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Two-Dimensional Generalized Toda Lattice

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Abstract. The zero curvature representation is obtained for the two-dimensional generalized Toda lattices connected with semisimple Lie algebras. The reduction group and conservation laws are found and the mass spectrum is calculated.

1. Introduction

In recent work [1] it was shown that the two-dimensional generalization of the classical periodic Toda lattice (TL) is solved by the inverse scattering method and the reduction from the complete Zakharov-Shabat equations was found. On the other hand, Bogoyavlensky constructed the generalized TL connected with the root systems of the semisimple Lie algebras [2]; the classical TL then corresponds to the root system of the type $A_{\ell-1}$. The purpose of the present paper is to generalize the results obtained in [1] on arbitrary root systems, in other words, to give a two-dimensialization of the lattices constructed in [2]. This generalization has some new features when compared with the system of type $A_{\ell-1}$ [1]. The results obtained are given in the most general form possible that enables one to understand the invariant meaning of the formulae in [1].

The plan of the paper is as follows: in Sect. 2 we describe the generalized TL and give a brief introduction to systems of roots. In Sect. 3 we construct a reduction group from a complete Lie algebra. In Sect. 4 we compute the mass spectrum of our systems and in Sect. 5 we investigate conservation laws.

2. The Description of the Systems

We shall investigate the relativistic systems with Lagrangians

$$L=\sum_{k=1}^{\ell}\partial_\mu\varphi^k\partial^\mu\varphi^k-\tfrac{1}{2}U(\varphi^1,\dots,\varphi^\ell)$$
$$\partial_\mu=\partial/\partial x_\mu \qquad \mu=0,1\,, \tag{2.1}$$

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where the potential U has the following form. We denote

$$V_k=\sum_{j=1}^{k-1}\exp 2(\varphi^j-\varphi^{j+1}).$$

There are five infinite series $(A_\ell, B_\ell, C_\ell, D_\ell, BC_\ell)$ and five exceptional systems $(G_2, F_4, E_6, E_7, E_8)$:

$$\begin{aligned}
A_{\ell-1}&: U=V_\ell+\exp 2(\varphi^\ell-\varphi^1) && \ell\geqq 2,\\
B_\ell&: U=V_\ell+\exp 2\varphi^\ell+\exp(-2(\varphi^1+\varphi^2)) && \ell\geqq 2,\\
C_\ell&: U=V_\ell+\exp 4\varphi^\ell+\exp(-4\varphi^1) && \ell\geqq 3,\\
D_\ell&: U=V_\ell+\exp 2(\phi^{\ell-1}+\varphi^\ell)+\exp(-2(\varphi^1+\varphi^2)) && \ell\geqq 4,\\
BC_\ell&: U=V_\ell+\exp 2\varphi^\ell+\exp(-4\varphi^1) && \ell\geqq 1,\\
G_2&: U=\exp 2(\varphi^3-\varphi^2)+\exp 2(\varphi^1+\varphi^2-2\varphi^3)\\
&\qquad +\exp 2(\varphi^2+\varphi^3-2\varphi^1),\\
F_4&: U=\exp 2(\varphi^2-\varphi^3)+\exp 2(\varphi^3-\varphi^4)\\
&\qquad +\exp 2\varphi^4+\exp(\varphi^1-\varphi^2-\varphi^3-\varphi^4)+\exp(-2(\varphi^1+\varphi^2)),\\
E_6&: U=V_5+\exp(-\varphi^1+\varphi^2+\ldots+\varphi^7-\varphi^8)\\
&\qquad +\exp 2(-\varphi^1-\varphi^2)+\exp 2(-\varphi^7+\varphi^8),\\
E_7&: U=V_6+\exp(-\varphi^1+\varphi^2+\ldots+\varphi^7-\varphi^8)\\
&\qquad +\exp 2(-\varphi^1-\varphi^2)+\exp 2(-\varphi^7+\varphi^8),\\
E_8&: U=V_7+\exp(-\varphi^1+\varphi^2+\ldots+\varphi^7-\varphi^8)\\
&\qquad +\exp 2(-\varphi^1-\varphi^2)+\exp 2(\varphi^7+\varphi^8).
\end{aligned} \tag{2.2}$$

These types of potentials are constructed using some finite sets of vectors in ℓ-dimensional Euclidean space – the so called root systems. Now we shall give some formal definitions [3].

Let $\mathfrak{H}$ be an Euclidean ℓ-dimensional space and α be a vector orthogonal to some hyperplane in $\mathfrak{H}$. Then the reflection s_α with respect to the hyperplane is of the form

$$s_\alpha(x)=x-\frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha.$$

Consider a finite set of nonzero vectors $R=\{\alpha\}\subset\mathfrak{H}$ generating $\mathfrak{H}$ as a linear space and satisfying the following conditions:

1. There is reflection s_α for every α which conserves R:

$$s_\alpha R=R.$$

2. The integrality condition

$$2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z},\qquad \alpha,\beta\in R.$$

The system of vectors R is called the root system and the dimension of the space $\mathfrak{H}$ is called the rank of the root system. Note that for a root α the vector $-\alpha$ (and possibly $\pm 2\alpha$) is also a root.

We shall consider only the irreducible sets in the following sense. Let the space $\mathfrak{H}$ be the direct sum of the subspaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_r$. For any i let R_i be a root system in $\mathfrak{H}_i$. Then the union $R = R_1 \cup \ldots \cup R_r$ will be a root system in $\mathfrak{H}$. If such a decomposition is impossible, the root system is said to be irreducible. All such systems were classified by Cartan. There are five infinite series and five exceptional systems which have been listed previously.

