector with labels $[m]_n$. The indices l and u designate the fact that the Wigner operators act, respectively, on the lower and upper Gelfand patterns of an arbitrary vector of $U_n * U_n$:

$$\left| \begin{pmatrix} (\mu)_{n-1} \\ [m]_n \\ (m)_{n-1} \end{pmatrix} \right\rangle = \left| \begin{matrix} \begin{pmatrix} (\mu)_{n-1} \\ [m]_n \end{pmatrix} \\ \hline \begin{pmatrix} [m]_n \\ (m)_{n-1} \end{pmatrix} \end{matrix} \right\rangle . \tag{19}$$

Note that when we apply the *individual* Wigner operators in Eq. (18) to an arbitrary basis vector (19), we should consider the common labels $[m]_n$ to be two identical sets of labels as indicated in Eq. (19). Note also that the two Wigner operators in Eq. (18) commute since they act in different spaces, and that the application of a *single* Wigner operator carries a vector *outside* of $U_n * U_n$, in the general case.

The matrix element of Eq. (18) between the initial state (19) and the final state

$$\left|\begin{pmatrix}(\mu')_{n-1}\\ [m]_n+[\Delta(\Gamma)]_n\\ (m')_{n-1}\end{pmatrix}\right\rangle ; \tag{20}$$

— where $(\Gamma)_n$ denotes a particular, but arbitrary, operator pattern appearing on the right-hand side of Eq. (18) — is as follows:

$$\begin{aligned}\left\langle\begin{pmatrix}(\mu')_{n-1}\\ [m]_n+[\Delta(\Gamma)]_n\\ (m')_{n-1}\end{pmatrix}\right| B\begin{pmatrix}(M')_{n-1}\\ [M]_n\\ (M)_{n-1}\end{pmatrix}\left|\begin{pmatrix}(\mu)_{n-1}\\ [m]_n\\ (m)_{n-1}\end{pmatrix}\right\rangle \\ = [\mathcal{M}([m]_n+[\Delta(\Gamma)]_n)/\mathcal{M}([m]_n)]^{1/2} \\ \times\sum_{(\Gamma')_{n-1}}\left\langle\begin{pmatrix}[m]_n+[\Delta(\Gamma)]_n\\ (m')_{n-1}\end{pmatrix}\right|\left\langle\begin{matrix}(\Gamma')_{n-1}\\ [M]_n\\ (M)_{n-1}\end{matrix}\right\rangle\left|\begin{pmatrix}[m]_n\\ (m)_{n-1}\end{pmatrix}\right\rangle \\ \times\left\langle\begin{pmatrix}[m]_n+[\Delta(\Gamma)]_n\\ (\mu')_{n-1}\end{pmatrix}\right|\left\langle\begin{matrix}(\Gamma')_{n-1}\\ [M]_n\\ (M')_{n-1}\end{matrix}\right\rangle\left|\begin{pmatrix}[m]_n\\ (\mu)_{n-1}\end{pmatrix}\right\rangle,\end{aligned} \tag{21}$$

where the sum is over all $(\Gamma')_{n-1}$ such that $(\Gamma')_n$ has $[\Delta(\Gamma')]_n = [\Delta(\Gamma)]_n$.

In particular, if

$$\left\langle\begin{matrix}(\Gamma)_{n-1}\\ [M]_n\\ (M)_{n-1}\end{matrix}\right\rangle$$

is non-degenerate, the right-hand side of Eq. (21) reduces to a *single term*. Since the left-hand side is, in principle, a known quantity, *Eq.* (21) *already determines,* except for choice of phase, *all non-degenerate Wigner coefficients.*

The present paper may be viewed as a detailed exposition of the consequences of this last remark; all of the results to follow are but deductions from Eqs. (12) and (21), put in a rather more transparent language.

III. The Fundamental Wigner Operators

In this section, the reduced Wigner coefficients of the fundamental $U(n)$ Wigner operators $\langle 1\,\dot{0}\rangle$ and their conjugates $\langle \dot{1}\,0\rangle$ are given in explicit form by the use of a diagrammatic technique, which we call "the pattern calculus". In the subsequent sections, the method is generalized to larger classes of Wigner operators.

First, let us introduce a simplified notation for the reduced Wigner operators

$$\begin{bmatrix}(\Gamma)_{n-1}\\ [1\,\dot{0}]_n\\ (\gamma)_{n-1}\end{bmatrix}$$

associated with the fundamental operators $\langle 1\ \dot{0}\rangle$ — since it is easily seen that operator patterns are highly redundant for such simple cases. Notice that the Δ-pattern of the upper (lower) operator pattern of $\langle 1\ \dot{0}\rangle$ is always of the form $\Delta_n(i) = [0 \ldots 0\ 1\ 0 \ldots 0]$, where 1 appears in position i $(i = 1, 2, \ldots, n)$. Note, also, that each Δ-pattern, $\Delta_n(i)$, *uniquely* determines the corresponding upper (lower) operator pattern. We may, therefore, map the operator patterns appearing in the notation (9) one-to-one onto the integers $1, 2, \ldots, n$. That is:

$$\begin{bmatrix} (\Gamma)_{n-1} \\ [1\ \dot{0}]_n \\ (\gamma)_{n-1} \end{bmatrix} \leftrightarrow \begin{bmatrix} i \\ [1\ \dot{0}]_n \\ j \end{bmatrix} \text{(for } i, j = 1, 2, \ldots, n) \tag{22}$$

denotes the (unique) reduced Wigner operator with the upper (lower) operator pattern which has Δ-pattern $\Delta_n(i)$ $(\Delta_n(j))$.

It is our goal now to demonstrate a pattern calculus for the matrix elements of $\langle 1\ \dot{0}\rangle$. To present the method most simply we shall first assert the rules, attempt to make them clear by examples, and afterwards verify the correctness of the result. The first step in the diagrammatic procedure is a prescription for writing out the *square* of the reduced matrix element

$$\left(\left\langle \begin{pmatrix} [m]_n + \Delta_n(i) \\ [m]_{n-1} + \Delta_{n-1}(j) \end{pmatrix} \right| \begin{bmatrix} i \\ [1\ \dot{0}]_n \\ j \end{bmatrix} \left| \begin{pmatrix} [m]_n \\ [m]_{n-1} \end{pmatrix} \right\rangle \right)^2 \equiv \begin{bmatrix} i \\ [1\ \dot{0}]_n \\ j \end{bmatrix}^2 . \tag{23}$$

As discussed in Section II, we also call the reduced matrix element in Eq. (23) a projection.

Let us now give the prescription whereby the square of the matrix element (23) may be written out. There are six rules; [15]:

Preliminary Rules

Rule 1. Write out two rows, n and $n-1$, of dots (n dots in each row) in the manner displayed below:

$$\begin{matrix} \bullet\ \bullet\ \bullet & & \bullet\ \bullet\ \bullet\ \bullet & \text{row } n \\ & \cdots & & \\ \bullet\ \bullet\ \bullet & & \bullet\ \bullet\ \bullet\ \bullet & \text{row } n-1\ . \end{matrix}$$

Rule 2. In row n, assign a 0 or a 1 to each dot according to whether 0 or 1 appears in the corresponding position in $\Delta_n(i)$; do the same in row $n-1$ using $\Delta_n(j)$.

Rule 3. Draw an arrow from each 1 (called "tail") to each 0 (called "head"). This produces what is called an *arrow pattern*.

Rule 4. In the arrow pattern obtained (the 1's and 0's assigned to the dots are now deleted) we assign to each dot i in row n the element (called a "partial hook") $p_{in} \equiv m_{in} + n - i$, where m_{in} is the element

i, n in the Gelfand pattern belonging to the initial vector in Eq. (23). Similarly assign $p_{j,n-1}$ to dot j in row $n-1$. (Note that we define $p_{n,n-1}$ as a formal element assigned to dot n in row $n-1$; this allows a formally larger symmetry, which proves very useful.)

Rule 5. Assign to each arrow the algebraic factor

$$p(\text{tail}) - p(\text{head}) + e(\text{tail}),$$

where the p_{ij} are the partial hooks of Rule 4, and

$$e(\text{tail}) = \begin{cases} 0 & \text{if tail in row } n \\ 1 & \text{if tail in row } n-1\,. \end{cases}$$

Rule 6. Write out the products:

$N^2 \equiv$ product of all factors for arrows going between rows,

$D^2 \equiv$ product of all factors for arrows going within rows.

Then

$$\begin{bmatrix} \overset{i}{[1\;\dot{0}]_n} \\ j \end{bmatrix}^2 = N^2/D^2\,. \qquad (24)$$

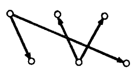
In this result $p_{n,n-1}$ is interpreted as a formally infinite element. All such factors cancel out in the final result. *Indeed, the* n^{th} *dot in row* $n-1$ *may simply be omitted in the procedure* (1)—(6), — and we shall do so in the final version of these rules, Section VI.

Before writing out the general result (24) and verifying its correctness, let us illustrate these results with an example in $U(3)$. Consider the reduced Wigner operator:

$$\begin{bmatrix} 1 \\ 1\;0 \\ 1\;0\;0 \\ 1\;0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\;0\;0 \\ 2 \end{bmatrix}.$$

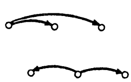
The upper pattern has $\Delta = [1\;0\;0]$ and the lower pattern has $\Delta = [0\;1\;0]$. It is convenient to display the numerator and denominator arrow patterns separately:

Numerator Arrow Pattern



$$N^2 = (p_{13} - p_{12})\,(p_{13} - p_{32})\,(p_{22} - p_{23} + 1)\,(p_{22} - p_{33} + 1)$$

Denominator Arrow Pattern



$$D^2 = (p_{13} - p_{23})\,(p_{13} - p_{33})\,(p_{22} - p_{12} + 1)\,(p_{22} - p_{32} + 1)\,.$$

Thus,

$$\begin{bmatrix} & 1 & \\ 1 & & 0 \\ 1 & 0 & 0 \\ 1 & & 0 \\ & 0 & \end{bmatrix}^2 = \left(\frac{N^2}{D^2}\right)_{p_{32}\to\infty} = \frac{(p_{13}-p_{12})(p_{22}-p_{23}+1)(p_{22}-p_{33}+1)}{(p_{13}-p_{23})(p_{13}-p_{33})(p_{22}-p_{12}+1)} \,.$$

Note that this is just the result obtained for the arrow pattern for

$$\Delta_3 = [1\ 0\ 0]$$

$$\Delta_2 = [0\ 1]\,.$$

It is clear therefore that we may suppress the formal element p_{32} from the beginning and write Δ_3 and Δ_2 patterns as part of a composite Δ-pattern $\left(\text{in this example the composite pattern is } \Delta = \begin{smallmatrix} 1 & 0 & 0 \\ & 0 & 1 \end{smallmatrix}\right)$, which (unlike the geometrically similar Gelfand pattern) does not obey betweenness.

The general result (24) is easy to write out using these rules: It is

$$N^2 = \prod_{\substack{j'=1\\ j'\neq i}}^{n-1} (p_{in} - p_{j',n-1}) \prod_{\substack{i'=1\\ i'\neq j}}^{n} (p_{j,n-1} - p_{i',n} + 1) \tag{25}$$

$$D^2 = \prod_{\substack{i'=1\\ i'\neq i}}^{n} (p_{in} - p_{i'n}) \prod_{\substack{j'=1\\ j'\neq j}}^{n-1} (p_{j,n-1} - p_{j',n-1} + 1)\,. \tag{26}$$

The validity of Eq. (24) with these factors above is now established by comparing with the same result obtained earlier ([6]). *Note, in particular, that for $j = n$, the last product factor of Eqs.* (25) *and* (26) *cancel in N^2/D^2, justifying in the general case our elimination of the formal element $p_{n,n-1}$.*

The preceding result would be just a novel observation if it were not a fact that the same method extends to a very much larger class of reduced Wigner operators, as will be clear in the sequel.

