Commun. math. Phys. 61, 69—85 (1978)

Communications in Mathematical Physics
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Moufang Plane and Octonionic Quantum Mechanics*

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Abstract. It is shown that the usual axioms of one-particle Quantum Mechanics can be implemented with projection operators belonging to the exceptional Jordan algebra J_3^8 over real octonions. Certain lemmas on these projection operators are proved by elementary means. Use is made of the Moufang projective plane. It is shown that this plane can be orthocomplemented and that there exists a unique probability function. The result of successive, compatible experiments is shown not to depend on the order in which they are performed, in spite of the non-associativity of octonion multiplication. The algebra of observables and the action of the exceptional group F_4 is studied, as well as a possible relation with the color group SU(3) and quark confinement.

1. Introduction

Following Jordan's formulation [1] of Quantum Mechanics in terms of anti-commutators, Jordan, von Neumann and Wigner (JNW) [2] showed that all realizations of Quantum Mechanics in terms of anti-commutators (Jordan product) are equivalent to realizations in terms of commutators (Lie product) except in the case of 3×3 Hermitian octonionic matrices which form the exceptional Jordan algebra. Since this one possible generalization via the Jordan formulation was found too narrow for explaining the then observed nuclear phenomena this approach was abandoned. However, Jordan algebras have since been extensively studied by the mathematicians who established some deep connections between the exceptional Lie groups and the exceptional Jordan algebra [3, 4].

After the sixties octonions made their appearance in physics again. Pais, Gamba and others tried to relate the octonions to various internal symmetry schemes in elementary particle physics [5, 6] Goldstine, Horwitz, and Biedenharn studied a Clifford algebra made of octonionic multiplication operators [7].

Günaydin and Gürsey suggested the extension of the underlying field of Quantum Mechanics from complex numbers to the octonion algebra in order to

* Work supported by the Swiss National Science Foundation

0010-3616/78/0061/0069/$03.40

explain the charge space properties of elementary particles on a fundamental level [8–13]. They showed that space-time groups like the Poincaré group can be unitarily implemented on an octonionic space with complex scalar products. The space-time symmetries and quantum mechanical considerations then select out an SU(3) subgroup of the automorphism group G_2 of octonions as an intrinsic exact symmetry group of the space, which they identified with the color SU(3) group of Gell-Mann and Fritzsch [14]. By taking octonionic quark fields, making the ansatz that the non-associative and non-commutative components of the space be unobservable, they could give an algebraic explanation of the observability of color singlet states only, and the unobservability of quarks and other colored states.

In this paper we answer the following question: is it possible to implement the usual axioms of quantum mechanics with objects belonging to a Jordan algebra? It turns out that this is not possible with the usual Hilbert space formulation of Quantum Mechanics, because the octonion algebra is non associative. So one has to go to a more abstract level [15], starting with the concept of proposition (yes-no experiment). Propositions correspond in the usual case to projection operators, and the structure of the propositional system of Quantum Mechanics is equivalent to the structure of an orthocomplemented projective geometry. At this point one finds the link with octonions. Ruth Moufang [16] has constructed a projective plane coordinated by octonions, and which turns out to be non-Desarguian.

Our aim in this paper is to study the quantum mechanical properties of this non-Desarguian geometry by using Jordan's formulation of it in terms of the exceptional Jordan algebra [17, 18]. We show that it can be orthocomplemented and that there exists a unique probability function satisfying Gleason's axioms. Successive compatible experiments yield a result which is independent of the order in which they are performed. Hence, on the one-particle level all the axioms of Quantum Mechanics are fullfilled. We further study the action of the automorphism group of the Moufang plane on states and the structure of the algebra of observables.

Physical interpretation of the octonionic Quantum Mechanics poses many problems. An important problem has to do with the possible product states, which is crucial for the algebraic explanation of the unobservability of colored states. Main-difficulty stems from the fact that all irreducible projective geometries of dimension greater than two are Desarguian. One possible way out is to have an imbedding of Desarguian subgeometries. Another possibility is to imbed this projective geometry into a non-irreducible higher geometry. In any case, we hope that the study of these problems may shed further light into the problem of unobservability of colored states.