We shall construct the potentials (2.2) by the root systems. The subset of roots $\{\alpha_j\} = A \subset R$ is called *admissible* provided the vectors $\alpha_j - \alpha_k$ are not roots for all $\alpha_j, \alpha_k \in R$ [2]. All subsets of an admissible set of roots are also admissible. A subset B in R such that

1. The vectors $\alpha_j \in B$ are linear independent.
2. Every $\alpha \in R$ is a linear combination of roots from B in which the coefficients are all positive or all negative, is called the set of simple roots.

The number of simple roots is equal to the rank of a system R.

Let ω be a *maximal root* in R, i.e. $\omega + \sum_{\alpha \in B} k_\alpha \alpha$ is not the root when all $k_\alpha \geqq 0$. Then the set $A = B \cup (-\omega)$ is admissible. These admissible sets are listed in [3] in an orthogonal basis $(\varphi^1, \ldots, \varphi^\ell)$ in space $\mathfrak{H}$ for all irreducible root systems [for the systems of type $A_{\ell-1}$, E_6, E_7, and G_2 the extended basis $(\varphi^1 \ldots \varphi^\ell, \varphi^{\ell+1})$ in the extended space is taken].

The potentials (2) may now be rewritten in the following universal form. Denote $\varphi_\alpha = (\varphi, \alpha) = \sum_{j=1}^{\ell} \varphi^j \alpha^j$ for $\varphi \in \mathfrak{H}$. Then all ten formulae in (2.2) expressed via roots belonging to admissible set $A = B \cup (-\omega)$ can be written as

$$U = \sum_{\alpha \in A} \exp 2\varphi_\alpha . \tag{2.3}$$

In conical variables $\xi = \frac{1}{2}(x_1 - x_0)$, $\eta = \frac{1}{2}(x_1 + x_0)$ equations of motion take the form

$$2\varphi^j_{\xi\eta} = \sum_{\alpha \in A} \alpha^j \exp 2\varphi_\alpha . \tag{2.4}$$

It follows from (2.2) that there are two nonequivalent systems of rank one (scalar systems) corresponding to root systems A_1 and BC_1. They give the sinh-Gordon equation

$$\varphi_{\xi\eta} = \sinh 2\varphi \tag{2.5}$$

and the so-called Bullough-Dodd (BD) equation

$$\varphi_{\xi\eta} = e^{2\varphi} - 2e^{-4\varphi} \tag{2.6}$$

(see [4, 5]). In [1] the latter equation was obtained by an additional reduction from the system of type A_2. These two systems should be added by the Liouville equation

$$\varphi_{\xi\eta} = \tfrac{1}{2} e^{2\varphi} . \tag{2.7}$$

Since a subset of an admissible set is again admissible then we can consider A as a set of simple roots B and drop the maximal root. Equation (2.7) corresponds to the unique simple root in A_1 or BC_1. Equations (2.5)–(2.7) are the complete list of scalar equations whose integrability follows from the group-theoretical approach.

In the following we shall only consider admissible root systems which contain the maximal root. The systems obtained by the two-dimensionalization of finite nonperiodic TL have been considered in [6].

3. The Reduction Group

Let U and V be two functions $g(\xi,\eta,\lambda)$, $\xi,\eta\in\mathbb{R}$, $\lambda\in\mathbb{C}$, with values in the complex Lie algebra $\mathfrak{G}$. We consider the Zakharov-Shabat equation (the representation of zero curvature):

$$\partial_\xi U-\partial_\eta V+[U,V]=0 \qquad \partial_\xi=\partial/\partial_\xi,\ \partial_\eta=\partial/\partial_\eta. \tag{3.1}$$

The purpose of [1] was to find additional conditions (reductions) on U and V which are consistent with Eq. (3.1) (see also [7]). To handle this problem in our context, we recall some properties of the complex Lie algebras.

Let $\mathfrak{H}$ be a Cartan subalgebra in the algebra $\mathfrak{G}$ and $(\mathfrak{K}_1,\ldots,\mathfrak{K}_\ell)$ be a canonical basis in $\mathfrak{H}$. There exists such a root system $R=\{\alpha\}$ in $\mathfrak{H}$, that for every root $\alpha\in R$ there is an element $E_\alpha\in\mathfrak{G}\backslash\mathfrak{H}$ such that elements $(\mathfrak{K}_1,\ldots,\mathfrak{K}_\ell)$ and $\{E_\alpha,\alpha\in R\}$ form the basis for the whole Lie algebra (the Cartan-Weyl basis). This basis satisfies the following commutation relations

$$[\mathfrak{K}_k,E_\alpha]=\alpha^k E_\alpha \qquad \alpha^k=(\alpha,\mathfrak{K}_k), \tag{3.2}$$

$$[\mathfrak{K}_k,\mathfrak{K}_j]=0, \tag{3.3}$$

$$[E_\alpha,E_\beta]=\begin{cases} N_{\alpha\beta}E_{\alpha+\beta}, & \alpha+\beta\in R\\ \sum\limits_k \alpha^k\mathfrak{K}_k, & \alpha+\beta=0,\\ 0,\alpha+\beta\neq0,\ \alpha+\beta\bar{\in}R, & \end{cases} \tag{3.4}$$

where $N_{\alpha\beta}$ are some numbers.
It follows from the definition of a set B of simple roots that every root β can be uniquely decomposed as

$$\beta=\sum_{\alpha\in B} n_\alpha\alpha \qquad n_\alpha\in\mathbb{Z}. \tag{3.5}$$

The number

$$a(\beta)=\sum_{\alpha\in B} n_\alpha \tag{3.6}$$

is called the *height* of the root β. Evidently, this definition depends on selecting a set of simple roots B. But the height of the maximal root does not depend on selecting a set of simple roots. It is an invariant of the system R and is determined via the Coxeter number h. The height of maximal roots is equal to $h-1$. The

Coxeter number is given in Table 1 for all irreducible systems (see also [3]). In what follows we fix some set B. The elements of the subset

$$R_b = \{\alpha \in R \mid a(\alpha) = b (\operatorname{mod} h)\}$$

are called homogeneous with a height b.