Let us illustrate this generality, in this section, by demonstrating the applicability of the same rules to the Wigner operators $\langle \dot{1}\ 0\rangle_n$, i.e., to the Wigner operators belonging to the irrep $[1\ldots 1\ 0]$, the conjugate irrep to $[1\ 0\ldots 0]$. The reduced Wigner operator 'conjugate' to

$$\begin{bmatrix} i \\ [1\ \dot{0}]_n \\ j \end{bmatrix}$$

may also be denoted by the simplified notation

$$\begin{bmatrix} i \\ [\dot{1}\ 0]_n \\ j \end{bmatrix}, \tag{27}$$

where now this expression denotes the reduced Wigner operator which has the unique upper and lower operator patterns corresponding to $\Delta_n(i) = [1 \ldots 1\,0\,1 \ldots 1]$ (0 in position i) and $\Delta_n(j) = [1 \ldots 1\,0\,1 \ldots 1]$ (0 in position j), respectively. [This specifies the precise sense — $1 \leftrightarrow 0$ in the Δ-pattern — in which the term 'conjugate' has been used.]

Applying rules (1)—(6) we obtain

$$\begin{bmatrix} i \\ [\dot{1}\ 0]_n \\ j \end{bmatrix}^2 = N^2/D^2\,, \tag{28a}$$

where

$$N^2 = \prod_{\substack{i'=1 \\ i' \neq i}}^{n} (p_{i'n} - p_{j,n-1}) \prod_{\substack{j'=1 \\ j' \neq j}}^{n-1} (p_{j',n-1} - p_{in} + 1) \tag{28b}$$

$$D^2 = \prod_{\substack{i'=1 \\ i' \neq i}}^{n} (p_{i'n} - p_{in}) \prod_{\substack{j'=1 \\ j' \neq j}}^{n-1} (p_{j',n-1} - p_{j,n-1} + 1)\,. \tag{28c}$$

This result may also be verified by referring to earlier results, [6].

We may summarize this section very briefly as follows: To every fundamental reduced Wigner operator $\begin{bmatrix} (\Gamma) \\ [1\ \dot{0}]_n \\ (\gamma) \end{bmatrix}$ we may associate a two-rowed composite Δ-pattern: $\begin{pmatrix} [\Delta(\Gamma)]_n \\ [\Delta(\gamma)]_{n-1} \end{pmatrix}$. Applying rules (1) to (6) we obtain an explicit algebraic formula for the associated reduced matrix element, *to within a sign* ($\pm$).

Using the iterative decomposition of Eq. (8) one thus determines, step-by-step, the reduced matrix elements for each projection:

$$[U_n : U_{n-1}],\ [U_{n-1} : U_{n-2}],\ \ldots,\ [U_2 : U_1].$$

The rules thereby determine explicitly (to within a sign) *all matrix elements of the fundamental operators* $\langle 1\,\dot{0} \rangle$ *in* $U(n)$, *for arbitrary* n, *in a remarkably simple way.*

The reader is invited to apply these rules to determine all spin-1/2 Wigner coefficients, in order to convince himself of the utility and economy of the pattern calculus.

The determination of the sign will be deferred until Section V. The essential idea which we develop next is based on the observation that the same rules actually work for the conjugate operators $\langle \dot{1}\,0 \rangle$. The rules, in fact, work for all elementary operators, which are defined and discussed in the following section.

IV. The Elementary Operators

In discussing the construction of an arbitrary irrep of $U(n)$ two types of 'building block' may be distinguished on the basis of CARTAN's work: (a) the fundamental irrep $[1\ \dot{0}]$ (or equivalently its conjugate $[\dot{1}\ 0]$) from which (by the Stone-Wigner proof of the Peter-Weyl theorem) all irreps may be built by reducing direct products, and (b) the elementary irreps $[1\ \dot{0}]$, $[1\ 1\ \dot{0}]$, ... $[\dot{1}\ 0]$, $[\dot{1}]$ from which an arbitrary irrep may be built by taking the highest weight vector in a direct product.

These results may be extended, *mutatis mutandis*, to Wigner operators in $U(n)$, from the corollaries to lemma 7 of Ref. (4). It is useful to note that $\langle\dot{1}\rangle = 1 \cdot \prod_{i=1}^{j} \prod_{j=1}^{n} \delta_{m_{ij}^{\text{final}}}^{m_{ij}^{\text{initial}}+1}$, and hence the construction of $U(n)$ Wigner operators effectively reduces to that of $SU(n)$ Wigner operators.

In particular, we define the elementary Wigner operators to be the set of tensor operators $\{\langle 1\ \dot{0}\rangle, \langle 1\ 1\ \dot{0}\rangle, \ldots, \langle\dot{1}\ 0\rangle\}$ — that is, the set of tensor operators $\{\langle\dot{1}_k\ \dot{0}_{n-k}\rangle\}$ for which the general operator transforms as the irrep $[\dot{1}_k\ \dot{0}_{n-k}]$ having k 1's and $n - k$ 0's, $k = 1 \ldots n$. This is clearly the analog to the concept of elementary irrep, and includes the fundamental irrep (and its conjugate) as special cases.

The reduced operator corresponding to an elementary Wigner operator $\langle\dot{1}_k\ \dot{0}_{n-k}\rangle$ is denoted by:

$$\begin{bmatrix} (\Gamma) \\ [\dot{1}_k\ \dot{0}_{n-k}] \\ (\gamma) \end{bmatrix}.$$

Once again, for brevity, it is convenient to use a simplified notation for the upper and lower operator patterns, namely:

$$\begin{bmatrix} (\Gamma) \\ [\dot{1}_k\ \dot{0}_{n-k}] \\ (\gamma) \end{bmatrix} \leftrightarrow \begin{bmatrix} (i_1 i_2 \ldots i_k) \\ [\dot{1}_k\ \dot{0}_{n-k}] \\ (j_1 j_2 \ldots j_k) \end{bmatrix}. \tag{29}$$

Here the $i_1 < i_2 < i_3 \cdots < i_k$ denote the k places $i_1 i_2 \ldots i_k$ in the Δ_n pattern where the 1's occur (with 0's in all other places). Analogously the numbers $(j_1 \ldots j_k)$ have the same significance with respect to Δ_{n-1} in the lower operator pattern. (Note that we have ordered these entries so as to have a unique correspondence.)

To every reduced elementary operator we can associate then two rows (Δ_n and Δ_{n-1}) of the Δ-pattern.

The essential remark now is this: *Rules* (1) *to* (6) *apply to the matrix elements of all reduced elementary operators (to within a* $\pm$ *sign).*

If we apply these rules, we obtain the explicit results:

$$N^2 = \prod_{s=1}^{k} \prod_{\substack{j'=1 \\ j' \neq (j_1 j_2 \ldots j_k)}}^{n} (p_{i_s,n} - p_{j',n-1}) \prod_{s=1}^{k} \prod_{\substack{i'=1 \\ i' \neq (i_1 i_2 \ldots i_k)}}^{n} (p_{j_s,n-1} - p_{i',n} + 1) \quad (30\text{a})$$

$$D^2 = \prod_{s=1}^{k} \prod_{\substack{j'=1 \\ j' \neq (j_1 \ldots j_k)}}^{n} (p_{j_s,n-1} - p_{j',n-1} + 1) \prod_{s=1}^{k} \prod_{\substack{i'=1 \\ i' \neq (i_1 \ldots i_k)}}^{n} (p_{i_s,n} - p_{i',n}) \,. \quad (30\text{b})$$

(Note that in these equations the pattern consists of n dots in both rows.) We defer the proof of these results to Section V.

At this point we might summarize by saying that rules (1) to (6) are a very convenient — and easily remembered — prescription for comprehending all elementary operators. This, while true, is not the full import of the rules: rather one should view these rules as assigning an arrow pattern to a projective operator, and more importantly, *arrow patterns in consequence have a natural multiplication.* Multiplication of projected Wigner operators corresponds to multiplication of arrow patterns, which in turn leads to the possibility of superposing arrow patterns. It is convenient to introduce, in this multiplication, the numerator and denominator arrow patterns as distinct entities, although of course they are intimately related. When these ideas are carried out (in Section VI), we arrive at a *pattern calculus*, i.e., a set of rules for multiplying arrow patterns to produce product arrow patterns which correspond to the product of the projective operators.

V. Proof of the Δ-Pattern Rules for Elementary Operators

In this section, two proofs are given that the rules (1)—(6) of Section III yield the reduced matrix elements of all elementary Wigner operators. The first proof is the simpler, but yields no information about the phase; accordingly we merely sketch the essential ideas. The second proof is straightforward, but lengthy and tedious; it establishes, however, the phase of all elementary reduced matrix elements (up to an arbitrary convention). In both proofs, the important results, Eqs. (12) and (21), play decisive roles.

The first proof is based on the properties of the matrix elements of the fundamental Wigner operators under permutations of the partial hooks. Namely, we observe that the projection

$$\begin{bmatrix} \overset{i}{\cdot} \\ [1\,\dot{0}]_n \\ \underset{j}{\cdot} \end{bmatrix} \quad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n \end{matrix}$$

can be obtained, except for phase, from the projection

$$\begin{bmatrix} 1 \\ [1\ \dot{0}]_n \\ 1 \end{bmatrix}$$

simply by interchanging p_{1n} with p_{in} and $p_{1,n-1}$ with $p_{j,n-1}$.

This result is in fact a direct consequence of the pattern calculus, since it is obviously true from the rules — an arbitrary permutation within each row of n dots takes an arrow pattern into an arrow pattern of the 'same kind'. [Recall that for the fundamental Wigner operators the rules have been demonstrated to be correct.]

It is important to note, however, that we have included the case $j = n$ in this symmetry; for this to be correct it is necessary to take the limit $p_{n,n-1} \to \infty$ *after* all permutations have been carried out.

From Eqs. (12) and (21), we see that this property extends to the matrix elements of the boson operator a^i_j. (The limit $p_{n,n-1} \to \infty$ is to be effected after the permutations are completed.)

Since every boson operator in $U(n) * U(n)$ is a polynomial in a^i_j it is clear that this permutational symmetry generalizes, provided only that any necessary limiting process can be properly treated. [This is always the case for the elementary operators, and the present heuristic argument is fully justified in the more detailed second proof given below.]

The significance of this permutational symmetry is readily appreciated in the calculation of the matrix elements of the reduced elementary Wigner operators:

Except for phase, we can obtain the projection

$$\begin{bmatrix} (i_1 i_2 \ldots i_k) \\ [\dot{1}_k\ \dot{0}_{n-k}] \\ (j_1 j_2 \ldots j_k) \end{bmatrix} \tag{31}$$

from the projection

$$\begin{bmatrix} (1\ 2 \ldots k) \\ [\dot{1}_k\ \dot{0}_{n-k}] \\ (1\ 2 \ldots k) \end{bmatrix} \tag{32}$$

by the transformation:

$$\begin{aligned} p_{i,n} &\to p_{i_i,n} \\ p_{j,n-1} &\to p_{j_j,n-1}\,. \end{aligned} \tag{33}$$

Observe, however, that the reduced matrix elements calculated from the rules satisfy precisely the above transformation property. Accordingly, to show that the rules work for all elementary operators, we need only show that they correctly give the result for (32).

The direct calculation of (32) is a relatively simple matter. Thus, from Eqs. (12) and (21), we have the result

$$\left\langle \begin{pmatrix} (\max)_{n-1} \\ [m]_n + \Delta(1\,2\ldots k) \\ [m]_{n-1} + \Delta(1\,2\ldots k) \\ (\max)_{n-2} \end{pmatrix} \middle| a^{1\,2\ldots k}_{1\,2\ldots k} \middle| \begin{pmatrix} (\max)_{n-1} \\ [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle = \{\mathcal{M}([m]_n + \Delta(1\,2\ldots k))/\mathcal{M}([m]_n)\}^{1/2} \cdot \begin{bmatrix} (1\,2\ldots k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ (1\,2\ldots k) \end{bmatrix}. \tag{34}$$

(Here we have used the fact that the Wigner operators $\begin{bmatrix} (1\,2\ldots k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ (1\,2\ldots k) \end{bmatrix}$ have unity for their matrix elements between maximal initial and final states.)