Before we begin our study of the octonion Quantum Mechanics we should stress the fact that the mathematics underlying our work has long been thoroughly developed by the mathematicians. The lemmas we use or prove may be found in the mathematical literature in some form or other. Our purpose is to reformulate these results in a physicists' language and lay the ground for future work on the subject.

2. Octonions and the Exceptional Jordan Algebra

Real octonion algebra $\mathbb{O}$ is an 8-dimensional division algebra whose elements can in general be decomposed as

$$O = r_0 e_0 + \sum_{A=1}^{7} r_A e_A , \tag{2.1}$$

where r_0 and r_A are real numbers, e_0 is the identity element and e_A are the seven imaginary units which obey the multiplication rule

$$e_A e_B = -\delta_{AB} + \sum_{C=1}^{7} \eta_{ABC} e_C . \tag{2.2}$$

η_{ABC} are totally anti-symmetric and their non-vanishing components, apart from permutations, are [9]

$$\eta_{123} = \eta_{246} = \eta_{435} = \eta_{367} = \eta_{651} = \eta_{572} = \eta_{471} = 1 . \tag{2.3}$$

The octonion algebra is not associative, but it satisfies a weaker property, namely alternativity, which implies that the associator (O_1, O_2, O_3) of any 3 octonions is an alternating function of its arguments:

$$(O_1, O_2, O_3) \equiv (O_1 O_2) O_3 - O_1 (O_2 O_3) = (O_3, O_1, O_2) = -(O_2, O_1, O_3) . \tag{2.4}$$

As a consequence of this property one has

$$(O_1 O_1) O_2 = O_1 (O_1 O_2) ; \quad (O_1 O_2) O_2 = O_1 (O_2 O_2) \tag{2.5}$$

and the Moufang identity

$$(O_1 O_2)(O_3 O_1) = O_1 (O_2 O_3) O_1 . \tag{2.6}$$

The conjugate octonion is defined by

$$\bar{O} = r_0 e_0 - \sum_{A=1}^{7} r_A e_A \tag{2.7}$$

and the real bilinear product by [19]

$$(O_1, O_2) \equiv \tfrac{1}{2} (\bar{O}_1 O_2 + \bar{O}_2 O_1) \tag{2.8}$$

which induces the usual octonionic norm

$$\begin{aligned} &n(O) = (O, O) = \bar{O} O = O \bar{O} \\ &n(O_1 O_2) = n(O_1) n(O_2) \end{aligned} \tag{2.9}$$

and has the invariance group $O(8)$.

The real exceptional Jordan algebra J_3^8 is a Jordan algebra whose elements are the 3×3 hermitian real octonionic matrices

$$J = \begin{pmatrix} \alpha_1 & O_3 & \bar{O}_2 \\ \bar{O}_3 & \alpha_2 & O_1 \\ O_2 & \bar{O}_1 & \alpha_3 \end{pmatrix} , \tag{2.10}$$

where α_i are real numbers and O_i are real octonions. These matrices form an intrinsically non-associative algebra under the Jordan product

$$J_1 \circ J_2 = \tfrac{1}{2}(J_1 J_2 + J_2 J_1), \tag{2.11}$$

where $J_1 J_2$ is the usual matrix product, and satisfy the Jordan identity:

$$(J_1 \circ J_2) \circ J_1^2 = J_1 \circ (J_2 \circ J_1^2). \tag{2.12}$$

In the classic work of Jordan et al. [2], it was shown that the algebra J_3^8 is the only Jordan algebra which has no realizations in terms of associative matrices. A one-dimensional projection operator P is a Jordan matrix which satisfies the conditions

$$P^2 = P \circ P = P; \quad \operatorname{tr} P = 1. \tag{2.13}$$

Any one-dimensional projection operator P belonging to J_3^8 can be brought to the following form [17]

$$P = \begin{pmatrix} a \\ b \\ c \end{pmatrix} (\bar{a}\bar{b}\bar{c}) = \begin{pmatrix} a\bar{a} & a\bar{b} & a\bar{c} \\ b\bar{a} & b\bar{b} & b\bar{c} \\ c\bar{a} & c\bar{b} & c\bar{c} \end{pmatrix}, \tag{2.14}$$