Proposition 1. *All homogeneous root-subsystems are admissible.*

The inverse statement is, generally speaking, wrong. There are admissible subsystems which are not homogeneous. But the important admissible subsystem $A = B \cup (-\omega)$ which determines the potentials (2.2) is homogeneous of height 1 ($A = R_1$). It is possible to construct a two-dimensional integrable system for every admissible set of roots. Because a subset of an admissible set is also admissible, it is important to classify all maximal admissible subsystems of roots. However, for the sake of brevity, we restrict ourselves here with studying the maximal homogeneous subsystems of roots only.

Now we shall define the $\mathbb{Z}_h$ grading in the Lie algebra $\mathfrak{G}$ as follows

$$\mathfrak{G} = \bigoplus_{a=0}^{h-1} \mathfrak{G}_a, \tag{3.7}$$

where

$$\mathfrak{G}_0 = \mathfrak{H}, \ \mathfrak{G}_b = \sum_{\alpha \in R_b} c_\alpha E_\alpha, \qquad c_\alpha \in \mathbb{C}. \tag{3.8}$$

Evidently

$$[\mathfrak{G}_a, \mathfrak{G}_b] \subseteqq \mathfrak{G}_{a+b} (\operatorname{mod} h). \tag{3.9}$$

We fix the element $Q \in \mathfrak{H}$ by its coordinates on the basis of simple roots

$$Q_\alpha = (Q, \alpha) = \frac{2\pi i}{h}, \qquad \alpha \in B, \tag{3.10}$$

where h is the Coxeter number. Let us denote by $\mathrm{Ad}_{\exp Q}$ the adjoint representation of $\exp Q \in G$ in the algebra $\mathfrak{G}$ where G is a complex Lie group with Lie algebra $\mathfrak{G}$.

Proposition 2. *Let a be an integer with $-h < a < h$, denote $q = \frac{2\pi i}{h}$ and let $\mathfrak{y} \in \mathfrak{G}$ be such one-parameter family*

$$\mathfrak{y}(\lambda) \stackrel{\mathrm{def}}{=} \lambda^a \mathfrak{y}, \qquad \lambda \in \mathbb{C} \tag{3.11}$$

that

$$\mathrm{Ad}_{\exp Q} \mathfrak{y}(\lambda) = \mathfrak{y}(\lambda q). \tag{3.12}$$

Then the family (3.11) *is homogeneous and belongs to* $\mathfrak{G}_a$. The proofs of Propositions 1 and 2 are left to the reader as an easy exercise.

The transformation $\mathrm{Ad}_{\exp Q}$ generates cyclic group $\mathbb{Z}_h$ and it singles out via the condition (3.12) the subset $\mathfrak{G}_a$ in the whole algebra $\mathfrak{G}$. We call this group the reduction group.

Table 1

The root systems	$\mathfrak{G}$	$\mathfrak{G}^{\mathbb{R}}$	The Coxeter number	Minimum dim. irred. repr.
$A_{n-1} n \geqq 2$	$sl(n, \mathbb{C})$	$sl(n, \mathbb{R})$	n	h
$B_n \quad n \geqq 2$	$so(2n+1, C)$	$so(n+1, n)$	$2n$	$h+1$
$C_n \quad n \geqq 3$	$sp(n, C)$	$sp(n, R)$	$2n$	h
$D_n \quad n \geqq 4$	$so(2n, C)$	$so(n, n)$	$2n-2$	$h+2$
$BC_n \; n \geqq 1$	–	$su(n+1, n)$	–	$2n+1$
G_2	$G_2^{\mathbb{C}}$	$G_2^{\mathbb{R}}$	6	$h+1$
F_4	$F_4^{\mathbb{C}}$	$F_4^{\mathbb{R}}$	12	$2h+2$
E_6	$E_6^{\mathbb{C}}$	$E_6^{\mathbb{R}}$	12	$2h+3$
E_7	$E_7^{\mathbb{C}}$	$E_7^{\mathbb{R}}$	18	$3h+2$
E_8	$E_8^{\mathbb{C}}$	$E_8^{\mathbb{R}}$	30	$8h+8$

Now we shall point out how the Zakharov-Shabat equation (3.1) is reduced to the TL equation. Let the action of the reduction group leave solutions of Eq. (3.1) unchanged. If we choose U and V in the form

$$\begin{aligned} U(\lambda, \xi, \eta) &= U_0(\xi, \eta) + \lambda U_1(\xi, \eta) \\ V(\lambda, \xi, \eta) &= V_0(\xi, \eta) + \lambda^{-1} V_{-1}(\xi, \eta) \end{aligned} \tag{3.13}$$

then Eq. (3.1) turns into

$$\partial_\xi U_0 - \partial_\eta V_0 + [U_0, V_0] + [U_1, V_{-1}] = 0, \tag{3.14}$$

$$\partial_\xi U_1 + [U_1, V_0] = 0, \qquad \partial_\eta V_{-1} - [U_0, V_{-1}] = 0. \tag{3.15}$$

From Proposition 2 we get

$$U_0, V_0 \in \mathfrak{G}_0 = \mathfrak{H}, \qquad U_1 \in \mathfrak{G}_1, V_{-1} \in \mathfrak{G}_{-1}. \tag{3.16}$$

For arbitrary function φ with values in $\mathfrak{H}$ let

$$U_0 = -\partial_\eta \varphi \qquad V_0 = \partial_\xi \varphi, \tag{3.17}$$

$$U_1 = \sum_{\alpha \in A} e^{\varphi_\alpha} E_\alpha, \qquad V_{-1} = \sum_{\alpha \in A} e^{\varphi_\alpha} E_{-\alpha}. \tag{3.18}$$

Then Eqs. (3.15) are satisfied and Eq. (3.14) is just the TL Eq. (2.4) [see (3.2), (3.4)].