The state vectors that appear in the matrix element on the left-hand side of Eq. (34) are maximal in the upper $U(n)$ group and semi-maximal (that is, maximal in $U(n-1)$) in the lower $U(n)$ group. These are precisely the same state vectors used earlier [6] to determine the matrix elements of the generators; the importance of the semi-maximal condition lies in the fact that the state vectors involve a single ('monomial') boson operator and are easily given explicitly:

$$\left| \begin{pmatrix} (\max)_{n-1} \\ [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle = (M)^{-1/2} \prod_{l=1}^{n} (a^{1\,2\ldots l}_{1\,2\ldots l})^{m_{l,n-1}-m_{l+1,n}} \times (a^{1\,2\ldots l}_{1\,2\ldots l-1\,n})^{m_{ln}-m_{l,n-1}} |0\rangle . \tag{35}$$

Here M denotes the measure of the semi-maximal state; this factor may be written down directly from the hook structure of the Weyl basis tableau. It has the explicit form

$$M = \prod_{j=1}^{n} \frac{(p_{jn})!}{(p_{jn}-p_{nn})!} \prod_{i \leq j=1}^{n-1} \frac{(p_{i,n-1}-p_{j+1,n})!\,(p_{in}-p_{j,n-1}-1)!}{(p_{i,n-1}-p_{j,n-1})!\,(p_{in}-p_{jn})!} . \tag{36}$$

To evaluate the matrix element on the left-hand side of Eq. (34) most easily observe that the operator $a^{1\ldots k}_{1\ldots k}$ causes precisely the shifts $\Delta(1\,2\ldots k)$ in both the m_{in} and $m_{i,n-1}$ labels. Hence the square of this matrix element is just the ratio of the final to the initial measure: $[M(\text{final})/M(\text{initial})]$. (In this very literal sense the Δ-pattern may be interpreted as *a generalized difference operator acting on the hook measure.*)

We *define* the phase of the projection $\begin{bmatrix} (1\,2\ldots k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ (1\,2\ldots k) \end{bmatrix}$ to be $+1$; this is in accord with the conventional (Condon-Shortley-Wigner) phase for $SU(2)$.

It follows that for the desired projection we obtain the explicit result:

$$\begin{bmatrix} (1\,2\,\ldots\,k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ (1\,2\,\ldots\,k) \end{bmatrix} = + \left[\prod_{i=1}^{k} \left(\prod_{j=k+1}^{n} \frac{(p_{i,n-1} - p_{jn} + 1)}{(p_{in} - p_{jn})} \prod_{j=k+1}^{n-1} \frac{(p_{in} - p_{j,n-1})}{(p_{i,n-1} - p_{j,n-1} + 1)} \right) \right]^{1/2}. \tag{37}$$

This is precisely the result given by the rules (1)—(6). *Thus, we conclude that the rules correctly yield all reduced matrix elements of the elementary Wigner operators.*

The careful reader will note that in proving this result we have actually obtained an intermediate result of far-reaching significance, namely that *there exists a generalization of the Weyl group which applies to the projective structures (reduced Wigner operators) of* $U_n : U_{n-1}$. The operations of this symmetry group acting on reduced elementary operators interchange the partial hooks and carry the various projections into each other to within phases (which are determined below). The existence of this generalized Weyl symmetry group is significant for determining tensor operator structures; for example, this symmetry is essential to understanding 'symmetry-vanishings' of particular matrix elements, as noted earlier [8] in discussing the Clebsch-Gordan series.

In the second proof, we establish the explicit expression for the reduced matrix element (31) by induction. The result to be proved is:

$$\left\langle \begin{pmatrix} [m]_n + \Delta(i_1 i_2 \ldots i_k) \\ [m]_{n-1} + \Delta(j_1 j_2 \ldots j_k) \end{pmatrix} \middle| \begin{bmatrix} (i_1 i_2 \ldots i_k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ (j_1 j_2 \ldots j_k) \end{bmatrix} \middle| \begin{pmatrix} [m]_n \\ [m]_{n-1} \end{pmatrix} \right\rangle$$

$$= (-1)^{k(k-1)/2} \prod_{l,s=1}^{k} S(j_l - i_s) \tag{38}$$

$$\times \left[\prod_{l=1}^{k} \left\{ \prod_{\substack{j=1 \\ j \neq (j_1 j_2 \ldots j_k)}}^{n} \frac{(p_{i_l,n} - p_{j,n-1})}{(p_{j_l,n-1} - p_{j,n-1} + 1)} \cdot \prod_{\substack{i=1 \\ i \neq (i_1 i_2 \ldots i_k)}}^{n} \frac{(p_{j_l,n-1} - p_{in} + 1)}{(p_{i_l,n} - p_{in})} \right\} \right]^{\frac{1}{2}}$$

for $i_1 < i_2 < \cdots i_k$ and $j_1 < j_2 < \cdots < j_k$, where $S(j - i) \equiv \begin{cases} +1 & j \geqq i \\ -1 & j < i \end{cases}$, and the square root is defined to be positive.

An arbitrary phase convention has been adopted in Eq. (38): the phase of the matrix element of the reduced operator

$$\begin{bmatrix} (i_1 \ldots i_k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ (i_1 \ldots i_k) \end{bmatrix}$$

has been defined to be $+1$. This choice is arbitrary, but, for $k = 1$, it reduces to the choice made previously for the fundamental Wigner operators, and it agrees with the usual convention for $SU(2)$.

We first observe that Eq. (38) is correct for general n and $k = 1$ ([6]). Our induction proof then proceeds as follows: We assume the result above is correct for fixed n and all integers $1, 2, 3, \ldots, k-1$; we then prove it holds for k. Since, however, the result is known to be correct for general n and $k = 1$, this induction on k actually establishes the result for general n and k.

In carrying out the proof, use is made of the boson operator identity:

$$a^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} = \sum_{s=1}^{k} (-1)^{s+1} a^{i_2 i_3 \ldots i_k}_{j_1 \ldots j_{s-1} j_{s+1} \ldots j_k} a^{i_1}_{j_s}. \tag{39}$$

It may be helpful at this point to indicate the ideas behind the rather technical (and involved) manipulations to follow. The main idea is to apply the factorization lemma to both sides of the operator identity above, and then to take matrix elements between very particular initial and final state vectors in order to simplify the calculations.

Let us first apply the factorization lemma to the left-side of the identity; we can reduce the result to a single term by choosing the final state labels $[m']_n = [m]_n + \Delta(i_1 \ldots i_k)$, where $[m]_n$ denotes the initial state labels. Moreover if we choose the upper $U(n)$ labels to be maximal, the upper $U(n)$ matrix element is in its simplest possible form. We must choose the $[m]_{n-1}$ labels in lower $U(n)$ to be arbitrary to obtain a general answer and hence we impose the shift $\Delta(j_1 \ldots j_k)$ on these labels. There is no loss in generality if we take the $(m)_{n-2}$ sub-group labels (in lower space) to be maximal. Thus we obtain, using Eq. (21):

$$\begin{aligned}
&\left\langle \begin{pmatrix} (\max)_{n-1} \\ [m]_n + \Delta(i_1 i_2 \ldots i_k) \\ [m]_{n-1} + \Delta(j_1 j_2 \ldots j_k) \\ (\max)_{n-2} \end{pmatrix} \middle| a^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} \middle| \begin{pmatrix} (\max)_{n-1} \\ [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle \\
&= [\mathcal{M}(\text{final})/\mathcal{M}(\text{initial})]^{1/2} \\
&\times \left\langle \begin{pmatrix} [m]_n + \Delta(i_1 i_2 \ldots i_k) \\ (\max)_{n-1} \end{pmatrix} \middle| \left\langle \begin{matrix} (i_1 i_2 \ldots i_k) \\ [\dot{1}_k \dot{0}_{n-k}] \\ (i_1 i_2 \ldots i_k) \end{matrix} \right\rangle \middle| \begin{pmatrix} [m]_n \\ (\max)_{n-1} \end{pmatrix} \right\rangle_u \qquad (40) \\
&\times \left\langle \begin{pmatrix} [m]_n + \Delta(i_1 i_2 \ldots i_k) \\ [m]_{n-1} + \Delta(j_1 j_2 \ldots j_k) \\ (\max)_{n-2} \end{pmatrix} \middle| \left\langle \begin{matrix} (i_1 i_2 \ldots i_k) \\ [\dot{1}_k \dot{0}_{n-k}] \\ (j_1 j_2 \ldots j_k) \end{matrix} \right\rangle \middle| \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle_l .
\end{aligned}$$

The next step is to introduce the tensor operator decomposition, into projections, by means of Eq. (12). In this step we are allowed to evaluate the projections, using the assumed result Eq. (38), provided k is smaller than the running value. Consider the lower space matrix element in (40);

it reduces to:

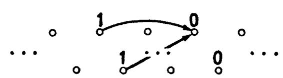
$$\left\langle \begin{pmatrix} [m]_n + \Delta(i_1 i_2 \dots i_k) \\ [m]_{n-1} + \Delta(j_1 j_2 \dots j_k) \\ (\max)_{n-2} \end{pmatrix} \middle| \left\langle \begin{matrix} (i_1 i_2 \dots i_k) \\ [\dot{1}_k\, \dot{0}_{n-k}] \\ (j_1 j_2 \dots j_k) \end{matrix} \right\rangle \middle| \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle$$
$$= \begin{bmatrix} (i_1 i_2 \dots i_k) \\ [\dot{1}_k\, \dot{0}_{n-k}] \\ (j_1 j_2 \dots j_k) \end{bmatrix} \cdot \left\langle \begin{pmatrix} [m]_{n-1} + \Delta(j_1 \dots j_k) \\ (\max)_{n-2} \end{pmatrix} \middle| \left\langle \begin{matrix} (j_1 j_2 \dots j_k) \\ [\dot{1}_k\, \dot{0}_{n-k-1}] \\ (j_1 j_2 \dots j_k) \end{matrix} \right\rangle \middle| \begin{pmatrix} [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle . \tag{41}$$

The first factor in the above equation is the desired general projection which we seek to determine; the second factor refers to $U(n-1)$ and may be evaluated from the assumed relation, Eq. (38). [Remark: To be completely precise, the case given above requires $j_k \neq n$. If $j_k = n$, then the U_{n-1} operator on the RHS has the partition (irrep) labels $[\dot{1}_{k-1}\, \dot{0}_{n-k}]$. This latter case can be treated completely analogously and the final result — Eq. (38) — does not distinguish the two cases. Hence we omit the proof for the case $j_k = n$.]

This evaluation affords an interesting application of the pattern calculus rules, and provides a nice example of the practical value of the arrow patterns. The evaluation proceeds by using the tensor operator decomposition; we obtain the sequence of projections $[U_{n-1}: U_{n-2}], \dots, [U_2: U_1]$ taken between *maximal* states. Consider an arbitrary projection in this sequence; it has the form:

$$\begin{bmatrix} (j_1 \dots j_{k'}) \\ [\dot{1}_{k'}\, \dot{0}_{n'-k'}] \\ (j_1 \dots j_{k'}) \end{bmatrix}_{\text{maximal}} ,$$

where $j_{k'} \leqq n'$, $n' \leqq n - 1$. This projection has the property that the shifts in the rows n' and $n' - 1$ are identical; thus a typical element in the arrow pattern will have the form:



Since the initial state vector is maximal, the contribution of these two arrows to the projection exactly *cancel*. It is easily recognized from the pattern calculus that the only projections not equal to $+1$ in the decomposition are precisely the k' projections $[U_{j_{k'}}: U_{j_{k'}-1}], [U_{j_{k'-1}}, U_{j_{k'-1}-1}], \dots [U_{j_1}: U_{j_1-1}]$. It is now not difficult to give the complete answer *(written for general n and k)*:

$$\left\langle \begin{pmatrix} [m]_n + \Delta(j_1 j_2 \dots j_k) \\ (\max)_{n-1} \end{pmatrix} \middle| \left\langle \begin{matrix} (j_1 j_2 \dots j_k) \\ [\dot{1}_k\, \dot{0}_{n-k}] \\ (j_1 j_2 \dots j_k) \end{matrix} \right\rangle \middle| \begin{pmatrix} [m]_n \\ (\max)_{n-1} \end{pmatrix} \right\rangle$$
$$= +1 \cdot \left[\prod_{l=1}^{k} \left[\prod_{\substack{s=1 \\ s \neq (j_1 \dots j_k)}}^{j_l} \frac{(p_{j_l,n} - p_{s,n} + 1)}{(p_{j_l,n} - p_{s,n})} \right] \right]^{\frac{1}{2}} . \tag{42}$$

The pattern calculus makes the curious structure of this result quite understandable (in particular, the reason why the product "avoids" the points $s = (j_1 \dots j_k)$.

