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On the Structure of Symmetry Generators

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Abstract. In a field theoretic framework we investigate generators of symmetry transformations induced by conserved local, not necessarily translationally covariant currents. Assuming the invariance of the vacuum and a mass gap, it is shown that the generator on one-particle states in general can be any polynomial of the generators of the Poincaré group and the internal symmetries. We give an example showing that the generator, defined as an integral over a conserved current, in spite of leaving the vacuum invariant, need not be self-adjoint.

1. Introduction

We consider a symmetry of a Wightman field theory induced by a conserved local current $j_\mu(x)$ which, in the general case, is not translationally covariant. We assume that $j_\mu(x)$ commutes with the basic fields for space-like separations and hence currents inducing supersymmetries are not considered here. The corresponding symmetry generator Q has been investigated in [1], [2] and papers quoted therein. In particular, it has been shown (under the assumption of a mass gap and the invariance of the vacuum) that Q extends to an operator on asymptotic scattering states. Its action on asymptotic fields is given by

$$i[Q(x+y), \psi_\kappa(x)] = \sum_\lambda \check{P}_{\kappa\lambda}(y, \partial_x)\psi_\lambda(x). \tag{1.1}$$

Here $Q(\xi)$ denotes the generator Q translated by $\xi \in \mathbb{R}^4$, $\check{P}_{\kappa\lambda}(y, \partial_x) = \check{\Xi}_{\kappa\lambda}(y, \partial_x) + \check{\Lambda}_{\kappa\lambda}(y, \partial_x)\partial_{x_0}$. $\check{\Xi}$, $\check{\Lambda}$ are polynomials in $y \in \mathbb{R}^4$ and spatial derivatives ∂, vanishing if the fields ψ_κ, ψ_λ have different mass. $\{\psi_\kappa\}$ is a (countable) complete set (cyclic with respect to the vacuum Ω) of linearly independent asymptotic incoming fields (We could also have chosen outgoing fields: Q does not depend on the choice in or out).

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The free fields ψ_κ of arbitrary spin are assumed to be in the Schwartz normal form [3] with (anti-) commutator

$$[\psi_k(x), \psi_{\kappa'}(y)]_\pm = -i\eta_{\kappa\kappa'} \Delta_{M_\kappa}(x-y). \tag{1.2}$$

κ, κ' are multi-indices, $\kappa = (j, \lambda_1, \ldots, \lambda_{n(j)})$ in the notation of [3], $\lambda_i \in \{0, 1\}$, and ψ_κ transforms under undotted spinor representations $(n(j), \bar{o})$ of the Lorentz group. All quantum numbers (mass, charges) carried by the field are specified by j. The $\eta_{\kappa\kappa'}$, (vanishing for $j \neq j'$) form an invertable hermitean matrix, real (imaginary) for integer (half integer) spin.

The advantage of having the fields in this normal form is that no derivatives occur in front of the Δ-function in the case of higher spin fields. The fields are, in general, not hermitean. However, they fulfil a reality condition [3].

The $\check{P}_{\kappa\lambda}$ fulfil some restrictions, e.g., one has

$$\sum_\lambda \check{P}_{\kappa\lambda}(y, \partial_x) \eta_{\lambda\varrho} \Delta_{M_\lambda}(x-y) = \pm \sum_\lambda \check{P}_{\varrho\lambda}(x, \partial_y) \eta_{\lambda\kappa} \Delta_{M_\lambda}(y-x). \tag{1.3}$$

The sign depends on the statistics. This equation was given for scalar fields in [1] and will be rederived for the general case in the next section.

The vanishing of $\check{P}_{\kappa\lambda}$ for different masses implies that the commutator of Q with the mass operator $P_\mu P^\mu$ vanishes on one particle states,

$$[Q, P_\mu P^\mu] \psi_\kappa(x) \Omega = 0. \tag{1.4}$$

This collection of previously obtained results will be sufficient for the following. It will suffice to consider only one mass multiplet. We therefore drop the mass index. For convenience, we assume now that any mass multiplet is of at most finite multiplicity (otherwise, e.g., (1.1) would hold only as an equation for forms).

