Commun. math. Phys. 65, 83—90 (1979)

Communications in
Mathematical
Physics

On a New Derivation of the Navier-Stokes Equation

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Abstract. The Navier-Stokes equation is derived by 'adding' the effect of the Brownian motion to the Euler equation. This is an example suggesting the 'equation': 'Reversible phenomena' ⊕ 'Probability' = 'Irreversible phenomena'.

§1. Introduction

As a model equation representing the motion of viscous incompressible fluid (resp. incompressible perfect fluid), the Navier-Stokes equation (resp. the Euler equation) in a domain $D \subset R^n$ was introduced physically in the 19th century. They are formulated as below:

$$(N.S.)\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } Q_T = (0,T) \times D, \\ \operatorname{div} u = 0 & \text{in } Q_T, \\ u|_{\partial D} = 0 & \text{on } (0,T) \times \partial D, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

and

$$(E)\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = f & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{in } Q_T, \\ u \cdot \mathbb{n} = 0 & \text{on } (0,T) \times \partial D, \\ u(0,\cdot) = u_0(\cdot), \end{cases}$$

Here $u = u(t,x) = (u^1(t,x), \dots, u^n(t,x))$ denotes the unknown velocity field at a point $(t,x) \in Q_T$, $p = p(t,x)$ is the unknown pressure at (t,x), $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{(\partial x^j)^2}$, $\Delta p = \left(\frac{\partial p}{\partial x^1}, \dots, \frac{\partial p}{\partial x^n}\right)$, $\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u^i}{\partial x^i}$, the i-th component of $(u \cdot \nabla) u = \sum_{j=1}^{n} u^j \frac{\partial u^i}{\partial x^j}$, $\mathbb{n}$ is the unit exterior normal, $u_0 = u_0(x) = (u_0^1(x), \dots, u_0^n(x))$ is the given initial velocity and $f=f(t,x)=(f^1(t,x),\dots,f^n(t,x))$ is the given exterior force: μ is a positive constant called the kinematic viscosity.

Concerning the Euler equation, Arnold [1] derived it as the geodesic spray of the group of diffeomorphisms. (See also the works of Ebin-Marsden [3], Omori [11].) So the geometrical foundation of the Euler equation is given. In another word, we may formulate it in a variational problem.

On the other hand, the heat equation is the typical equation representing the irreversible phenomena and the Brownian motion is deeply connected with it. So, we want to derive the Navier-Stokes equation in $\mathbb{R}^n$ (not in D) from the Euler equation by 'adding' the effect of the Brownian motion. In some sense we try to explain the physical term 'internal friction' (Landau-Lifshitz [9], see also Serrin [12]) by a probabilistic concept.

In the above, we use the word 'try' because we use some 'divergent' integral in deriving the non-linear partial differential equation with random coefficient named $(E)_B$ in §3. Moreover, the solvability of the above equation $(E)_B$ is unfortunately open for the time being. We may derive the Navier-Stokes equation or rather the Reynolds equation assuming the existence of solutions of $(E)_B$.

But our theorem suggests at least implicitly, it seems our derivation is deeply related to the turbulent phenomena. Because even the turbulant flow is, believed, governed by the Navier-Stokes equation and its statistical feature is exhibited by the Reynolds equation (Hopf [6], Foias [5]).

In any way, this paper is an experiment to derive the equation of irreversible phenomena by combining the variational problem with the probabilistic idea.

§2. The Variational Formulation of the Euler Equation

Let $\Phi_t(\cdot)$ be a one parameter family of $\mathrm{Diff}_\sigma(\mathbb{R}^n)$, where $\mathrm{Diff}_\sigma(\mathbb{R}^n)$ stands for the group of orientation preserving diffeomorphisms of $\mathbb{R}^n$ whose jacobian equal 1. (A subscripted variable will never denote differentiation.)

We consider the following variational problem: Given Φ^0 and Φ^1 belonging to $\dot{\mathrm{D}}\mathrm{iff}_\sigma(\mathbb{R}^n)$, find the one parameter family of diffeomorphisms Φ_t such that

(i) $\Phi_0=\Phi^0$ and $\Phi_1=\Phi^1$,

and

(ii) Φ_t attains the stationary point of the following energy functional:

$$J(\Phi_t)=\int_0^1\int_{\mathbb{R}^n}\sum_{i=1}^n\left[\frac{\partial\Phi_t^i(x)}{\partial t}\right]^2 dxdt \tag{2.1}$$

where $\Phi_t(x)=(\Phi_t^1(x),\dots,\Phi_t^n(x))$.