The remaining matrix element in Eq. (40) is of precisely the same form as the matrix element just evaluated. Although at first glance we are not permitted to evaluate this matrix element by the formula above, in fact the evaluation *is* justified in every case except that for which $i_k = n$, and a special argument removes this restriction. (The reason is that the $[U_n : U_{n-1}]$ projection gives $+1$ from the general property that the direct product of maximal vectors is unique. For the case $i_k = n$ we leave this projection undetermined and include it along with the general projection being sought. This shows that evaluating this maximal matrix element at this stage is merely a convenience in arranging the recursion proof.)

Now let us apply the factorization lemma to the right-hand side of Eq. (39), again taking matrix elements. We obtain:

$$\begin{aligned}
&\langle \text{final}|\sum_{s=1}^{k} (-)^{s+1}\, a^{i_2 i_3 \ldots i_k}_{j_1 \ldots j_{s-1} j_{s+1} \ldots j_k}\, a^{i_1}_{j_s}\, |\text{initial}\rangle = [\mathcal{M}(\text{final})/\mathcal{M}(\text{initial})]^{1/2} \\
&\times \left\langle \begin{pmatrix} (m)_{\text{initial}} + \Delta(i_1 i_2 \ldots i_k) \\ \text{in all rows} \end{pmatrix} \middle| \left\langle \begin{matrix} (i_2 i_3 \ldots i_k) \\ [\dot{1}_{k-1}\, \dot{0}_{n-k+1}] \\ (i_2 i_3 \ldots i_k) \end{matrix} \right\rangle \middle| \begin{pmatrix} (m)_{\text{initial}} + \Delta(i_1) \\ \text{in rows } n \text{ to } i_1 \text{ only} \end{pmatrix} \right\rangle \\
&\times \left\langle \begin{pmatrix} (m)_{\text{initial}} + \Delta(i_1) \\ \text{rows } n \text{ to } i_1 \end{pmatrix} \middle| \left\langle \begin{matrix} i_1 \\ [1\ \dot{0}] \\ i_1 \end{matrix} \right\rangle \middle| \begin{pmatrix} (m)_{\text{initial}} \equiv \begin{matrix} [m]_n \\ (\max)_{n-1} \end{matrix} \end{pmatrix} \right\rangle \\
&\times \sum_{s=1}^{k} (-)^{s+1} \sum_{q=1}^{s} \left\langle \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} + \begin{pmatrix} \Delta(i_1 i_2 \ldots i_k) \\ \Delta(j_1 j_2 \ldots j_k) \\ \text{in all rows } 1 \text{ to } n-1 \end{pmatrix} \right| \qquad (43) \\
&\times \left\langle \begin{matrix} (i_2 i_3 \ldots i_k) \\ [\dot{1}_{k-1}\, \dot{0}_{n-k+1}] \\ (j_1 j_2 \ldots j_{s-1} j_{s+1} \ldots j_k) \end{matrix} \right\rangle \left| (\text{intermediate}) \right\rangle \\
&\times \left\langle (\text{intermediate}) \middle| \left\langle \begin{matrix} i_1 \\ [1\ \dot{0}] \\ j_s \end{matrix} \right\rangle \middle| \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle ,
\end{aligned}$$

where

$$|(\text{intermediate})\rangle \equiv \left| \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} + \begin{pmatrix} \Delta(i_1) \\ \Delta(j_q) \\ \text{in rows } n-1 \text{ to } j_s \end{pmatrix} \right\rangle .$$

Before evaluating these matrix elements let us note various features of this result. Firstly the invariant operator $\mathcal{M}(\ldots)$ occurs in such a way that only the initial and final irrep labels enter; in fact this term cancels out from both sides of Eq. (39). Next one sees that the upper space matrix element — since the states are maximal — allows but a single intermediate state between the operators $\langle \dot{1}_{k-1}\, \dot{0}_{n+1-k} \rangle$ and $\langle 1\ \dot{0} \rangle$. The really essential point is to note that for the analogous step involving the lower space matrix element, the Gelfand pattern conditions allow *only* the intermediate states shown.

All matrix elements in Eq. (43) can be evaluated from the assumed validity of Eq. (38). This evaluation, although a bit tedious, is nevertheless clearly possible; we give only the final results, written in the form most convenient to the purpose at hand. For the upper space matrix element we find:

$$\left\langle \begin{pmatrix} [m_n] + \Delta(i_1 i_2 \ldots i_k) \\ (\max)_{n-1} \end{pmatrix} \right| \left\langle \begin{matrix} (i_2 i_3 \ldots i_k) \\ [\dot{1}_{k-1}\, \dot{0}_{n-k+1}] \\ (i_2 i_3 \ldots i_k) \end{matrix} \right\rangle \left\langle \begin{matrix} i_1 \\ [1\ \dot{0}] \\ i_1 \end{matrix} \right\rangle \left| \begin{pmatrix} [m]_n \\ (\max)_{n-1} \end{pmatrix} \right\rangle$$

$$= \left[\prod_{l=2}^{k} \frac{(p_{i_1,n} - p_{i_l,n})}{(p_{i_1,n} - p_{i_l,n} + 1)} \right]^{1/2} \tag{44}$$

$$\times \left\langle \begin{pmatrix} [m]_n + \Delta(i_1 i_2 \ldots i_k) \\ (\max)_{n-1} \end{pmatrix} \right| \left\langle \begin{matrix} (i_1 i_2 \ldots i_k) \\ [\dot{1}_k\, \dot{0}_{n-k}] \\ (i_1 i_2 \ldots i_k) \end{matrix} \right\rangle \left| \begin{pmatrix} [m]_n \\ (\max)_{n-1} \end{pmatrix} \right\rangle .$$

For the two lower space matrix elements (the two terms in the sum over q in Eq. 43) we first reduce each into a $[U(n) : U(n-1)]$ projection and a $U(n-1)$ matrix element. That is:

$$\left\langle \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} + \begin{pmatrix} \Delta(i_1 i_2 \ldots i_k) \\ \Delta(j_1 j_2 \ldots j_k) \\ \text{in all rows 1 to } n-1 \end{pmatrix} \right| \left\langle \begin{matrix} (i_2 i_3 \ldots i_k) \\ [\dot{1}_{k-1}\, \dot{0}_{n-k+1}] \\ (j_1 j_2 \ldots j_{s-1} j_{s+1} \ldots j_n) \end{matrix} \right\rangle \left| (\text{intermediate}) \right\rangle$$

$$= \left\langle \begin{pmatrix} [m]_n + \Delta(i_1 i_2 \ldots i_k) \\ [m]_{n-1} + \Delta(j_1 j_2 \ldots j_k) \end{pmatrix} \right| \left[\begin{matrix} (i_2 i_3 \ldots i_k) \\ [\dot{1}_{k-1}\, \dot{0}_{n-k+1}] \\ (j_1 \ldots j_{q-1} j_{q+1} \ldots j_k) \end{matrix} \right] \left| \begin{pmatrix} [m]_n + \Delta(i_1) \\ [m]_{n-1} + \Delta(j_q) \end{pmatrix} \right\rangle$$

$$\times \left\langle \begin{pmatrix} [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} + \begin{pmatrix} \Delta(j_1 j_2 \ldots j_k) \\ \text{in all rows} \end{pmatrix} \right| \left\langle \begin{matrix} (j_1 \ldots j_{q-1} j_{q+1} \ldots j_k) \\ [\dot{1}_{k-1}\, \dot{0}_{n-k}] \\ (j_1 \ldots j_{s-1} j_{s+1} \ldots j_k) \end{matrix} \right\rangle \tag{45a}$$

$$\times \left| \begin{pmatrix} [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} + \begin{pmatrix} \Delta(j_q) \text{ in} \\ \text{rows } n-1 \\ \text{to } j_s \end{pmatrix} \right\rangle \equiv A \cdot B .$$

$$\left\langle (\text{intermediate}) \right| \left\langle \begin{matrix} i_1 \\ [1\ \dot{0}] \\ j_s \end{matrix} \right\rangle \left| \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} [m]_n + \Delta(i_1) \\ [m]_{n-1} + \Delta(j_q) \end{pmatrix} \right| \left[\begin{matrix} i_1 \\ [1\ \dot{0}] \\ j_q \end{matrix} \right] \left| \begin{pmatrix} [m]_n \\ [m]_{n-1} \end{pmatrix} \right\rangle$$

$$\times \left\langle \begin{pmatrix} [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} + \begin{pmatrix} \Delta(j_q) \text{ in rows} \\ n-1 \text{ to } j_s \end{pmatrix} \right| \left\langle \begin{matrix} j_q \\ [1\ \dot{0}] \\ i_s \end{matrix} \right\rangle \left| \begin{pmatrix} [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle \equiv C \cdot D . \tag{45b}$$

The product of the two $U(n-1)$ matrix elements (B from Eq. (45a), D from Eq. (45b) can be explicitly evaluated using Eq. (38). We find:

$$B \cdot D = \frac{(-)^{k-s} \prod_{l=s+1}^{k} (p_{j_l,n-1} - p_{j_q,n-1}) \cdot \prod_{l=1}^{s-1} (p_{j_l,n-1} - p_{j_q,n-1} - 1)}{\left[\prod_{\substack{l=1\\ l\neq q}}^{k} (p_{j_l,n-1} - p_{j_q,n-1})(p_{j_l,n-1} - p_{j_q,n-1} - 1)\right]^{1/2}} \tag{46a}$$
$$\times \left\langle \begin{pmatrix} [m]_{n-1} + \Delta(j_1 \ldots j_k) \\ (\max)_{n-2} \end{pmatrix} \middle| \left\langle \begin{matrix} (j_1 \ldots j_k) \\ [\dot{1}_k \, \dot{0}_{n-k-1}] \\ (j_1 \ldots j_k) \end{matrix} \right\rangle \middle| \begin{pmatrix} [m]_{n-1} \\ (\max)_{n-2} \end{pmatrix} \right\rangle .$$

The product of the two $[U(n) : U(n-1)]$ projections $(A \cdot C)$ may also be evaluated explicitly in a similar way. The result is:

$$A \cdot C = \frac{(-1)^{k-1} \prod_{l=2}^{k} (p_{j_q,n-1} - p_{i_l,n} + 1) \cdot \prod_{\substack{l=1\\ l\neq q}}^{k} (p_{j_l,n-1} - p_{i_1,n})}{\left[\prod_{l=2}^{k} (p_{i_1,n} - p_{i_l,n})(p_{i_1,n} - p_{i_l,n} + 1) \times \prod_{\substack{l=1\\ l\neq q}}^{k} (p_{j_l,n-1} - p_{j_q,n-1})(p_{j_l,n-1} - p_{j_q,n-1} - 1)\right]^{1/2}}$$
$$\times \left\langle \begin{pmatrix} [m]_n + \Delta(i_1 \ldots i_k) \\ [m]_{n-1} + \Delta(j_1 \ldots j_n) \end{pmatrix} \middle| \begin{bmatrix} (i_1 \ldots i_k) \\ [\dot{1}_k \, \dot{0}_{n-k}] \\ (j_1 \ldots j_k) \end{bmatrix} \middle| \begin{pmatrix} [m]_n \\ [m]_{n-1} \end{pmatrix} \right\rangle . \tag{46b}$$

It is necessary to point out explicitly that the use of the $[U(n) : U(n-1)]$ projection in Eqs. (44) and (46b) above is a *definition* (modelled, of course, on Eq. (38)) — we write it in this way solely for convenience.