The purpose of this note is to show what Q may look like. We prove that Q on one-particle states is a polynomial in generators of Poincaré transformations and internal symmetries (Sect. 4). In a theory of free fields, any such Q indeed occurs (Sect. 3). However, Q need not be selfadjoint (Sect. 5).

For an interacting theory, it is believed [4, 5] that Q may be only a linear combination of those generators as one already knows in classical mechanics. We will discuss this question in a subsequent paper.

2. Restrictions on the Generator

For $y = -x$, we write instead of (1.1)

$$i[Q, \psi_\kappa(x)] = P_{\kappa\lambda}(x, \partial_x) \psi_\lambda(x) \tag{2.1}$$

(Summation convention!) with a corresponding decomposition of P in Ξ, Λ.

The property that Q on one-particle states commutes with the mass operator (1.4) evidently is equivalent to

$$\partial^\mu \partial_\mu (P_{\kappa\lambda}(x, \partial_x) \psi_\lambda(x)) = P_{\kappa\lambda}(x, \partial_x) \partial^\mu \partial_\mu \psi_\lambda(x). \tag{2.2}$$

By the assumptions on the current from which Q is derived, all fields on the right of (2.1) have half-integer spin if the field ψ_k has half integer spin, and correspondingly for integer spin. Depending on the statistics of ψ_κ consider $[[Q, \psi_\kappa(x)], \psi_\sigma(y)]_\pm$ (we continue to write $[\ ,\ ]$ occasionally for $[\ ,\ ]_-$). Since the (anti-)commutators of the free fields ψ are c-numbers,

$$[[Q, \psi_\kappa(x)], \psi_\sigma(y)]_\pm = \mp [[Q, \psi_\sigma(y)], \psi_\kappa(x)]_\pm \tag{2.3}$$

or

$$P_{\kappa\lambda}(x, \partial_x)\eta_{\lambda\sigma}\Delta(x-y) = \varepsilon_\kappa P_{\sigma\lambda}(y, \partial_y)\eta_{\lambda\kappa}\Delta(y-x)\,, \tag{2.4}$$

$\varepsilon_\kappa = +1$ or -1 if ψ_κ obeys Bose or Fermi statistics respectively.

In general, besides (2.2) and (2.4), there are no further restrictions on $P_{\kappa\lambda}$ (apart from a reality property due to the hermiticity of Q). This follows from the construction of the next section.

3. Associated Currents

For any polynomial $P_{\kappa\lambda}$ obeying (2.2) and (2.4), we construct for a theory of free fields a conserved current density which is local and bilinear in the fields and which induces (2.1).

Define

$$j_\mu(x) = -\tfrac{1}{2}\varepsilon_\varrho \eta_{\kappa\varrho}^{-1} : \psi_\varrho(x)\overleftrightarrow{\partial}_\mu P_{\kappa\lambda}(x, \partial_x)\psi_\lambda(x) : \tag{3.1}$$

(Summation over multiple indices. $\varphi(x)\overleftrightarrow{\partial}_\mu(x) := \varphi(x)\partial_\mu\varphi'(x) - (\partial_\mu\varphi(x))\varphi'(x)$. The double dots denote Wick ordering). Note that in (3.1) $\eta_{\kappa\varrho}^{-1}$ as well as $P_{\kappa\lambda}$ connects fields of the same statistics only. Then

$$\partial^\mu j_\mu(x) = 0$$

due to (2.2).