where a, b, c are octonions, one of them being pure real, and satisfy

$$a\bar{a} + b\bar{b} + c\bar{c} = \operatorname{tr} P = 1. \tag{2.15}$$

Following Freudenthal [20] one can define the following X product among the Jordan matrices

$$J_1 \times J_2 \equiv J_1 \circ J_2 - \tfrac{1}{2} J_1 \operatorname{tr} J_2 - \tfrac{1}{2} J_2 \operatorname{tr} J_1 + \tfrac{1}{2} I(\operatorname{tr} J_1 \operatorname{tr} J_2 - \operatorname{tr} J_1 J_2), \tag{2.16}$$

where I is the 3×3 identity matrix.

For Jordan matrices on real octonions one has the property that if $J \times J = 0$ then J is a scalar multiple of a projection operator i.e.

$$J \times J = 0 \Rightarrow J^2 = \lambda J; \quad \operatorname{tr} J = \lambda, \tag{2.17}$$

where λ is a non-zero real number if $J \neq 0$.

Using the X product one can define a completely symmetric trilinear form:

$$[J_1, J_2, J_3] \equiv \operatorname{tr}(J_1 \times J_2) \circ J_3 \equiv (J_1 \times J_2, J_3) \tag{2.18}$$

which induces the determinental form i.e.

$$\det J = \tfrac{1}{3}[J, J, J] = \tfrac{1}{3} \operatorname{tr} J^3 - \tfrac{1}{2} \operatorname{tr} J^2 \operatorname{tr} J + \tfrac{1}{6}(\operatorname{tr} J)^3. \tag{2.19}$$

3. Group Theoretic Properties of J_3^8

The exceptional Jordan algebra has, as the automorphism group, the exceptional group F_4 [4]. The 52 infinitesimal anti-hermitian generators D of F_4 act as derivations of J_3^8 i.e. they satisfy the property:

$$D(J_1 \circ J_2) = (DJ_1) \circ J_2 + J_1 \circ (DJ_2) \tag{3.1}$$

for any elements J_1, J_2 of J_3^8. This implies that the derivations annihilate the identity matrix:

$$DI=0.$$

The 26 traceless Jordan matrices form the basis vectors for an irreducible representation of F_4. F_4 can also be uniquely characterized as the simultaneous invariance group of the quadratic and cubic forms of J_3^8 i.e. as the invariance group of

$$(J_1, J_2)=\mathrm{tr}\, J_1 \circ J_2 \tag{3.2}$$

and of

$$[J_1, J_2, J_3]=\mathrm{tr}\,(J_1 \times J_2)\circ J_3 .$$

The derivations can be represented by the action of two traceless Jordan matrices X, Y on J_3^8:

$$D_{X,Y}J=(X, J, Y)=-(Y, J, X), \tag{3.3}$$

where the RHS is the associator

$$(X, J, Y)\equiv(X\circ J)\circ Y - X\circ(J\circ Y). \tag{3.4}$$

The derivations satisfy the following commutation relations:

$$[D_{X,Y}, D_{Z,W}]=D_{D_{X,Y}Z,W}+D_{Z,D_{X,Y}W}. \tag{3.5}$$

The automorphisms of the 2×2 Jordan matrices form an SO(9) subgroup of F_4. There are three of them in F_4. If we denote the elements X, Y of J_3^8 as

$$X=\begin{pmatrix} \alpha_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_3 \end{pmatrix}=\sum_{i=1}^{3}\alpha_i E_i+(x_3)_{12}+(x_2)_{31}+(x_1)_{23} \tag{3.6}$$

then the generators of the SO(9) subgroup leaving the indempotent E_3 invariant are given by

$$D_{(x_3)_{12},(y_3)_{12}}, \qquad D_{(x_3)_{12},E_1-E_2}. \tag{3.7}$$

The other SO(9) subgroups' generators can be obtained by a cyclic permutation of 1, 2, 3.