The root system of type A_{n-1} corresponds to the Lie algebra SL(n, $\mathbb{C}$). In this case the Coxeter number h is equal to n and the reduction group $\mathbb{Z}_n$ was considered in [1].

One can consider the auxiliary constraint on the elements U and V besides the special action of the reduction group (3.12). Namely, let U and V belong to some semisimple subalgebra $\mathfrak{G}$. In [1] this was SL(n, $\mathbb{R}$) a subalgebra of SL(n, $\mathbb{C}$). In other words U and V were real matrices. In the general case we shall consider the so-called normal forms of complex Lie algebras which are singled out by the condition of reality or, equivalently, $\mathbb{Z}_2$ group action (see [8]). So the generalized TL singles out by the direct product $\mathbb{Z}_h \otimes \mathbb{Z}_2$.

Table 1 contains the information needed in what follows about irreducible root systems, the corresponding complex Lie algebras, their normal forms, the Coxeter numbers and the minimal dimensions of nontrivial irreducible representation.

Let us note that the root system of type BC_n is a unique so-called nonreduced root system. It contains roots α and 2α. This root system corresponds to symmetric space with the transformation group $SU(n+1,n)$ but not to any other simple Lie group[1]. The reduction group (nonabelian) was constructed for this root system in [1] by a deep reduction of the system of type A_{2n}. The usual reduction of the Lie algebra $bc(n+1,n)$ leads to the same result. But (for brevity) we reject the consideration of real forms.

Now we describe which sort of elements U_0, V_0, U_1, V_{-1} enter the Zakharov-Shabat representation (3.14) and (3.15) for the classical normal forms of type B_n, C_n, D_n, BC_n and for the system of type G_2[2]. Consider first the system of type B_n. The fundamental representation has dimension $2n+1$, and we get

$$U_0=-\partial_\eta\varphi, \qquad V_0=\partial_\xi\varphi$$

$$\varphi=\operatorname{diag}(\varphi^1,\ldots,\varphi^n,0,-\varphi^n,\ldots,-\varphi^1)$$

$$U_1=\begin{pmatrix} A & B & C \\ D & 0 & -\tilde{B} \\ F & -\tilde{D} & -\tilde{A}\end{pmatrix}, \qquad \tilde{A}=JA^TJ, \qquad J=\begin{pmatrix} 0 & \ldots & 1 \\ \vdots & \iddots & \\ 1 & & 0\end{pmatrix}$$

$$A_{jk}=\delta_{j+1,k}\exp(\varphi^j-\varphi^{j+1})$$

$$\tilde{B}=(\exp\varphi^n,0,\ldots,0), \qquad B=J\tilde{B}^T$$

$$D=(0,\ldots,0)$$

$$C_{jk}=0$$

$$F_{jk}=(-1)^{k-1}\delta_{j-n+2,k}\exp(-\varphi^1-\varphi^2)$$

$$V_{-1}=U_1^T.$$

The fundamental representation of systems C_n has dimension $2n$, so

$$U_0=-\partial_\eta\varphi, \qquad V_0=\partial_\xi\varphi$$

$$\varphi=\operatorname{diag}(\varphi^1,\ldots,\varphi^n,-\varphi^n,\ldots,-\varphi^1)$$

$$U_1=\begin{pmatrix} A & B \\ C & -\tilde{A}\end{pmatrix}, \qquad V_{-1}=U_1^T$$

$$A_{jk}=\delta_{j+1,k}\exp(\varphi^j-\varphi^{j+1}), \qquad \tilde{A}=JA^TJ$$

$$B_{jk}=\delta_{j-n+1,k}\exp 2\varphi^n$$

$$C_{jk}=\delta_{j-n+1,k}\exp(-2\varphi^1)$$

1 There are other symmetric spaces which correspond to this root system
2 The root system of type A_{n-1} has been considered in [1]

for series D_n the matrices U_0 and V_0 are the same and

$$U_1=\begin{pmatrix} A & B \\ C & -\tilde{A}\end{pmatrix}, \qquad V_{-1}=U_1^T,$$

$$A_{jk}=\delta_{j+1,k}\exp(\varphi^j-\varphi^{j+1}),$$

$$B_{jk}=(-1)^{k-1}\delta_{j-n+2,k}\exp(\varphi^{n-1}-\varphi^n),$$

$$C_{jk}=\delta_{j-n+2,k}(-1)^{k-1}\exp(-\varphi^1-\varphi^2).$$

For system of type BC_n we consider the algebra $\mathrm{SU}(n+1,n)$:

$$U_0=-\partial_\eta\varphi, \qquad V_0=\partial_\xi\varphi,$$

$$\varphi=\mathrm{diag}(\varphi^1,\ldots,\varphi^n,0,-\varphi^n,\ldots,-\varphi^1),$$

$$U_1=\begin{pmatrix} A & B & C \\ D & 0 & -\tilde{B} \\ F & -\tilde{D} & -\tilde{A}\end{pmatrix}, \qquad V_{-1}=U_1^T,$$

$$A_{jk}=\delta_{j+1,k}\exp(\varphi^j-\varphi^{j+1}),$$

$$\tilde{B}=(\exp\varphi^n,0,\ldots,0),$$

$$C_{jk}=0, \qquad D=(0,\ldots,0),$$

$$F_{jk}=\delta_{j-n+1,k}\sqrt{2}\exp(-2\varphi^1).$$

The matrices of fundamental representation of the algebra G_2 have the size 7 by 7; then

$$U_0=-\partial_\eta\varphi, \qquad V_0=\partial_\xi\varphi,$$

$$\varphi=\mathrm{diag}(\varphi^1-\varphi^2,\varphi^1-\varphi^3,\varphi^3-\varphi^2,0,\varphi^2-\varphi^3,\varphi^3-\varphi^1,\varphi^2-\varphi^1)$$

$$\{U_1\}_{1,2}=-\{U_1\}_{6,7}=\sqrt{2}\{U_1\}_{3,4}=-\sqrt{2}\{U_1\}_{4,5}=\exp(\varphi^3-\varphi^2)$$

$$\{U_1\}_{2,3}=-\{U_1\}_{5,6}=\exp(\varphi^1+\varphi^2-2\varphi^3)$$

$$\{U_1\}_{6,1}=-\{U_1\}_{7,2}=\exp(\varphi^2+\varphi^3-2\varphi^1)$$

$$V_{-1}=U_1^+.$$

4. The Mass Spectrum

We determine here the mass spectrum of the vacuum excitarions, i.e. of the states which are near the minimum of the potential energy. Note that for systems of type A_{n-1} (the usual TL) this was performed in [9, 1].