We have chosen to write the terms in this form since it is then clear that upon introducing these results into Eq. (39) all projections and matrix elements cancel nicely on both sides, leaving us with the following sum:

$$\mathcal{S} = (-1) \cdot \sum_{s=1}^{k} \sum_{q=1}^{s} \left[\prod_{l=2}^{k} \frac{(p_{j_q,n-1} - p_{i_l,n} + 1)}{(p_{i_1,n} - p_{i_l,n} + 1)} \times \frac{\prod_{\substack{l=1\\ l\neq q}}^{k} (p_{j_l,n-1} - p_{i_1,n})}{\prod_{\substack{l=1\\ l\neq q}}^{s} (p_{j_l,n-1} - p_{j_q,n-1}) \cdot \prod_{l=s}^{k} (p_{j_l,n-1} - p_{j_q,n-1} - 1)} \right] . \tag{47}$$

The induction proof will have been accomplished if we can demonstrate that the formidable appearing sum above is precisely *unity*. This final step in the proof is of little intrinsic interest, and is carried out in the Appendix.

In summary, one sees that a direct application of the factorization lemma to the boson operator identity (Eq. (39)) demonstrates, by induction, that the result, Eq. (38), for the explicit matrix elements, including phase, of all elementary projections is correct. The sole arbitrariness in this result lies in the phase convention which assigns $+1$ to the projections

$$\begin{bmatrix} (i_1 \ldots i_k) \\ [\dot{1}_k \, \dot{0}_{n-k}] \\ (i_1 \ldots i_k) \end{bmatrix}$$

in accord with the Condon-Shortley-Wigner convention for $SU(2)$.

It is perhaps unnecessary to remark that the modulus of this result (Eq. (38)) (that is, the projections omitting phase) is in agreement with the result established by the pattern calculus rules (1) to (6). Since these rules necessarily incorporate permutational symmetry, it thus established that: *all elementary projections satisfy a generalized Weyl (permutational) symmetry.*

VI. Generalization of the Δ-Pattern Rules to Extremal Operators

The possibility of extending the rules for determining explicit matrix elements to a true pattern calculus — in which patterns may be multiplied — has already been noted in Section IV. It is the purpose of the present section to examine this possibility in some detail. In so doing we are led to a natural generalization of the Δ-pattern rules and to the concept of extremal patterns. It is shown that the matrix elements of all extremal operators can then be explicitly given.

The elementary operators have the property that their *operator patterns* were equivalent to their *Δ-patterns*, in that each implies the other. Let us investigate this property further, assuming (somewhat vaguely at the moment) that given the $U(n)$ operator labels $[M]_n$, then $[\Delta(\Gamma)]_n$ and $[\Delta(\gamma)]_n$ denote, respectively, any Δ-patterns which uniquely determine $(\Gamma)_{n-1}$ and $(\gamma)_{n-1}$ in the operator patterns:

$$\begin{pmatrix} [M]_n \\ (\Gamma)_{n-1} \end{pmatrix}, \begin{pmatrix} [M]_n \\ (\gamma)_{n-1} \end{pmatrix}.$$

The $U(n)$ Wigner operator

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{matrix} \right\rangle$$

is then uniquely determined by $[\Delta(\Gamma)]_n$, and is therefore non-degenerate (in $U(n)$). Furthermore, the $U(n-1)$ Wigner operator

$$\left\langle\begin{matrix}(\gamma)_{n-2}\\ [\gamma]_{n-1}\\ (M)_{n-2}\end{matrix}\right\rangle$$

is uniquely determined by $[\Delta(\gamma)]_n$, and is likewise non-degenerate in $U(n-1)$ — hence the $[U(n):U(n-1)]$ projections are unique. We therefore obtain the following result from Eq. (12):

$$\left\langle\begin{pmatrix}[m]_n+[\Delta(\Gamma)]_n\\ [m]_{n-1}+[\Delta(\gamma)]_{n-1}\\ (m')_{n-2}\end{pmatrix}\right|\left\langle\begin{matrix}(\Gamma)_{n-1}\\ [M]_n\\ (M)_{n-1}\end{matrix}\right\rangle\left|\begin{pmatrix}[m]_n\\ [m]_{n-1}\\ (m)_{n-2}\end{pmatrix}\right\rangle$$

$$=\left\langle\begin{pmatrix}[m]_n+[\Delta(\Gamma)]_n\\ [m]_{n-1}+[\Delta(\gamma)]_{n-1}\end{pmatrix}\right|\begin{bmatrix}(\Gamma)_{n-1}\\ [M]_n\\ (\gamma)_{n-1}\end{bmatrix}\left|\begin{pmatrix}[m]_n\\ [m]_{n-1}\end{pmatrix}\right\rangle \tag{48}$$

$$\times\left\langle\begin{pmatrix}[m]_{n-1}+[\Delta(\gamma)]_{n-1}\\ (m')_{n-2}\end{pmatrix}\right|\left\langle\begin{matrix}(\gamma)_{n-2}\\ [\gamma]_{n-1}\\ (M)_{n-2}\end{matrix}\right\rangle\left|\begin{pmatrix}[m]_{n-1}\\ (m)_{n-2}\end{pmatrix}\right\rangle.$$

The first term on the right in the equation above is the reduced matrix element of the projection in $[U(n):U(n-1)]$. Earlier, in Section II, we have discussed the formal development whereby we may write this projection as the projective operator:

$$\begin{bmatrix}(\Gamma)_{n-1}\\ [M]_n\\ (\gamma)_{n-1}\end{bmatrix}.$$

This operator, *by definition*, carries an arbitrary vector with $U(n)$ labels $[m]_n$ and $U(n-1)$ labels $[m]_{n-1}$ into a vector with $U(n)$ labels $[m]_n+$ $+[\Delta(\Gamma)]_n$ and $U(n-1)$ labels $[m]_{n-1}+[\Delta(\gamma)]_{n-1}$; that is,

$$\begin{bmatrix}(\Gamma)_{n-1}\\ [M]_n\\ (\gamma)_{n-1}\end{bmatrix}\left|\begin{pmatrix}[m]_n\\ [m]_{n-1}\end{pmatrix}\right\rangle=\#\left|\begin{pmatrix}[m]_n+[\Delta(\Gamma)]_n\\ [m]_{n-1}+[\Delta(\gamma)]_{n-1}\end{pmatrix}\right\rangle, \tag{49a}$$

where # is given by the reduced matrix element

$$\#=\left\langle\begin{pmatrix}[m]_n+[\Delta(\Gamma)]_n\\ [m]_{n-1}+[\Delta(\gamma)]_{n-1}\end{pmatrix}\right|\begin{bmatrix}(\Gamma)_{n-1}\\ [M]_n\\ (\gamma)_{n-1}\end{bmatrix}\left|\begin{pmatrix}[m]_n\\ [m]_{n-1}\end{pmatrix}\right\rangle. \tag{49b}$$

In other words we have defined by abstraction, operators acting in the space $U(n):U(n-1)$. Let now

$$\begin{bmatrix}(\Gamma')_{n-1}\\ [M']_n\\ (\gamma')_{n-1}\end{bmatrix}$$

denote a second operator in which $[\Delta(\Gamma')]_n$ and $[\Delta(\gamma')]_n$ uniquely determine $(\Gamma')_{n-1}$ and $(\gamma')_{n-1}$, respectively. Application of this operator to Eq. (49a) yields

$$\begin{bmatrix} (\Gamma')_{n-1} \\ [M']_n \\ (\gamma')_{n-1} \end{bmatrix} \begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix} \left| \begin{pmatrix} [m]_n \\ [m]_{n-1} \end{pmatrix} \right\rangle = (\#')(\#) \left| \begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n + [\Delta(\Gamma')]_n \\ [m]_{n-1} + [\Delta(\gamma)]_{n-1} + [\Delta(\gamma')]_{n-1} \end{pmatrix} \right\rangle, \tag{50a}$$

where

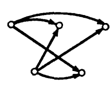
$$\#' = \left\langle \begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n + [\Delta(\Gamma')]_n \\ [m]_{n-1} + [\Delta(\gamma)]_n + [\Delta(\gamma')]_{n-1} \end{pmatrix} \right| \begin{bmatrix} (\Gamma')_{n-1} \\ [M']_n \\ (\gamma')_{n-1} \end{bmatrix} \left| \begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n \\ [m]_{n-1} + [\Delta(\gamma)]_{n-1} \end{pmatrix} \right\rangle. \tag{50b}$$

It should be noted that this result really demonstrates that the product of *any* two projections has a well-defined meaning for vectors of $[U(n):U(n-1)]$; the specialization to operator patterns equivalent to Δ-patterns is not necessary. The difficulty in defining products of projections lies rather in determining the circumstances under which a product of projections is equivalent to a projection. A sufficient condition is that the product of projections, for each of which $(\gamma) \Leftrightarrow \Delta(\gamma)$, is again a projection if $(\gamma_{\text{product}}) \Leftrightarrow \Delta(\gamma)_{\text{product}}$. We shall discuss this topic in more detail presently.

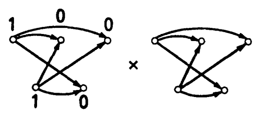
For the moment, let us consider a few examples of products of projections in order to indicate the necessity of generalizing the Δ-pattern rules. Consider the product of two $[U(3):U(2)]$ projections:

$$\begin{bmatrix} & 1 & \\ 1 & 0 & 0 \\ & 1 & \end{bmatrix} \times \begin{bmatrix} & 1 & \\ 1 & 0 & 0 \\ & 1 & \end{bmatrix}.$$

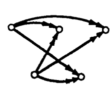
The arrow pattern for each of the two projections is given by



For the product, we note that the first operator (reading from *right* to *left*) shifts the state labels seen by the second operator: we indicate this by the shift $\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 0 & \end{smallmatrix}$ entered on the second arrow pattern below:



Let us next try to write this product of arrow patterns *by superposing* them:



We have seen that the product of two projections is associated with a well-defined algebraic expression (determined by the rules). If we compare this expression with the superposed arrow pattern we immediately get a hint for extending the rules: *k-multiple arrows correspond to factorials of k terms. The shifts induced by the operators to the right in the product accomplish the indexing for the factorial.*

This example suffices to motivate the following generalization of the rules. First we repeat the construction of the Δ-pattern.

The Δ-Pattern Rules:

Given a projective operator denoted by

$$\begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix}.$$

To this projection we associate a Δ-pattern of two rows written in the form of a Gelfand pattern, but without the betweenness condition. The entries Δ_{in} $(i = 1, \ldots n)$ are determined by:

$$\Delta_{in}(\Gamma) \equiv \sum_{j=1}^{i} \Gamma_{ji} - \sum_{j=1}^{i-1} \Gamma_{j,i-1}, \quad (i = 1, 2, \ldots, n). \tag{6}$$

Note the entries $\Delta_{i,n-1}$ $(i = 1, \ldots, n-1)$ are determined by the same equation with (γ) replacing (Γ). Example:

$$\begin{bmatrix} 1 \\ 2\ 0 \\ 3\ 0\ 0 \\ 3\ 0 \\ 3 \end{bmatrix} \leftrightarrow \Delta = \begin{matrix} 1\ 1\ 1 \\ 3\ 0 \end{matrix}.$$

The Arrow-Pattern Rules:

Rule 1. Write out two rows of dots, as shown:

$$\begin{matrix} \bullet\bullet\bullet & \cdots & \bullet\bullet\bullet & n \text{ dots} \\ \bullet\bullet\bullet & & \bullet\bullet & n-1 \text{ dots}. \end{matrix}$$

Rule 2. Draw arrows between dots as follows: Select a dot i in row n and a dot j in row $n-1$. If $\Delta_{in}(\Gamma) > \Delta_{jn}(\gamma)$, draw $\Delta_{in}(\Gamma) - \Delta_{jn}(\gamma)$ arrows from dot i to dot j; if $\Delta_{in}(\Gamma) < \Delta_{jn}(\gamma)$, draw the arrows from j to dot i. Carry out this procedure for all dots in rows n and $n-1$. This yields a *numerator arrow pattern* with arrows going *between* rows.

Carry out this procedure for dots within row n and dots within row $n-1$. This yields a *denominator arrow pattern* with arrows going *within* rows.

Rule 3. In the arrow patterns, assign the partial hook $p_{i,n}$ to dot i $(i = 1, 2, \ldots, n)$ in row n; $p_{j,n-1}$ to dot j $(j = 1, 2, \ldots, n-1)$ in row $n-1$. $(p_{ij} \equiv m_{ij} + j - i)$.