To get $[Q, \psi_\sigma(y)]$ consider

$$\begin{aligned}
&[j_0(x), \psi_\sigma(y)] \\
&\quad = \frac{i}{2}\varepsilon_\varrho \eta_{\kappa\varrho}^{-1}\psi_\varrho(x)\overleftrightarrow{\partial}{}_0^x P_{\kappa\lambda}(x, \partial_x)\eta_{\lambda\sigma}\Delta(x-y) \\
&\qquad + \frac{i}{2}\varepsilon_\varrho\varepsilon_\sigma \eta_{\kappa\varrho}^{-1}\eta_{\varrho\sigma}\Delta(x-y)\overleftrightarrow{\partial}{}_0^x P_{\kappa\lambda}(x, \partial_x)\psi_\lambda(x) \\
&\quad = \frac{i}{2}\psi_\varrho(x)\overleftrightarrow{\partial}{}_0^x P_{\sigma\varrho}(y, \partial_y)\Delta(y-x) \\
&\qquad + \frac{i}{2}\Delta(x-y)\overleftrightarrow{\partial}{}_0^x P_{\sigma\lambda}(x, \partial_x)\psi_\lambda(x)
\end{aligned}$$

where (2.4) has been used in the first term $\left(\partial_0^x := \dfrac{\partial}{\partial x^0}\right)$.

For free fields,

$$\begin{aligned}
i[Q, \psi_\sigma(y)] &= \int [j_0(x), \psi_\sigma(y)]_{x^0 = y^0} d^3\bar{x} \\
&= P_{\sigma\lambda}(y, \partial_y)\psi_\lambda(y)
\end{aligned}$$

(For the last step it is convenient to decompose $P_{\kappa\lambda}(x, \partial_x) = \Xi_{\kappa\lambda}(x, \partial_x) + \Lambda_{\kappa\lambda}(x, \partial_x)\partial_0^x$ where Ξ and Λ are polynomials in x and spatial derivatives only). Note that Q leaves the vacuum invariant.

For the construction of the current density we did not use the hermiticity of Q and the corresponding condition on the $P_{\kappa\lambda}$ which would have led us to a hermitean current.

This shows that in general there are no further restrictions on the generators apart from (2.2) and (2.4). For theories with interaction it is believed that there are further severe restrictions [4, 5].

4. Q as Function of Poincaré Generators

The $-iP_{\kappa\lambda}$ represent Q on one-particle states. It will be shown that the $P_{\kappa\lambda}$ can be written as a polynomial matrix in generators of the Poincaré group and internal symmetries. The latter correspond to constant matrices operating on field indices.

This statement follows from the fact that $P_{\kappa\lambda}$ commutes with $\partial^\mu\partial_\mu$ and that the Poincaré generators and the constant matrices form an irreducible algebra on each mass multiplet. In more rigorous detail, one can proceed as follows.

Commuting (2.2) with $\psi_\varrho(y)$ and making use of the invertibility of η,

$$0 = [\partial_\mu\partial^\mu, P_{\kappa\lambda}(x, \partial_x)]\,\Delta(x-y).$$

Hence, for fixed but arbitrary κ and λ

$$\begin{aligned} 0 &= [\partial_\mu\partial^\mu, P_{\kappa\lambda}(x, \partial_x)]\,f(x) \\ &= (\partial_\mu\partial^\mu P_{\kappa\lambda}(x, \partial_x))\,f(x) + 2\partial_\mu P_{\kappa\lambda}(x, \partial_x)\partial^\mu f(x) \end{aligned} \tag{4.1}$$

with every solution f of the Klein-Gordon equation for the mass $M = M_\kappa = M_\lambda$ considered.