Let

$$\omega=\sum_{\alpha\in B} n_\alpha\alpha \tag{4.1}$$

Table 2

The root system	The values of v_k
$A_{\ell-1}$	$v_k=4\sin^2\left(\frac{\pi}{\ell}k\right), \quad k=1,\dots,\ell$
$B_\ell \ell \geqq 2$	$v_k=8\sin^2\left(\frac{\pi}{2\ell}k\right), \quad k=1,\dots,\ell-1,\ v_\ell=2$
C_ℓ	$v_k=8\sin^2\left(\frac{\pi}{2\ell}k\right), \quad k=1,\dots,\ell$
D_ℓ	$v_k=8\sin^2\frac{\pi}{2(\ell-1)}k, \quad k=1,\dots,\ell-2,\ v_{\ell-1}=v_\ell=2$
G_2	$v_1=6, \quad v_2=18, \quad v_3=0$
F_4	$v_{1,2}=3\pm\sqrt{3} \quad v_{3,4}=2v_{1,2}$
E_6	$v_{1,2}=v_{3,4}=\frac{1}{4}(3\pm\sqrt{2}), \quad v_{5,6}=2v_{1,2}, \quad v_{7,8}=0$
E_7	$v_k=\begin{cases}4\left\{\cos\left(\frac{\pi}{9}+\frac{2\pi}{3}(k-1)\right)+1\right\}, & k=1,2,3, \quad v_7=6\\ 4\sqrt{3}\cos\left(\frac{\pi}{18}+\frac{2\pi}{3}(k-4)\right)+6, & k=4,5,6, \quad v_8=0\end{cases}$
E_8	$v_1=7.44, \quad v_2=19.48, \quad v_3=4.92, \quad v_4=12.89$ $v_5=3.37, \quad v_6=\ 8{,}82, \quad v_7=2.23, \quad v_8=\ 0.85$

be the decomposition of a maximal root on simple ones. We recall that the Coxeter number is given by

$$h=\sum_{\alpha\in B} n_\alpha+1\,. \tag{4.2}$$

Let us denote

$$N=\prod_{\alpha\in B}(n_\alpha)^{-n_\alpha/h}\,. \tag{4.3}$$

Proposition 3. *The vacuum states $\bar{\varphi}$ of the systems with Lagrangian* (2.1) *have the coordinates*

$$(\bar{\varphi},\alpha)=\bar{\varphi}^\alpha=\tfrac{1}{2}\ln(n_\alpha N)\,. \tag{4.4}$$

Proposition 4. *The mass spectrum m_j^2 corresponds to the eigenvalues of the matrix*

$$\tilde{\Omega}=2N\Omega\,, \tag{4.5}$$

where N is defined in (4.3) *and*

$$\Omega_{jk}=\sum_{\alpha\in R_1} n_\alpha \alpha^j\alpha^k\,, \tag{4.6}$$

α^j are the coordinates of roots; $-\omega\in R_1$, with $n_{-\omega}=1$. We did not succeed in solving this eigenvalue in general form. Note that only for the classical system, this problem is reduced to the search of the zeroes of the Chebyshev polynomials of the second type.

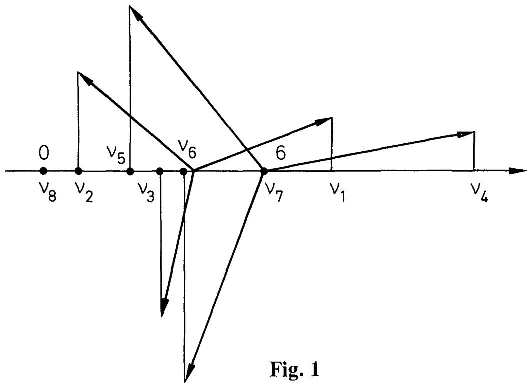

Fig. 1

For special cases we have used the fact that the characteristic polynomial of the matrix Ω is factorized due to the action of the group of symmetries of the extended Dynkin diagram. Note that this group coincides with the center of the universal covering group of the corresponding complex Lie group.

We present here the eigenvalues v_k of the matrix Ω (4.6). The genuine masses are equal to $(2Nv_k)^{1/2}$.

Unfortunately, for the system E_8 we have computer calculations results only. Figure 1 evidently illustrates the situation for the system E_7 on a complex plane.

5. Conservation Laws

To compute the conservation laws it is more convenient to make the gauge transformation in the Zakharov-Shabat equation (3.1) and to pass to the pole gauge, i.e. to reduce the matrix U_1 to the diagonal form. The purpose of this transformation is to reduce U to the form when the "diagonal" contains only spectral parameter.

We will make the transformation in two steps. Let us act first on Eq. (3.1) by the transformation $\mathrm{Ad}_{\exp\{-\varphi\}}$. It follows from (3.1), (3.17), and (3.18) that the elements U and V will be replaced by the elements Ξ and Θ, respectively

$$U \to \Xi = -2\partial_\eta \varphi + \lambda F, \qquad F = \sum_{\alpha \in A} E_\alpha \tag{5.1}$$

$$V \to \Theta = \lambda^{-1} \sum_{\alpha \in A} \exp(2\varphi^\alpha) E_{-\alpha} . \tag{5.2}$$

Equation (3.1) conserves, naturally, its form.