Rule 4. In general, there will now be several arrows going between two dots in the arrow patterns. Assign to the first arrow the factor

$$p(\text{tail}) - p(\text{head}) + e(\text{tail})\,;$$

to the second arrow, the factor

$$p(\text{tail}) - p(\text{head}) + e(\text{tail}) + 1\,;$$

etc., until all arrows going between the same two dots have been counted.

$$\left(\text{Recall that } e(\text{tail}) \equiv \begin{cases} 1 & \text{if tail of arrow on row } n-1 \\ 0 & \text{if tail of arrow on row } n \end{cases}\right).$$

Rule 5. Write out the products

N^2 = product of all factors for numerator arrow pattern,

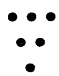
D^2 = product of all factors for denominator arrow pattern.

The net effect of these rules (which we call the "extended pattern rules") is to make the associations:

$$\begin{bmatrix} (\Gamma) \\ [M] \\ (\gamma) \end{bmatrix} \leftrightarrow \varDelta\text{-pattern} \leftrightarrow \begin{matrix}\text{arrow}\\ \text{pattern}\end{matrix} \leftrightarrow \begin{matrix}\text{algebraic}\\ \text{factor}\end{matrix} \equiv |N/D|\,.$$

Clearly the rules associate a well-defined algebraic factor to every projection, but this factor is, in general (that is, for arbitrary projective operators), *not* the modulus of the matrix element of the operator. However, there is a non-trivial class of operator patterns for which the rules do yield the square of the reduced matrix element. These are the *extremal patterns* which we now define.

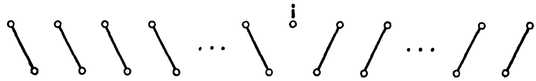
Consider any triangular array of n rows of dots, the dots being arranged in the manner of the entries of a Gelfand pattern, e.g., for $n = 3$,



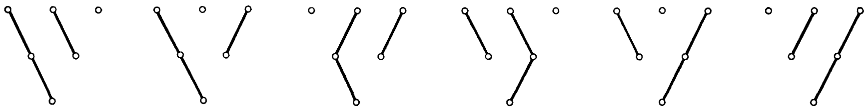
The point (i, j) $(i \leqq j = 1, 2, \ldots, n)$ is said to be *tied maximally* when it is identified with the point $(i, j + 1)$, and we indicate this by drawing a line, called a *tie*, between the two points. Similarly, the point (i, j) is said to be *tied minimally* when it is identified with the point $(i + 1, j + 1)$. The n-rowed array, with ties, is called an *extremally tied pattern in row* $k - 1$ if every point in row $k - 1$ which lies to the left (right) of any point i (called the free point of row k) is tied maximally (minimally). Thus, the most general appearance of row k and row $k - 1$ in an extremally tied pattern in row $k - 1$ is:

row k

row $k - 1$



The n-rowed array, with ties, is called an extremal pattern, or synonymously, extremal tie-pattern, if it is an extremally tied pattern in each of its rows $1, 2, \ldots, n-1$. Clearly, there are $n!$ distinct extremal patterns associated with every n-rowed triangular array of dots. For example, for $n = 3$, we have 6 extremal patterns:

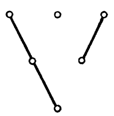


An operator pattern $(\Gamma)_n$ is said to be *extremal* if the only integers which appear in it are $\Gamma_{1n}, \Gamma_{2n}, \ldots, \Gamma_{nn}$, where $[\Gamma]_n = [\Gamma_{1n}\Gamma_{2n} \ldots \Gamma_{nn}]$ is the first row of $(\Gamma)_n$. Clearly, we obtain *all* extremal operator patterns by assigning Γ_{in} to point i ($i = 1, 2, \ldots, n$) of row n in an extremal tie-pattern. Notice that if $\Gamma_{1n} > \Gamma_{2n} > \cdots > \Gamma_{nn}$, we obtain in this manner $n!$ distinct operator patterns. However, if some of the Γ_{in} are equal, distinct extremal tie-patterns may yield the same operator pattern. Conversely, each extremal operator pattern has associated with it at least one extremal pattern.

Two extremal operator patterns $(\Gamma)_n$ and $(\Gamma')_n$ are said to have the same *tie-structure* if they can be obtained from the same extremal pattern. For example,

$$\begin{matrix} 3 & & 1 & & 0 \\ & 3 & & 0 & \\ & & 3 & & \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{matrix}$$

have the same tie structure since they can both be obtained from the extremal tie-pattern



The Δ-pattern of an extremal operator pattern is always a permutation of $[\Gamma_{1n}\Gamma_{2n} \ldots \Gamma_{nn}]$, and every permutation occurs when we consider all extremal operator patterns with first row $[\Gamma_{1n}\Gamma_{2n} \ldots \Gamma_{nn}]$. Conversely, every Δ-pattern which is a permutation of $[\Gamma_{1n}\Gamma_{2n} \ldots \Gamma_{nn}]$ uniquely determines an extremal operator pattern $\begin{pmatrix}(\Gamma)_{n-1} \\ [\Gamma]_n\end{pmatrix}$. Furthermore, if Δ is a permutation of $[\Gamma_{1n}\Gamma_{2n} \ldots \Gamma_{nn}]$ and Δ' is the same permutation of $[\Gamma'_{1n}\Gamma'_{2n} \ldots \Gamma'_{nn}]$, then the extremal operator patterns $(\Gamma)_n$ and $(\Gamma')_n$ have the same tie-structure.

In order to form a third Wigner operator from two given Wigner operators, the two tensor operators must, in general, be *coupled* not only in their lower Gelfand patterns, but also in their upper operator patterns

[8]. However if we consider not only extremal operator patterns $(\Gamma)_n$ and $(\Gamma')_n$ of the same tie-structure (upper pattern), but also lower Gelfand patterns $(\gamma)_n$ and $(\gamma')_n$ which are extremal and of the same tie-structure (lower pattern), then the coupling assumes the simple form:

$$\left\langle \begin{matrix} (\Gamma')_{n-1} \\ [M']_n \\ (\gamma')_{n-1} \end{matrix} \right\rangle \left\langle \begin{matrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{matrix} \right\rangle = \left\langle \begin{matrix} (\Gamma)_{n-1} + (\Gamma')_{n-1} \\ [M]_n + [M']_n \\ (\gamma)_{n-1} + (\gamma')_{n-1} \end{matrix} \right\rangle . \tag{51a}$$

Expressed in words, such Wigner operators 'multiply' by addition of their patterns.

The following property of the projective operators can now be deduced from the above result, Eq. (51a):

$$\begin{bmatrix} (\Gamma')_{n-1} \\ [M')_n \\ (\gamma')_{n-1} \end{bmatrix} \cdot \begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix} = \begin{bmatrix} (\Gamma)_{n-1} + (\Gamma')_{n-1} \\ [M]_n + [M']_n \\ (\gamma)_{n-1} + (\gamma')_{n-1} \end{bmatrix} . \tag{51b}$$

In this result, all upper and lower operator patterns are extremal, upper patterns have the same tie-structure, and lower patterns have the same tie-structure. An immediate consequence of Eq. (51b) is: *All extremal projective operators* (both upper and lower patterns extremal) *which have the same tie-structures commute.*

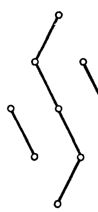
A second important result also follows: *Every extremal projective operator has a unique decomposition into an extremal projective operator of the same tie-structure times an elementary projective operator of the same tie-structure.* (Note that an elementary projective operator is always extremal):

$$\begin{bmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{bmatrix} = \begin{bmatrix} (\Gamma')_{n-1} \\ [M']_n \\ (\gamma')_{n-1} \end{bmatrix} \cdot \begin{bmatrix} (i_1 i_2 \ldots i_k) \\ [\dot{1}_k \, \dot{0}_{n-k}] \\ (j_1 j_2 \ldots j_k) \end{bmatrix} . \tag{52}$$

In this expression, k denotes the first integer such that $M_{kn} > M_{k+1,n}$. It immediately follows from this result that every extremal projection may be decomposed into a product of elementary projections of the same tie-structure (hence these projections commute). Let us illustrate this property with an example from $U(3)$:

$$\begin{bmatrix} & & 1 & & \\ & 1 & & 0 & \\ 3 & & 1 & & 0 \\ & 3 & & 1 & \\ & & 1 & & \end{bmatrix} = \begin{bmatrix} & & 1 & & \\ & 1 & & 0 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix} \cdot \begin{bmatrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} = \begin{bmatrix} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} \begin{bmatrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \begin{bmatrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} ,$$

where the common tie pattern is



Equation (52) is the basic result needed to give the proof that the extended rules yield the square of the matrix element denoted by # in Eq. (49b) for all extremal operator patterns. The proof is by induction. Assume the extended rules are correct for all extremal projections with labels $[M']_n$ such that

$$\sum_{i=1}^{n} M'_{in} < \sum_{i=1}^{n} M_{in} .$$

Then we must show that the number obtained from the rules upon application of the product on the right-hand side of Eq. (52) to an arbitrary initial state is precisely the number assigned by the rules to the operator on the left-hand side.

In order to give the proof of the aforementioned result, it is useful to introduce the notion of 'opposing arrows' in two arrow patterns. The arrow pattern for two extremal projections are said to have an opposing arrow if for an arrow going between two points in the first arrow pattern there is an arrow going in the opposite direction between the same two points in the second arrow pattern. In general, the two arrow patterns have a number of opposing arrows. *The arrow patterns for extremal projections with the same tie-structure have no opposing arrows.* The proof is rather obvious, and we omit it.

Now for the proof that the extended rules apply to all extremal projections. Consider a typical arrow going from row n to row $n-1$ in the arrow pattern of the elementary projection in Eq. (52); let the arrow start at point i_s of row n and terminate at point $j \neq (j_1 j_2 \ldots j_k)$ of row $n-1$. The Δ-patterns for row n and row $n-1$ of the $[M']$ projection[8] are of the form

$$\Delta'_n = [M'_{\tau_1 n} M'_{\tau_2 n} \ldots M'_{\tau_n n}] \quad \text{and} \quad \Delta'_{n-1} = [M'_{\varrho_1 n} M'_{\varrho_2 n} \ldots M'_{\varrho_{n-1}, n}],$$

respectively, where $\tau_1 \tau_2 \ldots \tau_n$ and $\varrho_1 \varrho_2 \ldots \varrho_n$ are permutations of $1\,2 \ldots n$. There are $N'_{i_s j} = M'_{\tau_{i_s} n} - M'_{\varrho_j n} \geqq 0$ arrows going from point i_s to point j in the arrow pattern for $[M']$. Similarly, there are $N_{i_s j} = M_{\tau_{i_s} n} - M_{\varrho_j n} = (M'_{\tau_{i_s} n} + 1) - M'_{\varrho_j n} = N'_{i_s j} + 1$ arrows going from point i_s to point j in the arrow pattern for $[M]$. These results are a consequence of the fact that all three operators have the same tie structure, hence, no opposing arrows. When applied to a state with $U(n)$ labels $[m]_n$ and $U(n-1)$ labels $[m]_{n-1}$, the elementary projection in Eq. (52) carries these labels, respectively, into $[m]_n + \Delta(i_1 i_2 \ldots i_k)$ and $[m]_{n-1} + \Delta(j_1 j_2 \ldots j_k)$; these labels, in turn, become the initial

[8] The square brackets designate a projective operator in which the upper and lower operator patterns have been suppressed. For brevity, this notation is used in the remainder of this section, and is not to be confused with a partition $[M']_n$.

labels for the second operator. Thus, the rules assign the factor

$$(p_{i_s n} - p_{j,n-1})(p_{i_s n} - p_{j,n-1} + 1)\dots(p_{i_s n} - p_{j,n-1} + N'_{i_s j}) \qquad (53)$$

to the arrows going between points i_s and j: the factor $(p_{i_s n} - p_{j,n-1})$ results from the action of the elementary projection on the state

$$\left|\begin{pmatrix}[m]_n \\ [m]_{n-1}\end{pmatrix}\right\rangle;$$

the remaining factors from the action of $[M']$ on the new state

$$\left|\begin{pmatrix}[m]_n + \varDelta(i_1 i_2 \dots i_k) \\ [m]_{n-1} + \varDelta(j_1 j_2 \dots j_k)\end{pmatrix}\right\rangle.$$