As $\mathbb{R}^n$ is not compact, (2.1) does not converge for general Φ_t. So we must confine ourselves to some class of $\mathrm{Diff}_\sigma(\mathbb{R}^n)$ for which (2.1) converges. Denoting this class by $\dot{\mathrm{D}}\mathrm{iff}_\sigma(\mathbb{R}^n)$, though we leave the problem of characterizing it for future study (i.e. characterizing the behavior at ∞; see Babenko [2], Finn [4]).

Leaving this ambiguity aside, we have the following result characterizing the desired solution Φ_t by corresponding velocity field $u(t,x)=\left(\frac{\partial \Phi_t^1(x)}{\partial t},\dots,\frac{\partial \Phi_t^n(x)}{\partial t}\right)$. (Here and after, we identify freely a vector field and its component vector.)

Theorem 2.1. *Let Φ_t be the desired solution of the above variational problem. Then the vector field u whose integral curve is Φ_t, satisfies the Euler equation [E] in $\mathbb{R}^n$ with $f=0$.*

Remark 2.2. (i) There exists some difference between the initial value problem and the two point fixed problem. But we connive at this difference as we do in deriving the geodesic equation.

(ii) Modifying if necessary, we have the rigorous result for compact manifold with or without boundary. See Arnold [1], Ebin-Marsden [3], Marsden-Ebin-Fisher [10]. Concerning $\mathrm{Diff}_\sigma(\mathbb{R}^2)$, see also Kato [8].

§3. Preparations from the Probability Theory

Let H be a real separable Hilbert space with an inner product (,) and let $B_t=\{B_t^j(\omega)\}_{j=1}^n$ be an $\mathbb{R}^n$-valued Brownian motion defined on a complete probability space (Ω, F, P). Denoting by F_t the σ-field generated by $\{B_s(\omega); 0\leqq s\leqq t\}$, we consider following classes of H-valued stochastic processes.

$B(H)=\{X_t; X_t$ is H-valued process such that (X_t,h) is $\{F_t\}$ adapted measurable process for all $h\in H\}$

$L^2(H)=\{\Phi_t\in B(H); \Phi_t\in L^2([0,T]\times\Omega; H)\}$ with norm given by

$$\|\Phi_t\|^2_{L^2(H)}=E\left[\int_0^T \|\Phi_t\|_H^2 dt\right]$$

$M(H)=\{M_t\in B(H); (M_t,h)$ is square integrable martingale relative to $\{F_t\}$ for all $h\in H\}$

$A(H)=\{A_t\in B(H); A_t$ has a strong derivative $\frac{dA}{dt}\in L^2(H)$ and $A_0=0\}$

$Q(H)=\{Q_t=M_t+A_t; M_t\in M(H)$ and $A_t\in A(H)\}$.

We call an element of $Q(H)$ as H-valued quasi-martingale relative to $\{F_t\}$, and every $X\in Q(H)$ is expressed uniquely as $X=M^X+A^X, M^X\in M(H)$ and $a^X\in A(H)$, called the martingale part and the absolute continuous part of X respectively.

Now, we enumerate some results.

1. For $\Phi_t\in L^2(H)$, we can define a stochastic integral $\int_0^t \Phi_s dB_s^j$ by the relation:

$$\left(\int_0^t \Phi_s dB_s^j, h\right)=\int_0^t (\Phi_s,h)dB_s^j \text{ for all } h\in H,$$

where right hand side of this equality is the usual stochastic integral of Itô. We note that $\int_0^t \Phi_s dB_s^j$ is a H-valued martingale and belongs to $L^2(\Omega, H)$ for each $t\in[0,T]$ and $C([0,T],H)$ for *a.s.ω*.