Next one should "diagonalize" the element $F = \sum_{\alpha \in A} E_\alpha$. For this, consider the fundamental representation T of the algebra $\mathfrak{G}$ of the least dimension. Let the dimension of this representation equal d and $F(\mu)$ denote the characteristic determinant

$$F(\mu) = \det(\mu I - T(F)) ,$$

where I is the unit matrix of order d.

It follows from invariance F with respect to the reduction group, that

$$F(\mu)=\exp\left(\frac{2\pi i}{h}d\right)F\left(\exp\left(-\frac{2\pi i}{h}\right)\mu\right). \tag{5.3}$$

Hence $F(\mu)$ has the form

$$F=\mu^d+a_1\mu^{d-h}+\ldots+a_k\mu^{d-kh},$$

where $k=[d/h]$. Thus, the matrix $T(F)$ has $d-kh$ zero eigenvalues.

The rest of eigenvalues μ_j are

$$\mu_j=(v_j)^{1/h}, \tag{5.4}$$

where v_j are the roots of the polynomial

$$v^k+a_1v^{k-1}+\ldots+a_k=0.$$

For instance, in the case of series A_{n-1}, BC_n these eigenvalues are the roots of unity: $\mu_k=\exp\left(\frac{2\pi i}{h}k\right)$.

Thus, making a gauge transformation for (3.1) we arrive at the compatibility condition of the form

$$\begin{aligned}&\partial_\eta\chi=(-2W^{-1}(\partial_\eta\varphi)W+\lambda Q)\chi,\\&\partial_\xi\chi=\lambda^{-1}K\chi,\end{aligned} \tag{5.5}$$

where matrix W diagonalizes $T(F)$, i.e.

$$\begin{aligned}&Q=W^{-1}T(F)W, \qquad Q=\operatorname{diag}(\mu_1,\ldots,\mu_d),\\&K=W^{-1}\sum_{\alpha\in A}\exp(2\varphi^\alpha)E_{-\alpha}W.\end{aligned} \tag{5.6}$$

Let us represent χ as

$$\chi=(I+R)\exp\left(\lambda Q\eta-\lambda^{-1}Q\xi+\int_{-\infty}^{\eta}S(\eta^1)\,d\eta^1\right), \tag{5.7}$$

where $\operatorname{diag}R=0$, S is a diagonal matrix, and I is a unit matrix.

Substitute (5.7) into (5.5) and separate diagonal and offdiagonal parts

$$RS+\partial_\eta R+\lambda[R,Q]=\{-2W^{-1}(\partial_\eta\varphi)W\cdot R-W^{-1}(\partial_\eta\varphi)W\}, \tag{5.8}$$

$$S=\operatorname{diag}\{2W^+(\partial_\eta\varphi)WR\}, \tag{5.9}$$

$$\partial_\xi S=\lambda^{-1}\operatorname{diag}\partial_\eta(KR+Q^++K). \tag{5.10}$$

The first two relations enable one to determine the asymptotic expansion of the matrices R, S in inverse powers of λ and the third formula is the generalized conservation law. Let us represent S and R as

$$S=\sum_{k=1}^{\infty}S_k\lambda^{-k}, \qquad R=\sum_{k=1}^{\infty}R_k\lambda^{-k}. \tag{5.11}$$

Substituting (5.11) into (5.8) and (5.9) we get recurrent formulae for the coefficients R_k and S_k:

$$S_n = -\operatorname{diag}(2W^{-1}\varphi_\eta W R_n), \tag{5.12}$$

$$\begin{aligned} [R_1, Q] &= -2W^{-1}\varphi_\eta W, \\ [R_{n+1}, Q] &= -2\{W^{-1}\varphi_\eta W R_n\}_{nd} - \partial_\eta R_n - \sum_{m+k=n} R_m S_k. \end{aligned} \tag{5.13}$$

The conservation laws appear from (5.10) in differential writing as

$$\partial_\xi S_n + \partial_\eta \operatorname{diag}(K R_{n-1} + (K+Q^+)\delta_{n,1}) = 0. \tag{5.14}$$

We have gotten d infinite series of conservation laws, since the diagonal matrix S has d elements. However, just as the reduction group acts on the eigenfunctions χ, so the matrix elements $\{S_k\}_{jj}(\lambda)$ become linearly dependent. For instance, in the case of series A_{n-1} and BC_n the action of the reduction group is of the form

$$\chi_{j,k}(\lambda q) = \chi_{j+1,k+1}(\lambda),$$
$$q = \exp(2\pi i/h).$$

Since all matrix elements $\{S_k\}_{jj}(\lambda)$ are linearly dependent it is enough to calculate only one series, e.g. $\{S_k\}_{nn}$. Let us rewrite the recurrent formulae for this series in components:

$$\begin{aligned} \{S_k\}_{nn} &= -\frac{2}{n}\sum_{\ell=1}^{n}\sum_{m=1}^{n} \varphi_\eta^\ell \{R_k\}_{mn} q^{\ell m} \\ (1-q^k)\{R_{n+1}\}_{kn} &= -\frac{2}{n}\sum_{j,s=1}^{n} q^{s(j-k)}\varphi_\eta^s \{R_n\}_{jn}, \\ (1-q^k)\{R_1\}_{kn} &= -\frac{2}{n}\sum_{s=1}^{n} q^{-sk}\varphi_\eta^s. \end{aligned} \tag{5.15}$$