But the factor (53) is precisely the factor assigned by the rules to the $N_{i_s j} = N'_{i_s j} + 1$ arrows going between point i_s and j of $[M]$ when applied to the initial state. Thus, the rule assigns the correct factor to all arrows going between point i_s and $j \neq (j_1 j_2 \dots j_k)$. Clearly, a similar argument applies to all pairs of points which have an arrow going between them in the arrow pattern of the elementary projection. However, there may still be pairs of points in the arrow pattern for $[M']$ which have arrows going between them, but for which there are no corresponding arrows in the arrow pattern of the elementary projection. Such points then have either 1, 1 or 0, 0 at the two corresponding points of the elementary pattern. Accordingly, the effect of the elementary projection is to shift the $U(n)$ and $U(n-1)$ labels in such a way that the factor assigned by the rules to arrows of this type when $[M']$ acts on the new state is independent of the initial shifts. Finally, the arrows for all such pairs of points are exactly duplicated in the arrow pattern for $[M]$, and clearly the rules assign to these arrows the same factors that result from forming the product. It now follows that the extended rules (1)—(5) correctly yield $|N/D|$ for all extremal projections if they do so for all elementary projections. Since this latter result was proved in the previous section, we have then the desired result: *The pattern rules work for all extremal projections.*

Next, let us turn to the question of commutivity of two extremal projections. We have already noted that for commutivity it is sufficient for the two extremal projections to have the same tie structure. This condition is, however, *not* necessary as is evidenced by the fact that the two operators

$$\begin{bmatrix} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}$$

commute, but nevertheless have distinct tie structures in their lower patterns. These operators are peculiar in that the (denominator) arrow patterns for three dots in both row 3 *and* row 2 have opposing arrows, but the arrow patterns for three dots in row 3 and *two* dots in row 2 have *no* opposing arrows. Observe that these two operators do have the same tie structure in their upper patterns and the same tie structure in the bottom two rows of their lower patterns. It is this property that characterizes the general case.

Assertion. *Two extremal projections commute if and only if their upper patterns have the same tie-structure and their lower patterns have the same tie-structure in the lower* $n - 1$ *rows.*

In order to prove the above statement, we must first prove: A necessary and sufficient condition that two extremal projections commute is that they have no opposing arrows in their arrow patterns of n dots in row n and $n - 1$ dots in row $n - 1$. The proof of this is fairly straightforward, and we omit it. What we will prove is that this property of no opposing arrows implies that the extremal operators possess the alleged tie-structures, and conversely. The arrow pattern for the extremal projection $[M]$ which has Δ-patterns

$$\Delta_n = [M_{\tau_1 n} M_{\tau_2 n} \dots M_{\tau_n n}]$$

and

$$\Delta_{n-1} = [M_{\varrho_1 n} M_{\varrho_2 n} \dots M_{\varrho_{n-1} n}]$$

can be obtained from the arrow pattern which has Δ patterns

$$\Delta_n = [M_{1n} M_{2n} \dots M_{nn}] \quad \text{and} \quad \Delta_{n-1} = [M_{1n} M_{2n} \dots M_{n-1n}]$$

by applying the permutations $1 \to \tau_1, 2 \to \tau_2, \dots, n \to \tau_n$ and $1 \to \varrho_1$, $2 \to \varrho_2, \dots, n - 1 \to \varrho_{n-1}$ to rows n and $n - 1$, respectively. In carrying out these permutations, the heads and tails of arrows are to be carried along with the M_{in}. Now consider the arrow patterns associated with any two extremal projections $[M']$ and $[M]$. Also, consider the new arrow patterns obtained by applying a permutation P to row n and a permutation Q to row $n - 1$ of the original arrow patterns, respectively. If there are no opposing arrows in the original arrow patterns, there are none in the permuted arrow patterns, and conversely. It is, therefore, no restriction to take one of the extremal projections, say $[M']$, to be maximal in its upper and lower operator patterns. Its Δ-patterns are $\Delta'_n = [M'_{1n} M'_{2n} \dots M'_{nn}]$, $\Delta_{n-1} = [M'_{1n} M'_{2n} \dots M'_{n-1n}]$. If extremal $[M]$ is maximal in its upper pattern and semi-maximal in its lower pattern, its Δ-patterns are $\Delta_n = [M_{1n} M_{2n} \dots M_{nn}]$ and $\Delta_{n-1} = [M_{1n} \dots M_{i-1,n} M_{i+1,n} \dots M_{nn}]$ for some $i = 1, 2, \dots, n$. Clearly, the arrow patterns for $[M']$ and $[M]$ have no opposing arrows. Therefore, every two extremal projections $[M']$ and $[M]$ commute which have the same tie-structure in their upper patterns and in their lower $n-1$ rows. Now for the converse proof. If it happens that $M'_{1n} > M'_{2n} > \cdots > M'_{nn}$, the denominator arrow pattern for maximal $[M']$ has at least one arrow going from each particular point in row n (and row $n-1$) to every point lying to the right of the particular point. Since there are no opposing arrows in the denominator arrow pattern for $[M]$, we must have $M_{\tau_1 n} \geqq \geqq M_{\tau_2 n} \geqq \cdots \geqq M_{\tau_n n}$ in row n and $M_{\varrho_1 n} \geqq M_{\varrho_2 n} \geqq \cdots \geqq M_{\varrho_{n-1} n}$ in row $n-1$, i.e., $M_{\tau_j n} = M_{jn}$ $(j = 1, 2, \ldots, n)$ in row n and

$$[M_{\varrho_1 n} M_{\varrho_2 n} \ldots M_{\varrho_{n-1} n}] = [M_{1n} \ldots M_{i-1,n} M_{i+1,n} \ldots M_{nn}]$$

(for some i) in row $n-1$. Thus, $[M]$ is maximal in its upper pattern and in its lower $n-1$ rows. In general, some of the $[M']$ labels may be equal, and we must consider the general case $M'_{1n} = M'_{2n} = \cdots = M'_{k_1 n}$, $M'_{k_1+1,n} = \cdots = M'_{k_1+k_2,n}$, etc., where $k_1 + k_2 + \cdots = n$. When both numerator and denominator arrow patterns for $[M]$ are considered, the conclusion drawn from the condition of no opposing arrows is that the Δ-pattern for row n has the following structure: The first k_1 integers may be any permutation of $M_{1n} M_{2n} \ldots M_{k_1 n}$; the second k_2 integers any permutation of $M_{k_1+1,n} M_{k_1+2,n} \cdots M_{k_1+k_2,n}$; etc. Row $n-1$ has a similar structure in which the first k_1 integers are some permutation of the first k_1 integers from $[M_{1n} \ldots M_{i-1,n} M_{i+1,n} \ldots M_{nn}]$ (for some i); the second k_2 integers some permutation of the second k_2 integers from $[M_{1n} \ldots M_{i-1,n} M_{i+1,n} \ldots M_{nn}]$; etc. But this is precisely the structure required such that $[M']$ and $[M]$ have at least one tie-pattern in common in their upper operator patterns and at least one tie-pattern in common in their lower $n-1$ rows. Thus, the condition of no opposing arrows in extremal $[M']$ and $[M]$ requires that these projections have the same tie-structure in their upper patterns and the same tie-structure in their lower $n-1$ rows.

This proves the validity of our assertion above. It follows then that we have achieved our desired goal — the generalization of the pattern rules to all extremal projections, and the beginnings of a pattern calculus for projections.

Summary. The most general reduced Wigner operator for which the extended pattern rules apply are the *extremal projections*. From an extremal projection we determine a Δ-pattern of two rows; application of the arrow pattern rules determines an explicit matrix element, to within a $\pm$ sign.

The *sign* of the matrix element is determined from the sign of the elementary projections in the decomposition of the extremal operator implied by Eq. (52).

VII. Expansions of Reduced Wigner Operators in Terms of Elementary Projections

In this concluding section we extend the pattern calculus to obtain explicit matrix elements for the analog to the most general monomial boson operator.

From Eq. (52) we know already that every extremal projection can be expanded in terms of elementary ones. As a particular case, we have the result:

$$\begin{bmatrix} (\max)_{n-1} \\ [M]_n \\ (\max)_{n-1} \end{bmatrix} = \prod_{k=1}^{n} \left(\begin{bmatrix} (1\,2\,\ldots\,k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ (1\,2\,\ldots\,k) \end{bmatrix} \right)^{M_{kn}-M_{k+1,n}}, \tag{54}$$

which is in exact analogy to the boson operator result, Eq. (17).

This result generalizes to arbitrary extremal projections. In order to write this result in the notation (29) for elementary projections, we first observe that an operator pattern

$$\begin{pmatrix} [M]_n \\ (\mathrm{extr})_{n-1} \end{pmatrix} \tag{55}$$

which has

$$\Delta_n = [M_{\tau_1 n} M_{\tau_2 n} \ldots M_{\tau_n n}] \tag{56}$$

has M_{kn} in position i_k $(k = 1, 2, \ldots, n)$ where $(i_1 i_2 \ldots i_n)$ are the numbers obtained by rearranging the columns in the permutation $1 \to \tau_1, 2 \to \tau_2, \ldots, n \to \tau_n$:

$$\begin{pmatrix} 1\,2\,\ldots\,n \\ \tau_1\tau_2\ldots\tau_n \end{pmatrix} = \begin{pmatrix} i_1 i_2 \ldots i_n \\ 1\,2\,\ldots\,n \end{pmatrix}. \tag{57}$$

In particular, for $[M]_n = [\dot{1}_k\,\dot{0}_{n-k}]$, the Δ_n pattern (56) has 1 in positions $i_1, i_2, \ldots, i_k$. In general, these numbers are not ordered in increasing magnitude to the right. Let $\mathcal{O}(i_1 i_2 \ldots i_k)$ denote the arrangement of $i_1 i_2 \ldots i_k$ which is ordered, e.g., $\mathcal{O}(31) = (13)$. Then

$$\begin{pmatrix} [\dot{1}_k\,\dot{0}_{n-k}] \\ \mathcal{O}(i_1 i_2 \ldots i_k) \end{pmatrix}$$

is the elementary operator pattern which has the same tie-structure as the general extremal pattern (55) with Δ given by Eq. (56). Every extremal projection then has the following expansion (iterate Eq. (52)):

$$\begin{bmatrix} (\mathrm{extr})_{n-1} \\ [M]_n \\ (\mathrm{extr})'_{n-1} \end{bmatrix} = \prod_{k=1}^{n} \left(\begin{bmatrix} \mathcal{O}(i_1 i_2 \ldots i_k) \\ [\dot{1}_k\,\dot{0}_{n-k}] \\ \mathcal{O}(j_1 j_2 \ldots j_k) \end{bmatrix} \right)^{M_{kn}-M_{k+1,n}}, \tag{58}$$

where the Δ's for the upper and lower patterns of the projection on the left-hand side are respectively $\Delta_n = [M_{\tau_1 n} M_{\tau_2 n} \ldots M_{\tau_n n}]$ and $\Delta'_n = [M_{\varrho_1 n} M_{\varrho_2 n} \ldots M_{\varrho_n n}]$, and where the $(j_1 j_2 \ldots j_n)$ are determined from the permutation $1 \to \varrho_1, 2 \to \varrho_2, \ldots, n \to \varrho_n$ by the rule, Eq. (57). Again, Eq. (58), *is in exact analogy to the corresponding boson operator result.*

We now ask: Can Eq. (54) be extended to include the semi-maximal analogue to Eq. (35)? The explicit boson operator result is

$$B\begin{pmatrix}(\max)_{n-1}\\ [M]_n\\ [M]_{n-1}\\ (\max)_{n-2}\end{pmatrix} = N^{-1/2}\cdot\prod_{k=1}^{n-1}(a^{12\ldots k}_{12\ldots k})^{M_{k,n-1}-M_{k+1,n}} \times\prod_{k=1}^{n}(a^{12\ldots k}_{12\ldots k-1,n})^{M_{k,n}-M_{k,n-1}}, \tag{59a}$$

where

$$N\equiv M/\mathcal{M}=\prod_{\substack{i<j\\1}}^{n}\frac{(P_{i,n-1}-P_{j,n})!}{(P_{i,n}-P_{j,n}-1)!}\cdot\prod_{\substack{i\leqq j\\1}}^{n-1}\frac{(P_{i,n}-P_{j,n-1}-1)!}{(P_{i,n-1}-P_{j,n-1})!}, \tag{59b}$$

with the P_{ij} being partial hooks, $P_{ij}\equiv M_{ij}+j-1$. We will, in fact, now prove the corresponding result for projections:

$$\begin{bmatrix}(\max)_{n-1}\\ [M]_n\\ [M]_{n-1}\\ (\max)_{n-2}\end{bmatrix} = N^{-1/2}\cdot\prod_{k=1}^{n-1}\begin{bmatrix}(1\,2\ldots k)\\ [\dot{1}_k\,\dot{0}_{n-k}]\\ (1\,2\ldots k)\end{bmatrix}^{M_{k,n-1}-M_{k+1,n}} \times\prod_{k=1}^{n}\begin{bmatrix}(1\,2\ldots k)\\ [\dot{1}_k\,\dot{0}_{n-k}]\\ (1\,2\ldots k-1,n)\end{bmatrix}^{M_{kn}-M_{k,n-1}}. \tag{60}$$

First, observe that the elementary projections appearing on the right-hand side are all maximal in their upper patterns and either maximal or semi-maximal in their lower patterns. Hence, they all mutually commute, and no problem of ordering arises.