2. For $M_t \in M(H)$, there exist uniquely $\Phi_t^j \in L^2(H)(j=1,\dots,n)$ such that

$$M_t = M_0 + \sum_{j=1}^{n} \int_0^t \Phi_s^j dB_s^j \text{ and } M_0 \in H.$$

Moreover, for $Q_t \in Q(H)$, we have the following representation

$$Q_t = Q_0 + \sum_{j=1}^{n} \int_0^t \Phi_s^j dB_s^j + \int_0^t \Psi_s ds$$

with $\Phi_t^j, \Psi_t \in L^2(H)$ and $Q_0 \in H$. We say $\{\Phi_t^j\}_{j=1}^n$ the B-derivative of Q and denote $\Phi_t^j = \dfrac{\partial Q_t}{\partial B_t^j}$.

3. For $Q_t \in Q(H)$, we can define a stochastic integral $\int_0^t Q_s \circ dB_s^j$ by the relation:

$$\left(\int_0^t Q_s \circ dB_s^j, h\right) = \int_0^t (Q_s, h) \circ dB_s^j \text{ for all } h \in H,$$

where right hand side is defined by the stochastic integral of Stratonovich (Itô [7]). We have

$$\int_0^t Q_s \circ dB_s^j = \int_0^t Q_s dB_s^j + \frac{1}{2}\int_0^t \frac{\partial Q_s}{\partial B_s^j} ds.$$

Lemma 3.1. *If, for real valued* $\{F_t\}$ *adapted processes* $a_j(t)(j=1,\dots,n)$ *and* $b(t) \in L^2([0,T]\times\Omega)$, *the equality*

$$\sum_{j=1}^{n} \int_0^T \phi(t) a_j(t) dB_t^j + \int_0^T \phi(t) b(t) dt = 0$$

holds for each $\phi \in C_0^\infty((0,T))$, *then we have*

$$a_j(t) = b(t) = 0 \quad (j=1,\dots,n).$$

Proof. Since the equality in the statement of this lemma also holds for each $\phi \in L^2([0,T])$, taking $\phi(t) = X_{[0,s]}(t)$, we have

$$\sum_{j=1}^{n} \int_0^s a_j(t) dB_t^j + \int_0^s b(t) dt = 0 \quad \text{for all } s \in [0,T].$$

We see easily that this implies the conclusion.

§4. The Derivation of the Navier-Stokes Equation

Now we consider the following "variational problem affected by the Brownian motion".

For $a.s.\omega$, given $\Phi^0(\cdot\,;\omega)$ and $\Phi^1(\cdot\,;\omega)$ belonging to $\mathrm{Diff}_\sigma(\mathbb{R}^n)$, find the one parameter family of diffeomorphisms $\Phi_t(\cdot\,;\omega)$ such that

(i) $\Phi_0(\cdot\,;\omega)=\Phi^0(\cdot\,;\omega)$ and $\Phi_1(\cdot\,;\omega)=\Phi^1(\cdot\,;\omega)$

and

(ii) $\Phi_t(\cdot\,;\omega)$ attains the stationary point of the following functional:

$$J_B(\Phi_t)=\int_0^1\int_{\mathbb{R}^n}\sum_{i=1}^n\left[\frac{\partial\Phi_t^i}{\partial t}+\sqrt{2\mu}\frac{dB_t^i}{dt}\right]^2 dxdt \tag{4.1}$$

where B_t denotes the n-dimensional Brownian motion defined on a probability space (Ω,F,P).

Remark 4.1. Generally speaking, there is no meaning in (4.1) because $B_t(\cdot)$ is not differentiable in t in a usual sense. But this point is partly remedied by considering $\int_0^1 u_t^i\circ dB_t^i$ instead of $\int_0^1 u_t^i\dot{B}_t^i dt$. More serious problem is to give some meaning to the formal integral $\int_0^1\sum_{i,j=1}^n \dot{B}_t^i\dot{B}_t^j dt$. Taking into account that the only the variation of the functional $J_B(\Phi_t)$ is necessary, this difficulty will be evaded.

Let $\Phi_t(\cdot\,;\omega)$ be a desired solution of the above problem. Taking the first varia-variation of the functional $J_B(\Phi_t)$, we may show that the vector field $u(t,\cdot\,;\omega)$ $\cdot\left(u^i(t,\cdot\,;\omega)=\dfrac{\partial\Phi_t^i}{\partial t}\right)$ satisfies the following equation.

$$(E)_B\begin{cases}\dfrac{\partial(u^i+\sqrt{2\mu}\dot{B}_t^i)}{\partial t}+\displaystyle\sum_{j=1}^n\frac{\partial u^j}{\partial x^j}\circ(u^j+\sqrt{2\mu}\dot{B}_t^j)+\frac{\partial p}{\partial x^i}=0\\ \operatorname{div} u_t=0\end{cases}$$

Definition 4.2. A vector field u will be called a weak solution of the initial value problem (for brevity, I. V. P.) for $(E)_B$ with the initial value $u_0\in\mathbb{H}_\sigma^1(\mathbb{R}^n)$ (independent of ω) if the following conditions are satisfied.