These relations permit the calculation of the conservation laws. In the case of the BD equation [3] the density of the currents which are not full derivatives is of the form

$$\begin{aligned} S^{(2)} &= \tfrac{4}{3}\varphi_\eta^2 \\ S^{(6)} &= -\tfrac{4}{27}(\varphi_\eta\varphi_{\eta\eta\eta\eta\eta} + \tfrac{28}{3}\varphi_\eta\varphi_{\eta\eta}\varphi_{\eta\eta\eta} + \tfrac{4}{3}\varphi_\eta^2\varphi_{\eta\eta\eta\eta} \\ &\quad - \tfrac{40}{3}\varphi_\eta^3\varphi_{\eta\eta\eta} - 20\varphi_\eta^2\varphi_{\eta\eta}^2 + \tfrac{16}{3}\varphi_\eta^6 - \tfrac{8}{9}\partial_\eta\varphi_\eta^5). \end{aligned} \tag{5.16}$$

Note that $S^{(3)}$, $S^{(4)}$, $S^{(5)}$ are full derivatives and $S^{(6)}$, a modulo full derivative, is reduced to the form

$$S^{(6)} = -\tfrac{4}{27}(\varphi_{\eta\eta\eta}^2 - \tfrac{10}{3}\varphi_{\eta\eta}^3 + 20\varphi_\eta^2\varphi_{\eta\eta}^2 + \tfrac{16}{3}\varphi_\eta^6) \tag{5.16$'$}$$

(this form coincides with the corresponding integral calculated by Bullough and Dodd [4]). In the case of $A_n (n>1)$, the first nontrivial integral will already be $S^{(3)}$.

3 In what follows we shall give the conserved currents for the BD equation in the form $\varphi_{\xi\eta} = e^{2\varphi} - e^{-4\varphi}$

For instance, for $n=2$, the first nontrivial conservation law can be represented as

$$\begin{aligned}&\partial_\xi(\varphi_\eta^1\varphi_{\eta\eta}^2+\varphi_\eta^2\varphi_{\eta\eta}^3+\varphi_\eta^3\varphi_{\eta\eta}^1-6\varphi_\eta^1\varphi_\eta^2\varphi_\eta^3)\\&\quad=\partial_\eta(f_1\varphi_\eta^2+f_2\varphi_\eta^3+f_3\varphi_\eta^1),\\&f_1=2\exp(2\varphi^2-2\varphi^1)-2\exp2(\varphi^1-\varphi^3)\\&f_2=2\exp2(\varphi^3-\varphi^2)-2\exp2(\varphi^2-\varphi^1)\\&f_3=2\exp2(\varphi^1-\varphi^3)-2\exp2(\varphi^3-\varphi^2).\end{aligned}\tag{5.16''}$$

It is clear that at $\varphi^3=0$, $\varphi^1=-\varphi^2=\varphi$ the latter expression transforms into the full derivative.

To obtain the second series of integrals of motion one should re-gauge the compatibility problem to the form where the residue at the point $\lambda=0$ is a diagonal matrix. We make the calculations according to the above scheme.

Note that if eigenvalues of the matrix Q are degenerate, the expansion of the corresponding eigenvalues will lead to a series of nonlocal conservation laws and only the first integral of this series will be local[4].

Another method for finding the polynomial conserved quantities of $S^{(k)}$ can also be proposed. To this end, transform the matrix equation

$$\partial_\eta\psi=(F^0+\lambda F^1)\psi \tag{5.17}$$

to the "scalar form".[5]

Let us illustrate this transformation by an A_{n-1} model. In this case Eq. (5.17) is equivalent to the set of equations

$$(\partial_\eta-\varphi_\eta^1)\psi_1=\lambda\psi_2,\ \ldots,(\partial_\eta-\varphi_\eta^n)\psi_n=\lambda\psi_1. \tag{5.18}$$

Excluding the components $\psi_2,\ldots,\psi_n$ we get the equation for ψ_1

$$(\partial_\eta-\varphi_\eta^n)(\partial_\eta-\varphi_\eta^{n-1})\ldots(\partial_\eta-\varphi_\eta^1)\psi_1=\lambda^n\psi_1 \tag{5.19}$$

or

$$(\partial_\eta^n+a_2\partial_\eta^{n-2}-a_3\partial_\eta^{n-3}+\ldots+(-1)^n a_n)\psi_1=\lambda^n\psi_1, \tag{5.20}$$

where

$$\begin{aligned}&a_2(\varphi_\eta)=4\sum_{j<k}\varphi_\eta^j\varphi_\eta^k-2\sum_{k=1}^{n-1}(n-k)\varphi_{\eta\eta}^k\\&a_3(\varphi_\eta)=8\sum_{j<k<\ell}\varphi_\eta^j\varphi_\eta^k\varphi_\eta^\ell+\ldots\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&a_n(\varphi_\eta)=2n\varphi_\eta^1\varphi_\eta^2\ldots\varphi_\eta^n+\ldots\end{aligned}$$

and the dots denote the terms containing the second and higher derivatives of φ^j in η.

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4 The nonlocal integrals for systems with degenerate eigenvalues were originally considered by Manakov [10] when he studied the system of a multicomponent nonlinear Schrödinger equation

5 The connection between factorized operators and matrix operators was discussed also in recent paper [11]

Representing ψ_1 as $\psi_1=\exp\left(2\int\limits_{-\infty}^{\eta}\chi(\eta^1)\,d\eta^1\right)$ we come to the nonlinear equation for the function $\chi(\eta)$

$$(-1)^n a_n(\tilde{\varphi}^j_\eta)=(-1)^n a_n(\varphi_\eta-\chi)=\lambda^n\,. \tag{5.21}$$

Expanding function χ in a series of negative powers of λ

$$\chi=\lambda+\sum_{k=0}^{\infty}\lambda^{-k}S^{(k)}$$

we get the recurrent relations which enable one to find the coefficidnts $S^{(k)}$. They indeed are the integrals of motion we were looking for.