4.1. Lemma. *Any polynomial $P(x, \partial_x)$ obeying* (4.1) *can be written as a polynomial in $x_\nu\partial_\mu - x_\mu\partial_\nu$ and ∂_ϱ.*

Proof. The statement is obviously true for polynomials of degree zero in x. We prove it by induction on the degree. Consider a polynomial $P(x, \partial_x)$ of degree n in x.

$$P(x, \partial_x) = x_{\mu_1}\ldots x_{\mu_n} c^{\mu_1\ldots\mu_n}(\partial_x) + \hat{P}(x, \partial_x) \tag{4.2}$$

where $\hat{P}$ is of degree $n-1$ and the $c^{\mu_1\ldots\mu_n}$ are symmetric in $\mu_1\ldots\mu_n$. By assumption, P fulfills (4.1). This then also follows for all derivatives of P, too. Replacing P in (4.1) by $\partial_{\nu_1}\ldots\partial_{\nu_{n-1}}P$ gives

$$0 = c^{\nu_1\ldots\nu_{n-1}\mu}(\partial_x)\partial_\mu f.$$

By Fourier transformation

$$0 = c^{\nu_1\ldots\nu_{n-1}\mu}(ip)p_\mu \tilde{f}.$$

Since the c are symmetric,

$$0 = c^{\mu_1\ldots\mu_\kappa\ldots\mu_n}(ip)p_{\mu_\kappa} \tag{4.3}$$

on $p^2 = M^2$ for all κ. These polynomials have the following form:

Auxiliary Lemma. *Let $c^{\mu_1 \dots \mu_n}(ip)$ be a polynomial in $p \in \mathbb{R}^4$, symmetric in $\mu_1 \dots \mu_n$ which fulfills (4.3) for $p^2 = M^2 \neq 0$ and all κ. Then there exist polynomials $d^{(\nu_1\mu_1)\dots(\nu_n\mu_n)}(ip)$ symmetric in $\nu_1 \dots \nu_n$, symmetric in $\mu_1 \dots \mu_n$ and antisymmetric under the exchange of ν_i with μ_i for the same i such that*

$$c^{\mu_1 \dots \mu_n}(ip)|_{p^2=M^2} = ip_{\nu_1} \dots ip_{\nu_n} d^{(\nu_1\mu_1)\dots(\nu_n\mu_n)}(ip)|_{p^2=M^2}. \tag{4.4}$$

(For the proof see below). Inserting (4.4) into (4.2)

$$\begin{aligned} P(x, \partial_x) &= x_{\mu_1} \dots x_{\mu_n} \partial_{\nu_1} \dots \partial_{\nu_n} d^{(\nu_1\mu_1)\dots(\nu_n\mu_n)}(\partial) + \hat{P}(x, \partial_x) \\ &= 2^{-n}(x_{\mu_1}\partial_{\nu_1} - x_{\nu_1}\partial_{\mu_1}) \dots (x_{\mu_n}\partial_{\nu_n} - x_{\nu_n}\partial_{\mu_n}) d^{(\nu_1\mu_1)\dots(\nu_n\mu_n)}(\partial) + \hat{\hat{P}}(x, \partial_x) \end{aligned} \tag{4.5}$$

where $\hat{\hat{P}}$ is still a polynomial of degree $n-1$ differing from $\hat{P}$ by terms which arise from commuting ∂'s and x's. The terms in front of $\hat{\hat{P}}$ fulfil (4.1) (as P does by assumption). Hence $\hat{\hat{P}}$ is a polynomial of degree $n-1$ obeying (4.1) which has the asserted form by induction assumption. □

Proof of the Auxiliary Lemma.[1] For $n=1$, define

$$d^{\nu\mu}(ip) = -\frac{i}{M^2}(p^\nu c^\mu(ip) - p^\mu c^\nu(ip)).$$

Then $d^{\nu\mu}$ is antisymmetric, and for $p^2 = M^2 \neq 0$

$$ip_\nu d^{\nu\mu}(ip) = \frac{1}{M^2} p_\nu p^\nu c^\mu(ip) = c^\mu(ip).$$

To prove the statement for general n, consider in $c^{\mu_1 \dots \mu_n}$ the index μ_n as a fixed parameter. By induction assumption for $n-1$ there exist polynomials $d^{(\nu_1\mu_1)\dots(\nu_{n-1}\mu_{n-1}),\mu_n}(ip)$ with the required symmetry properties in $(\nu_1\mu_1)\dots(\nu_{n-1}\mu_{n-1})$ such that

$$c^{\mu_1 \dots \mu_n}(ip)|_{p^2=M^2} = ip_{\nu_1} \dots ip_{\nu_{n-1}} d^{(\nu_1\mu_1)\dots(\nu_{n-1}\mu_{n-1}),\mu_n}(ip)|_{p^2=M^2}. \tag{4.6}$$