(i) $u(t,x;\omega)\in Q(\mathbb{H}_\sigma^1(\mathbb{R}^n))$.

(ii) $$\int_0^T\int_{\mathbb{R}^n}\left[u^i\frac{\partial\theta}{\partial t}v^i+u^iu^j\frac{\partial v^i}{\partial x^j}\theta\right]dxdt$$
$$+\sqrt{2\mu}\int_0^T\int_{\mathbb{R}^n}u^i\frac{\partial v^i}{\partial x^j}\theta\circ dB_t^j dx=-\theta(0)\int_{\mathbb{R}^n}u_0^iv^idx\quad\text{for}\quad a.s.\omega$$

for any $\theta(t)\in C_0^\infty[0,T]$, $v(x)\in\mathbb{C}_{0,\sigma}^\infty(\mathbb{R}^n)$.

(We abbreviate the summation notation, i.e. u^iu^i means that $\sum_{i=1}^n u^iu^i$.)

Hereafter, the following function spaces are used.

$\mathbb{L}_\sigma^2(\mathbb{R}^n)=\{v\in(L^2(\mathbb{R}^n))^n\,;\operatorname{div}v=0\}$.

$\mathbb{C}_{0,\sigma}^\infty(\mathbb{R}^n)=\{v\in(C_0^\infty(\mathbb{R}^n))^n\,;\operatorname{div}v=0\}$.

$\mathbb{H}_\sigma^1(\mathbb{R}^n)=\{v\in(H^1(\mathbb{R}^n))^n\,;\operatorname{div}v=0\}$.

$\mathbb{H}^{-1}(\mathbb{R}^n)=(H^{-1}(\mathbb{R}^n))^n$.

where $H^l(\mathbb{R}^n)$ stands for the Sobolev space of order l.

Remark 4.3. (a) The pressure p does not appear explicity in the above definition. But as usual, we may recover $p(t,x;\omega)\in\mathscr{D}'((0,T)\times\mathbb{R}^n)$ for $a.s.\omega$ by

$$\int_{\mathbb{R}^n}\frac{\partial p}{\partial x^i}\frac{\partial\phi}{\partial x^i}=-\int_{\mathbb{R}^n}\frac{\partial u^i}{\partial x^j}\frac{\partial u^j}{\partial x^i}\phi dx \text{ for } \phi\in C_0^\infty(\mathbb{R}^n). \tag{4.2}$$

(b) Comparing with $(E)_B$, the term $\sqrt{2\mu}\dot{B}_t$ is dropped also. Because we have

$$\int_0^T\int_{\mathbb{R}^n}\dot\theta v^i dB_t^i=\int_{\mathbb{R}^n}v^i\left[\int_0^T\dot\theta dB_t^i\right]dx=0$$

for any $\theta\in C_0^\infty[0,T), v\in\mathbb{C}_{0,\sigma}^\infty(\mathbb{R}^n)$. (4.3)

Definition 4.4. (See Serrin [13]). A vector field u will be called a weak solution of the I.V.P. of the Navier-Stokes equation below

$$(N.S.)\begin{cases}\dfrac{\partial u}{\partial t}-\mu\Delta u+(u\cdot\nabla)u+\nabla p=f,\\ \operatorname{div} u=0,\\ u(0)=u_0,\end{cases}$$

if the following are satisfied.

(i) $u(t,x)\in L^\infty(0,T:\mathbb{L}_\sigma^2(\mathbb{R}^n))\cap L^2(0,T:\mathbb{H}_\sigma^1(\mathbb{R}^n))$.

(ii) $u_0\in\mathbb{H}_\sigma^1(\mathbb{R}^n), f\in L^2(0,T:\mathbb{H}^{-1}(\mathbb{R}^n))$.