To prove Eq. (60), we calculate the reduced matrix element directly from Eq. (21); noting that maximal Wigner operators between maximal states are by definition unity. The result is:

$$\left\langle\begin{pmatrix}(\max)_{n-1}\\ [m]_n+\Delta_n\\ [m]_{n-1}+\Delta_{n-1}\\ (\max)_{n-2}\end{pmatrix}\middle| B\begin{pmatrix}(\max)_{n-1}\\ [M]_n\\ [M]_{n-1}\\ (\max)_{n-2}\end{pmatrix}\middle|\begin{pmatrix}(\max)_{n-1}\\ [m]_n\\ [m]_{n-1}\\ (\max)_{n-2}\end{pmatrix}\right\rangle = (\mathcal{M}([m]_n+\Delta_n)/\mathcal{M}([m]_n))^{1/2} \times\left\langle\begin{pmatrix}[m]_n+\Delta_n\\ [m]_{n-1}+\Delta_{n-1}\end{pmatrix}\middle|\begin{bmatrix}(\max)_{n-1}\\ [M]_n\\ [M]_{n-1}\\ (\max)_{n-2}\end{bmatrix}\middle|\begin{pmatrix}[m]_n\\ [m]_{n-1}\end{pmatrix}\right\rangle, \tag{61}$$

where $\Delta_n=[M_{1n}M_{2n}\ldots M_{nn}]$ and $\Delta_{n-1}=[M_{1n-1}M_{2n-1}\ldots M_{n-1n-1}]$. The left-hand side of this equation can be evaluated explicitly by use of Eqs. (35) and (59). Using this evaluation in Eq. (61), we obtain

$$\begin{bmatrix}(\max)_{n-1}\\ [M]_n\\ [M]_{n-1}\\ (\max)_{n-2}\end{bmatrix} = N^{-1/2}\cdot(\Delta\text{-pattern factor}), \tag{62}$$

where N is defined by Eq. (59b). The second factor in Eq. (62) *is precisely the factor* which results from applying the rules to the Δ-patterns Δ_n and Δ_{n-1} *with phase assigned to be* $+1$. From this result we immediately deduce: *The extended pattern rules, when applied to the reduced Wigner operator*

$$\begin{bmatrix} (\mathrm{extr})_{n-1} \\ [M]_n \\ [M]_{n-1} \\ (\mathrm{extr})_{n-2} \end{bmatrix} \tag{63}$$

which has

$$\Delta_n = [M_{\tau_1 n} M_{\tau_2 n} \ldots M_{\tau_n n}]$$

and

$$\Delta_{n-1} = [M_{\varrho_1 n-1} M_{\varrho_2 n-1} \ldots M_{\varrho_{n-1} n-1}]$$

yield the reduced matrix element except for a factor, and this factor is the $N^{-1/2}$ *of Eq.* (59b) *times a phase.*

Equation (62) gives the explicit expression (including the phase) for the reduced matrix element in question. We next show that Eq. (60) also yields the same result upon applying the rules to the terms of the right-hand side (observe from Eq. (38) that the phase of the right-hand side of Eq. (60) is $+1$). This is somewhat more readily accomplished by recognizing that

$$\prod_{k=1}^{n} \begin{bmatrix} (1\,2 \ldots k) \\ [\dot{1}_k\, \dot{0}_{n-k}] \\ (1\,2 \ldots k-1\,n) \end{bmatrix}^{M_{kn}-M_{k,n-1}} = \begin{bmatrix} (\max)_{n-1} \\ [M'']_n \\ [\min]_{n-1} \\ (\max)_{n-2} \end{bmatrix},$$

where

$$M''_{in} = \sum_{k=i}^{n} (M_{kn} - M_{k,n-1}), \quad i = 1, 2, \ldots n,$$

$$(\text{Note: } M_{n,n-1} \equiv 0)$$

and

$$\prod_{k=1}^{n-1} \begin{bmatrix} (1\,2 \ldots k) \\ [\dot{1}_k\, \dot{0}_{n-k}] \\ (1\,2 \ldots k) \end{bmatrix}^{M_{k,n-1}-M_{k+1,n}} = \begin{bmatrix} (\max)_{n-1} \\ [M']_n \\ (\max)_{n-1} \end{bmatrix},$$

where

$$M'_{in} \equiv \sum_{k=i}^{n-1} (M_{k,n-1} - M_{k+1,n}), \quad i = 1, 2, \ldots n-1,$$

$$M'_{nn} = 0.$$

Thus, Eq. (60) may be written in the equivalent form

$$\begin{bmatrix} (\max)_{n-1} \\ [M]_n \\ [M]_{n-1} \\ (\max)_{n-2} \end{bmatrix} = N^{-1/2} \cdot \begin{bmatrix} (\max)_{n-1} \\ [M']_n \\ (\max)_{n-1} \end{bmatrix} \cdot \begin{bmatrix} (\max)_{n-1} \\ [M'']_n \\ [\min]_{n-1} \\ (\max)_{n-2} \end{bmatrix}. \tag{60'}$$

Observe that both projections are extremal, they commute, and

$$M'_{in} + M''_{in} = M_{in} \quad \text{for} \quad i = 1, 2, \ldots n$$

$$M'_{in} + M''_{i+1,n} = M_{i,n-1} \quad \text{for} \quad i = 1, 2, \ldots n-1 \,.$$

It is now tedious, but straightforward, to apply the rules to the product operators and verify that Eq. (60′) yields the same result as the direct calculation (62).

Summary: Matrix elements of the most general ("monomial") operator, Eq. (63), are given by the extended pattern rules to within a phase and normalization. This normalization is given by Eq. (59b).

VIII. Concluding Remarks

In preceding sections we have shown how the concepts of (a) a reduced Wigner operator (an operator acting on the space $U(n) : U(n-1)$) (b) the Δ-pattern, and (c) the arrow pattern of a Δ-pattern may be applied to a large class of operators: fundamental, elementary and extremal Wigner operators to yield explicit matrix elements in terms of the Δ-pattern rules. For this class of operators the rules are both simple and comprehensive. In fact, we feel that these rules almost eliminate the subject, since they can easily be incorporated in a computer to produce arbitrary algebraic and numerical tables at will.

Obviously the class of operators not covered by these rules now becomes of the most interest. Of this class, the analog to the semi-maximal boson monomial, given in Eq. (60), is the simplest: the rules require almost no modification. It is for the remaining class of operators that we feel the pattern calculus comes into its own, for these operators may be composed out of elementary projections *which no longer commute.* The pattern calculus applied to these non-commuting patterns is a subject of much intrinsic interest which we have explored so far very little. For $U(3)$ we can make one useful remark: By considering numerator and denominator arrow patterns separately, all $U(3)$ projections with extremal lower pattern may be written explicitly in terms of commuting elementary numerator projections. This in effect classifies all $U(3)$ projections completely.

One of the primary objectives of such a program is the task of explicitly defining the content of the canonical operator labelling. On this problem, the pattern calculus for $U(3)$ is definitive. We hope to discuss this matter, and the generalization to $U(n)$ further.

Acknowledgements. The authors would like to thank Drs. A. Giovannini, B. Gruber, and H. J. Weber for many helpful discussions.

Appendix

It is desired to prove that the following sum has the value unity:

$$\mathcal{S} \equiv (-1)\cdot \sum_{s=1}^{k} \sum_{q=1}^{s} \Bigg[\prod_{l=2}^{k} \frac{(p_{j_q,n-1} - p_{i_l,n} + 1)}{(p_{i_1,n} - p_{i_l,n} + 1)} \cdot \prod_{\substack{l=1 \\ l \neq q}}^{k} (p_{j_l,n-1} - p_{i_1,n})$$

$$\times \prod_{\substack{l=1 \\ l\neq q}}^{s} (p_{i_l,n-1} - p_{j_q,n-1})^{-1} \cdot \prod_{l=s}^{k} (p_{j_l,n-1} - p_{j_q,n-1} - 1)^{-1} \Bigg]. \quad \text{(A1)}$$

Exchanging the order of the sums on s and q gives a term with the following sum on s, which may be explicitly summed with the result shown:

$$(-1)\cdot \sum_{s=q}^{k} \prod_{\substack{l=1 \\ l\neq q}}^{s} (p_{j_l,n-1} - p_{j_q,n-1})^{-1} \cdot \prod_{l=s}^{k} (p_{j_l,n-1} - p_{j_q,n-1} - 1)^{-1}$$

$$= \prod_{\substack{l=1 \\ l\neq q}}^{k} (p_{j_l,n-1} - p_{j_q,n-1})^{-1}. \quad \text{(A2)}$$

With this partial summation accomplished the desired sum in Eq. (A1) takes the form

$$\mathcal{S} = \sum_{q=1}^{k} \prod_{l=2}^{k} \frac{(p_{j_q,n-1} - p_{i_l,n} + 1)}{(p_{i_1,n} - p_{i_l,n} + 1)} \cdot \prod_{\substack{l=1 \\ l\neq q}}^{k} \frac{(p_{j_l,n-1} - p_{i_1,n})}{(p_{j_l,n-1} - p_{j_q,n-1})}$$

$$= \frac{\prod_{l=1}^{k} (p_{j_l,n-1} - p_{i_1,n})}{\prod_{l=2}^{k} (p_{i_1,n} - p_{i_l,n} + 1)} \times \sum_{q=1}^{k} \frac{\prod_{l=2}^{k} (p_{j_q,n-1} - p_{i_l,n} + 1)}{(p_{j_q,n-1} - p_{i_1,n}) \prod_{\substack{l=1 \\ l\neq q}}^{k} (p_{j_l,n-1} - p_{j_q,n-1})}. \quad \text{(A3)}$$

The basic result now required is the identity:

$$\sum_{q=1}^{k+1} \frac{\prod_{l=2}^{k} (x_q - y_l)}{\prod_{\substack{l=1 \\ l\neq q}}^{k+1} (x_l - x_q)} = 0\,, \quad \text{(A4)}$$

for every set of numbers $x_1 \neq x_2 \neq \cdots \neq x_{k+1}$ and arbitrary numbers y_l. With the aid of this identity, we easily prove that:

$$\sum_{q=1}^{k} \frac{\prod_{l=2}^{k} (p_{j_q,n-1} - p_{i_l,n} + 1)}{(p_{j_q,n-1} - p_{i_1,n}) \prod_{\substack{l=1 \\ l\neq q}}^{k} (p_{j_l,n-1} - p_{j_q,n-1})} = \frac{\prod_{l=2}^{k} (p_{i_1,n} - p_{i_l,n} + 1)}{\prod_{l=1}^{k} (p_{j_l,n-1} - p_{i_1,n})}. \quad \text{(A5)}$$

Hence the sum on the right-hand side of Eq. (A1) is unity, as required.

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