Put

$$d^{(\nu_1\mu_1)\dots(\nu_n\mu_n)}(ip) = -\frac{i}{M^2}(p^{\nu_n} d^{(\nu_1\mu_1)\dots(\nu_{n-1}\mu_{n-1}),\mu_n} - p^{\mu_n} d^{(\nu_1\mu_1)\dots(\nu_{n-1}\mu_{n-1}),\nu_n}) \tag{4.7}$$

which is antisymmetric in ν_n and μ_n. For $p^2 = M^2$, (4.4) follows by computation using (4.6). The same is true if d in (4.7) is symmetrized with respect to $\nu_1 \dots \nu_n$ and with respect to $\mu_1 \dots \mu_n$. □

For scalar fields, this already proves the following

4.2. Theorem. *A generator Q on one-particle states is a polynomial in the generators $M_{\mu\nu}$ and P_ϱ of the Poincaré group.*

Proof. For scalar fields, $M_{\mu\nu}$ and P_ϱ on one-particle states are represented by $x_\mu\partial_\nu - x_\nu\partial_\mu$ and ∂_ϱ. In 4.1 we proved for fields with arbitrary spin that $P_{\kappa\lambda}$ can be

1 We thank E. Seiler for this simplified proof

written in terms of these,

$$P_{\kappa\lambda}(x,\partial_x)=\sum_{s=0}^{n}(x_{\mu_1}\partial_{\nu_1}-x_{\nu_1}\partial_{\mu_1})\dots(x_{\mu_s}\partial_{\nu_s}-x_{\nu_s}\partial_{\mu_s})2^{-s}d_{\kappa\lambda}^{(\nu_1\mu_1)\dots(\nu_s\mu_s)}(\partial_x) \tag{4.8}$$

In general for $M_{\mu\nu}$ we have

$$iM_{\mu\nu}\psi_\kappa(x)\Omega=((x_\mu\partial_\nu-x_\nu\delta_\mu)\delta_{\kappa\lambda}+(S_{\mu\nu})_{\kappa\lambda})\psi_\lambda(x)\Omega$$

($(S_{\mu\nu})_{\kappa\lambda}$ is a finite dimensional representation of the Lie algebra of SL(2, C)). We must prove that $P_{\kappa\lambda}$ can be written in terms of $(m_{\mu\nu})_{\varrho\sigma}$ with

$$i(m_{\mu\nu})_{\varrho\sigma}:=(x_\mu\partial_\nu-x_\nu\partial_\mu)\delta_{\varrho\sigma}+(S_{\mu\nu})_{\varrho\sigma}. \tag{4.9}$$

This, however, causes no problem:

Consider for fixed $\kappa\lambda$ the term of degree n in x

$$(x_{\mu_1}\partial_{\nu_1}-x_{\nu_1}\partial_{\mu_1})(x_{\mu_2}\partial_{\nu_2}-x_{\nu_2}\partial_{\mu_2})\dots2^{-n}d_{\kappa\lambda}^{(\mu_1\nu_1)\dots(\mu_n\nu_n)}.$$