(iii) $$-\int_0^T\int_{\mathbb{R}^n}\dot\theta u^i v^i dxdt+\mu\int_0^T\int_{\mathbb{R}^n}\theta\frac{\partial u^i}{\partial x^j}\frac{\partial v^i}{\partial x^j}dxdt-\int_0^T\int_{\mathbb{R}^n}\theta u^i u^j\frac{\partial v^i}{\partial x^j}dxdt$$
$$=\theta(0)\int_{\mathbb{R}^n}u_0^i v^i dx+\int_0^T\int_{\mathbb{R}^n}\theta f^i v^i dxdt,$$

for any $\theta\in C_0^\infty[0,T), v\in\mathbb{C}_{0,\sigma}^\infty(\mathbb{R}^n)$.

Now, we may state our theorem.

Theorem 4.5. *Let u be a weak solution of the $(I.V.P.)$ of $(E)_B$ with $u_0\in\mathbb{H}_\sigma^1(\mathbb{R}^n)$. Then the average*

$$\bar u=\int_\Omega u(\cdot;\omega)dP$$

gives the weak solution of the following equation.

$$(R)\begin{cases}\dfrac{\partial\bar u}{\partial t}-\mu\Delta\bar u+(\bar u\cdot\nabla)\bar u+\nabla\pi=-\overline{(v\cdot\nabla)v},\\ \operatorname{div}\bar u=0\end{cases}$$

where $v=u-\bar u$.

Proof. For $u\in Q(\mathbb{H}_\sigma^1(\mathbb{R}^n))$, by the representation theorem (§3, 2.),

$$u(t)=u(0)+\int_0^t \frac{\partial u}{\partial B^j} dB_s^j+\int_0^t \frac{\partial u^A}{\partial s} ds .$$

By Itô's formula,

$$\int_0^T \int_{\mathbb{R}^n} \dot{\theta} u^i v^i dx dt = -\theta(0) \int_{\mathbb{R}^n} u^i(0) v^i dx - \int_0^T \theta(t) \left[\int_{\mathbb{R}^n} v^i \frac{\partial u^i}{\partial B_t^j} dx \right] dB_t^j$$

$$- \int_0^T \theta(t) \left[\int_{\mathbb{R}^n} v^i \frac{du^{Ai}}{dt} dx \right] dt \quad a.s.\omega \tag{4.4}$$

for $\theta \in C_0^\infty[0,T), v \in \mathbb{C}_{0,\sigma}^\infty(\mathbb{R}^n)$.

By the rule 3° of §3, we have

$$\int_0^T \int_{\mathbb{R}^n} \theta u^i \frac{\partial v^i}{\partial x^j} \circ dB_t^j dx = \int_0^T \theta \left[\int_{\mathbb{R}^n} u^i \frac{\partial v^i}{\partial x^j} dx \right] dB_t^j$$

$$+ \tfrac{1}{2} \int_0^T \theta \left[\int_{\mathbb{R}^n} \frac{\partial u^i}{\partial B^j} \frac{\partial v^i}{\partial x^j} dx \right] dt. \tag{4.5}$$

Combining there with the relation (ii) of Definition 4.2, and applying the lemma, we have readily

(a) $\int_{\mathbb{R}^n} \frac{du^{Ai}}{dt} v^i dx = \int_{\mathbb{R}^n} u^i u^j \frac{\partial v^i}{\partial x^j} dx + \frac{\sqrt{2\mu}}{2} \int_{\mathbb{R}^n} \frac{\partial u^i}{\partial B_t^j} \frac{\partial v^i}{\partial x^j} dx$

(b) $\int_{\mathbb{R}^n} \frac{\partial u^i}{\partial B_t^j} v^i dx = \sqrt{2\mu} \int_{\mathbb{R}^n} u^i \frac{\partial v^i}{\partial x^j} dx \quad$ for $\forall_j = 1, 2, \ldots, n$.

Inserting the relation (b) into (a), we have

(a)′ $\int_{\mathbb{R}^n} \frac{du^{Ai}}{dt} v^i dx = \int_{\mathbb{R}^n} u^i u^j \frac{\partial v^i}{\partial x^j} dx + \mu \int_{\mathbb{R}^n} u^i \Delta v^i dx$

Taking the average of (a)′, we have the desired equation.

Q.E.D.

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Communicated by J.L. Lebowitz

Received December 19, 1977; in revised form July 13, 1978