Each SO(9) subgroup has an SO(8) subgroup that leaves the diagonal terms invariant. However, the 3 SO(8) subgroups' actions are not independent of each other. These 3 actions are related via the principle of triality [21]. The principle of triality states that given an action d_1 of SO(8) on the octonion algebra $\mathbb{O}$ then there exist actions d_2 and d_3, unique up to a sign, which satisfy the property

$$(d_1 O_1)(d_2 O_2)=\overline{d_3(\overline{O_1 O_2})}\,\forall O_1, O_2\in\mathbb{O} \tag{3.8}$$

and there is a cyclic symmetry of the actions d_1, d_2, d_3. The global action of the SO(8) subgroup on J_3^8 can then be written as:

$$\mathrm{SO}(8): J \to \begin{pmatrix} \alpha_1 & (d_3 O_3) & \overline{(d_2 O_2)} \\ \overline{(d_3 O_3)} & \alpha_2 & (d_1 O_1) \\ (d_2 O_2) & \overline{(d_1 O_1)} & \alpha_3 \end{pmatrix}. \tag{3.9}$$

The infinitesimal generators of SO(8) satisfy the principle of local triality (sometimes called infinitesimal triality):

$$(D_1 O_1) O_2 + O_1 (D_2 O_2) = \overline{D_3 (O_1 O_2)}, \tag{3.10}$$

where D_i belongs to the Lie algebra of SO(8). An example of triality action, which we shall use later, is given by

$$d_1 = L_a \quad d_2 = R_a \quad d_3 = L_{\bar{a}} R_{\bar{a}}, \tag{3.11}$$

where a is a unit octonion and L_a and R_a stand for multiplication from the left and the right by the octonion a.

We now prove four lemmas which we shall need in the following [22]. They all apply to one-dimensional projection operator P.

Lemma 1. *There exists always a transformation belonging to* F_4 *which brings* P *given by* (2.14) *to the form* E_1 *(see* (3.6)*).*

Lemma 2. *Given* P_1, P_2, P_3 *such that* $P_1 \circ P_2 = P_2 \circ P_3 = P_3 \circ P_1 = 0$ *then there exists always a transformation of* F_4 *which brings them to the form* E_1, E_2, E_3.

Lemma 3. *Given any two* P_1 *and* P_2 *then there exists always a transformation of* F_4 *which brings them into a real form.*

Lemma 4. $\mathrm{tr}\, P_1 \circ P_2 = 0$ *implies* $P_1 \circ P_2 = 0$. *For the proof of Lemma 1, we start with the general form* (2.14) *for* P. *Consider the* SO(9) *rotation*

$$\begin{aligned} R_{12}(\varphi) &= \exp(\varphi D_{(x)12, E_1 - E_2}) \\ x &= \frac{a\bar{b}}{[n(a\bar{b})]^{1/2}} \\ \sin\varphi &= \frac{2[n(a\bar{b})]^{1/2}}{n(a)+n(b)} \quad \cos\varphi = \frac{n(a)-n(b)}{n(a)+n(b)}. \end{aligned} \tag{3.12}$$

One then gets, using (3.3)

$$P' = R_{12} P = (n(a)+n(b)) E_1 + n(c) E_3 + \left(\frac{n(a)+n(b)}{n(a)}\right)^{1/2} (c\bar{a})_{31}. \tag{3.13}$$

Now, perform the SO(9) *rotation*

$$\begin{aligned} R_{31}(\psi) &= \exp(\psi D_{(y)31, E_3 - E_1}) \\ y &= \frac{c\bar{a}}{[n(c\bar{a})]^{1/2}} \\ \sin\psi &= -2[n(c)(n(a)+n(b))]^{1/2} / [n(a)+n(b)+n(c)]^{1/2} \\ \cos\psi &= -[n(c)-n(a)-n(b)] / [n(a)+n(b)+n(c)]^{1/2} \end{aligned} \tag{3.14}$$

The result is

$$P''=R_{31}P'=R_{31}R_{12}P=E_1\,. \tag{3.15}$$

With an appropriate permutation of indices and letters one could also get E_2 *and* E_3.