Replace each $x_\mu\partial_\nu-x_\nu\partial_\mu$ by $(x_\mu\partial_\nu-x_\nu\partial_\mu)\delta_{\kappa\lambda}+(S_{\mu\nu})_{\kappa\lambda}$. One obtains

$$\begin{aligned}((x_{\mu_1}\partial_{\nu_1}-x_{\nu_1}\partial_{\mu_1})\delta_{\kappa\kappa_1}+(S_{\mu_1\nu_1})_{\kappa\kappa_1})((x_{\mu_2}\partial_{\nu_2}-x_{\nu_2}\partial_{\mu_2})\delta_{\kappa_1\kappa_2}\\+(S_{\mu_2\nu_2})_{\kappa_1\kappa_2})\dots2^{-n}d_{\kappa_n\lambda}^{(\mu_1\nu_1)\dots(\mu_n\nu_n)}+\dots\end{aligned}$$

plus correction terms of degree $n-1$ and lower. The n^{th} order term is now a function of the one-particle operators $m_{\mu\nu}$ and $\partial_\varrho\cdot\mathbb{1}$, as wanted. The same holds for the lower order correction terms. In the next step, one considers the terms of degree $n-1$ which are not yet in the form wanted, etc. □

The motivation for formulating Theorem 4.2 is that it allows to express Q on asymptotic one-particle states in terms of quantities that are conserved on these.

Note that Q may connect different spin multiplets. Consider, e.g., a free vector field $\Phi_\nu(x)$ and a free scalar field $\Phi(x)$, both of the same mass $M>0$ and the current density

$$j_{\mu\nu}(x)=:\Phi_\nu(x)\overleftrightarrow{\partial}_\mu\Phi(x):$$

5. Concerning Selfadjointness of Q

Assume now that Q is hermitean and ask whether it is automatically selfadjoint. This is the case for translation-invariant Q which then acts on one-particle states as a real function of the momentum operator and as such is selfadjoint (on the natural domain) [6]. For arbitrary Q, one cannot hope that such a result holds in general. We demonstrate this by an example of a generator with unequal deficiency indices and thus having no selfadjoint extension. This generator arises from a conserved current corresponding to a polynomial $P(x,\partial_x)$ of degree one in x.

Consider one free real scalar field $\Phi(x)$ of mass $M>0$ and the polynomial

$$\begin{aligned}P(x,\partial_x)&=(\partial^0)^3\partial^i(x_0\partial_i-x_i\partial_0)+(x_0\partial_i-x_i\partial_0)\partial^i(\partial^0)^3\\&=i[m_{0i},d^{0i}(\partial)]_+,\qquad d^{0i}:=(\partial^0)^3\partial^i\end{aligned} \tag{5.1}$$

which fulfills (2.2) since m_{0i} and d^{0i} commute with $\partial^\mu\partial_\mu$ separately. In view of Sect. 3 we are left with checking (2.4) to see that the corresponding current

$$j^\mu(x) = -\tfrac{1}{2} : \Phi(x)\overleftrightarrow{\partial}_\mu P(x, \partial_x)\Phi(x) : \tag{5.2}$$

is conserved and induces a Q represented by $-iP_{\kappa\lambda}$ on one particle states.

Observe that $-im_{0i}$ (as representing one of the standard generators of a free scalar field) fulfils (2.4). Since d^{0i} is even in ∂_x,

$$d^{0i}(\partial_x)\left(x_0\frac{\partial}{\partial x^i} - x_i\frac{\partial}{\partial x^0}\right)\Delta(x-y) = d^{0i}(\partial_x)\left(y_0\frac{\partial}{\partial y^i} - y_i\frac{\partial}{\partial y^0}\right)\Delta(y-x)$$
$$= \left(y_0\frac{\partial}{\partial y^i} - y_i\frac{\partial}{\partial y^0}\right)d^{0i}(\partial_x)\Delta(y-x) = \left(y_0\frac{\partial}{\partial y^i} - y_i\frac{\partial}{\partial y^0}\right)d^{0i}(\partial_y)\Delta(y-x)$$

A similar relation holds for the other term in (5.1) so that the anticommutator fulfils (2.4).