For the BC_n model we must put in these formulae $n=2\ell+1$ and $\varphi_n=-\varphi_1$, $\varphi_{n-1}=-\varphi_2\ldots\varphi_{\ell+2}=-\varphi_\ell$, $\varphi_{\ell+1}=0$. After that the equation of interest will take the form

$$(\partial_\eta+2\varphi^1_\eta)(\partial_\eta+2\varphi^2_\eta)\ldots(\partial_\eta+2\varphi^\ell_\eta)\,\partial_\eta(\partial_\eta-2\varphi^\ell_\eta)\ldots(\partial_\eta-2\varphi^1_\eta)\psi_1=\lambda^{2\ell+1}\psi_1\,.$$

We present more detailed formulae for $n=3$: Eq. (5.19) takes the form

$$(\partial^3_\eta+a_2(\varphi_\eta)\partial_\eta-a_3(\varphi_\eta))\psi=\lambda^3\psi\,, \tag{5.22}$$

where

$$a_2(\varphi_\eta)=4(\varphi^2_\eta\varphi^1_\eta+\varphi^3_\eta\varphi^2_\eta+\varphi^1_\eta\varphi^3_\eta)-\varphi^2_{\eta\eta}-\varphi^1_{\eta\eta}\,,$$
$$a_3(\varphi_\eta)=8\varphi^1_\eta\varphi^2_\eta\varphi^3_\eta-4(\varphi^3_\eta\varphi^1_{\eta\eta}+\varphi^1_\eta\varphi^2_{\eta\eta}+\varphi^2_\eta\varphi^1_{\eta\eta})+2\varphi^1_{\eta\eta\eta}\,.$$

Hence we get the equation for the function $\chi-$ generating function of integrals of motion:

$$8(\chi-\varphi^3_\eta)(\chi-\varphi^2_\eta)(\chi-\varphi^1_\eta)+4[(\varphi^3_\eta-\chi)(\varphi^1_\eta-\chi)_\eta$$
$$+(\varphi^1_\eta-\chi)_\eta(\varphi^2_\eta-\chi)+(\varphi^1_\eta-\chi)(\varphi^2_\eta-\chi)_\eta]-2\varphi^1_{\eta\eta\eta}+2\chi_{\eta\eta}=\lambda^3$$

or

$$\chi^3-12\chi\chi_\eta-2\chi_{\eta\eta}+4\chi(2\varphi^1_{\eta\eta}+\varphi^2_{\eta\eta}$$
$$-2(\varphi^1_\eta\varphi^2_\eta+\varphi^1_\eta\varphi^3_\eta+\varphi^2_\eta\varphi^3_\eta))-a_3(\varphi_\eta)=\lambda^3\,. \tag{5.23}$$

Writing down an asymptotic expansion for χ

$$\chi=\lambda(1+\lambda^{-2}S^{(2)}+\lambda^{-3}S^{(3)}+\ldots),\ \lambda\to\infty$$

we immediately find the expressions for conserved currents of rank 2 and 3 which coincide with (5.17).

From Eq. (5.23) it also follows that some of the functions $S^{(k)}$ are full derivatives, i.e. the corresponding conservation laws are absent. This result can also be derived from the recent paper by Kaup who found the conservation laws connected with the linear operator of the third order

$$\partial^3+p\partial+q$$

[as it follows from (5.22), the conservation laws of interest can be obtained from these conservation laws]. Note, in particular, that the quantities $S^{(k)}$ in the conservation laws for the Sine-Gordon equation can be obtained from the known invariants for KdV equation by replacing u by $4\varphi_\eta^2 + 2\varphi_{\eta\eta}$. (This fact seems to be known to specialists in this field.)

Consider at last the BD equation. In this case we must put $\varphi^1 = \varphi$, $\varphi^3 = -\varphi$, $\varphi^2 = 0$ into Eq. (5.22). Then it takes the form

$$(\partial_\eta^3 - p\partial_\eta - q)\psi = \lambda^3\psi$$
$$p = 4\varphi_\eta^2 + 4\varphi_{\eta\eta}, \qquad q = \tfrac{1}{2}p_\eta$$

and the equation for χ is

$$\chi^3 - 12\chi\chi_\eta - 2\chi_{\eta\eta} + 8\chi\varphi_{\eta\eta} + 8\chi\varphi_\eta^2 - 4\varphi_{\eta\eta}\varphi_\eta - 2\varphi_{\eta\eta\eta} = \lambda^3 .$$

It can be easily seen that in this case all $S^{(2k+1)}$ are full derivatives and so do not give new conservation laws. Thus, the following nontrivial $S^{(k)}$ exist: $S^{(2)}$, $S^{(6)}$, $S^{(8)}$, $S^{(12)}, \ldots$.

Therefore, the $S^{(k)}$ with odd k as well as the values of the type $S^{(4+6k)}$ are absent here. We present the explicit form for the sixth conservation law

$$\begin{aligned}&\partial_\xi(3\varphi_{\eta\eta\eta}^2 - 10\varphi_{\eta\eta}^3 + 60\varphi_\eta^2\varphi_{\eta\eta}^2 + 16\varphi_\eta^6)\\ &\quad = \partial_\eta[6(\varphi_{\eta\eta}^2 + 4\varphi_{\eta\eta}\varphi_\eta^2 + 8\varphi_\eta^4)e^{2\varphi}\\ &\qquad - 12(\varphi_{\eta\eta}^2 - 8\varphi_{\eta\eta}\varphi_\eta^2 + 2\varphi_\eta^4)e^{-4\varphi}]\end{aligned}$$

($S^{(6)}$ and $S^{(8)}$ were calculated by Boullough and Dodd). It can be shown that the value $S^{(6)}$ exists if and only if $f(\varphi)$ has the form

$$f(\varphi) = Ae^{\alpha\varphi} + Be^{\beta\varphi}, \alpha = \pm\beta \quad \text{or} \quad \beta = -2\alpha, \beta = -\frac{\alpha}{2}.$$

Analogous results can also be obtained for the two-dimensional TL connected with other Lie groups. These results will be published elsewhere.

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