To prove Lemma 2 we first bring P_1 to the form E_1 leaving P_2 in the general form (2.14). Then

$$E_1\circ P_2=a\bar{a}E_1+\tfrac{1}{2}(a\bar{b})_{12}+\tfrac{1}{2}(c\bar{a})_{31}\,. \tag{3.16}$$

$E_1\circ P_2=0$ implies $a=0$, hence the most general form for P_2 is $P_2=b\bar{b}E_2+c\bar{c}E_3+(b\bar{c})_{23}$ with $b\bar{b}+c\bar{c}=1$. The transformation $R_{23}(\varphi)$ obtained from (3.12) by the cyclic permutation $1\to2\to3$ and $a\to b\to c$ brings P_2 to the form E_2, while leaving E_3 invariant. It is then obvious that the only P_3 that satisfies $P_3\circ E_1=P_3\circ E_2=0$ is $P_3=E_3$.

For Lemma 3 we bring P_1 to the form E_3 with P_2 given by (2.14). The SO(9) rotation $R_{12}(\varphi)$ given by (3.12) brings P_2 to the form (3.13). An SO(8) rotation via the triality action given by (3.11):

$$d_1=L_O\qquad d_2=R_O\qquad d_3=L_OR_O$$

with O chosen as $\dfrac{a\bar{c}}{|n(a\bar{c})|^{1/2}}$ brings P_2 into a real form.

For Lemma 4, first consider an F_4 transformation that brings P_1 into the form E_1 with P_2 being of the general form (2.14). Then from (3.16) we have

$$\mathrm{tr}\,E_1\circ P_2=a\bar{a}$$

therefore $\mathrm{tr}\,E_1\circ P_2=0$ implies $a=0$ and thus $E_1\circ P_2=0$.

Lemma 5. *Any element of* J_3^8 *can be brought to the diagonal form by an* F_4 *transformation.*

Consider a general element X of J_3^8 as given by (3.6) then by the triality action

$$d_1=L_{\frac{x_3}{|x_3|}}R_{\frac{x_3}{|x_3|}}\qquad d_2=L_{\frac{\bar{x}_3}{|x_3|}}\qquad d_3=R_{\frac{\bar{x}_3}{|x_3|}}\,.$$

it is brought to the form

$$X_1=\begin{pmatrix}\alpha_1 & |x_3| & \bar{x}_2'\\ |x_3| & \alpha_2 & x_1'\\ x_2' & \bar{x}_1' & \alpha_3\end{pmatrix}.$$

Then by a simple real rotation this can be brought to the form

$$X_2=\begin{pmatrix}\alpha_1' & 0 & \bar{x}_2''\\ 0 & \alpha_2 & x_1''\\ x_2'' & x_1'' & \alpha_3\end{pmatrix}$$

Again via the triality action this can be brought to the form

$$X_3=\begin{pmatrix}\alpha_1' & 0 & \bar{x}_2'''\\ 0 & \alpha_2'' & |x_1''|\\ x_2''' & |x_1''| & \alpha_3'\end{pmatrix}$$

Now under the rotation

$$TX_3T^{-1}=X_4,$$

where $T=\begin{pmatrix}1 & 0 & 0\\ 0 & & \\ 0 & & \mathrm{R}\end{pmatrix}$, with R being the real rotation matrix that diagonalizes $\begin{pmatrix}\alpha_2'' & |x_1''|\\ |x_1''| & \alpha_3'\end{pmatrix}$, we find

$$X_4=\begin{pmatrix}\alpha_1' & -\sin\alpha\bar{x}_2''' & \cos\alpha\bar{x}_2'''\\ -\sin\alpha x_2''' & \alpha_2''' & 0\\ \cos\alpha x_2''' & 0 & \alpha_3''\end{pmatrix}.$$

Then by the triality action

$$d_3=R_{\frac{x_2'''}{|x_2'''|}}\qquad d_2=L_{\frac{\bar{x}_2'''}{|x_2'''|}}\qquad d_1=L_{\frac{\bar{x}_2'''}{|x_2'''|}}R_{\frac{x_2'''}{|x_2'''|}}.$$

X_4 is brought to a real symmetric matrix which can always be diagonalized by a real rotation matrix. QED.

4. The Moufang Projective Plane

The representation theorem for projective spaces states that projective spaces of dimension $n>2$ can be represented by vector spaces over a skew field [23]. If the underlying division ring is commutative then Pappus-Pascal-theorem holds. The Desargues' theorem follows from the projective axioms for $n>2$.