It is convenient for the following to look at Q on one particle states in momentum representation, i.e., on $L^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2+M^2}}\right)$. There Q is represented by

$$\hat{D} = c((\mathbf{p}^2+M^2)(p_j i\partial^j + i\partial^j p_j)(\mathbf{p}^2+M^2) - i(\mathbf{p}^2+M^2)\mathbf{p}^2) \tag{5.3}$$

where the real constant c depends on normalization conventions for the relation between $\Phi(x)$ and one-particle states.

We next map by a unitary transformation V the space $L^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2+M^2}}\right)$ onto $L^2(\mathbb{R}^3, d^3\mathbf{p})$ on which the corresponding operator $D = V\hat{D}V^{-1}$ has a simpler form.

$$(V\varphi)(\mathbf{p}) = (2\sqrt{\mathbf{p}^2+M^2})^{-1/2}\varphi(\mathbf{p}), \qquad \varphi \in L^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2+M^2}}\right)$$
$$D = c(\mathbf{p}^2+M^2)(\mathbf{p}i\partial + i\partial\mathbf{p})(\mathbf{p}^2+M^2)$$
$$= c((\mathbf{p}^2+M^2)^2\, 2\mathbf{p}i\partial + 3i(\mathbf{p}^2+M^2)^2 + 4i\mathbf{p}^2(\mathbf{p}^2+M^2))$$

The domain of D certainly contains $\mathfrak{S}(\mathbb{R}^3)$ [1, 2].

The deficiency indices of D, the scaled $\frac{1}{c}D$ and the unitarily equivalent $\hat{D}$ are the same. We determine those of $\frac{1}{c}D$ by looking at solutions from $L^2(\mathbb{R}^3, d^3p)$ of

$$\frac{1}{c}D^* f = \pm if. \tag{5.4}$$

Any solution of (5.4) is a distribution solution of

$$((\mathbf{p}^2+M^2)^2\, 2\mathbf{p}\partial + 3(\mathbf{p}^2+M^2)^2 + 4\mathbf{p}^2(\mathbf{p}^2+M^2))g = \pm g \tag{5.5}$$

where g is a square integrable function.

In polar coordinates $\left(\mathbf{p}\to(r, \Theta, \chi), \mathbf{p}\partial\to r\frac{\partial}{\partial r}\right)$ (5.5) reads

$$2r\partial_r g = \frac{\pm 1 - 3(r^2+M^2)^2 - 4r^2(r^2+M^2)}{(r^2+M^2)^2} g$$

with the solution

$$g_\pm(r,\Theta,\chi) = C_\pm(\Theta,\chi)\frac{1}{r^{3/2}(r^2+M^2)}\exp\left[\pm\frac{1}{4M^4}\left(\log\left(\frac{r^2}{r^2+M^2}\right)+\frac{M^2}{r^2+M^2}\right)\right]$$

where $C_\pm(\Theta,\chi)$ is arbitrary. The r-dependent factor is square integrable with respect to r^2dr at ∞, however, for $r\to 0$ only in the case of the upper sign. Since $C_\pm$ is arbitrary square integrable on the sphere, $\frac{1}{c}D$ has deficiency indices 0 and ∞, and so has $\hat{D}$.

Q represented by $\hat{D}$ on one particle states and acting additively on multiparticle states, *has no selfadjoint extension as well*: Q^* leaves the one-particle subspace $\mathscr{H}_1$ invariant $\left(\text{for } g\in\mathscr{H}_1\cap D(Q^*) \text{ and } f\in\bigoplus_{n\geqq 2}\mathscr{H}_n\cap D(Q)\right.$ where $\mathscr{H}_n$ are the n-particle subspaces, one has that $Qf\in\bigoplus_{n\geqq 2}\mathscr{H}_n$ and $\left.(Q^*g|f)=(g|Qf)=0\right)$. Therefore, every selfadjoint extension would also leave $\mathscr{H}_1$ invariant and hence would give rise to a selfadjoint extension of D.

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