One example of a non-Desarguian projective plane ($n=2$) was given by Moufang [16]. She gave an affine coordination of this geometry in terms of octonions and showed that the harmonic lock incidence theorem (which is a weaker theorem than Desargues' theorem) is satisfied as a consequence of the alternativity of the underlying octonion algebra.

Jordan gave a construction of the Moufang plane in terms of the idempotents of J_3^8 [17]. The first rigorous study of the Moufang plane and its automorphism group was given by Freudenthal [24]. Below we shall use Jordan's method in our study of the Moufang plane and follow the work of Freudenthal [20], Springer [25], and the excellent review of Jacobson [3].

A projective plane is defined as a set of elements called points and a collection of subsets of points called lines such that the three following axioms are satisfied.

$G=$ Moufang plane

1) Any two distinct points are contained in one and only one line.
2) The intersection of any two distinct lines is one point.
3) There exist four points no three of which are in the same line:

as an immediate consequence, we can prove

4) Given three distinct lines which intersect in three points, the line defined by two other points of two of these lines intersects the third one.
5) Any line contains at least three points.

If we take 1), 4), and 5) as axioms we define an irreducible projective geometry. As a matter of fact, by 2) such a projective geometry is a plane. A linear variety is a subset of points such that, each time it contains two points, it contains the line defined by this two points. Here, for the projective plane the linear varieties are the void set, the points, the lines and plane itself.

Up to a real factor, one-dimensional projection operators (or irreducible idempotents) are given by the condition

$$P \times P = 0 . \tag{4.1}$$

As points in the plane we shall take the equivalence class P satisfying (4.1) and as a representative of this class take a P satisfying the trace condition

$$\mathrm{tr} P = 1 . \tag{4.2}$$

A line in this plane will be represented by a two-dimensional projection operator l

$$l^2 = l; \quad trl = 2 \tag{4.3}$$

A point P is said to be contained in the line l if and only if

$$P \circ l = P \quad \text{or} \quad P \circ (I - l) = 0 . \tag{4.4}$$

To prove that axioms 1, 2, and 3 are satisfied, one first needs to show that for P_1 and P_2 different one-dimensional projection operators,

$$L_{12} = \frac{P_1 \times P_2}{\mathrm{tr} P_1 \times P_2} \tag{4.5}$$

is a one-dimensional projection operator. This can be simply done by considering the case $P_1 = E_1$ and P_2 general and extending it to the general case. Or else, one can use the identity

$$\begin{aligned} &2(J_1 \times J_1) \times (J_2 \times J_2) + 4(J_1 \times J_2) \times (J_1 \times J_2) \\ &\quad = (J_1 \times J_1, J_2) J_2 + (J_1, J_2 \times J_2) J_1 . \end{aligned} \tag{4.6}$$

It then follows that

$$l_{12} = I - L_{12} = I - \frac{P_1 \times P_2}{\mathrm{tr} P_1 \times P_2} \tag{4.7}$$

is a two-dimensional projection operator and hence a line. The points P_1 and P_2 lay on the line l_{12} since

$$(P_1, L_{12}) = \frac{1}{\mathrm{tr} P_1 \times P_2} (P_1, P_1 \times P_2) = \frac{1}{\mathrm{tr} P_1 \times P_2} (P_1 \times P_1, P_2) = 0$$

and

$$(P_2, L_{12}) = \frac{1}{\mathrm{tr} P_1 \times P_2} (P_2, P_1 \times P_2) = \frac{1}{\mathrm{tr} P_1 \times P_2} (P_2 \times P_2, P_1) = 0 .$$

To show that l_{12} is the only line through P_1 and P_2, one needs:

Lemma 6. *If P_1 and P_2 are different one-dimensional projectors then the projector P_3 satisfies $P_1 \circ P_3 = P_2 \circ P_3 = 0$ if and only if P_3 is a multiple of $P_1 \times P_2$.*

It is enough to prove the lemma for $P_3 = E_1$, P_1 and P_2 being of the general form (2.14). Then $P_i \circ P_3 = 0$ $(i = 1, 2)$ implies

$$P_i = b_i \bar{b}_i E_2 + c_i \bar{c}_i E_3 + (b_i \bar{c}_i)_{23} .$$

Direct computation gives $P_1 \times P_2$ proportional to E_1. The converse has already been proven above. Therefore, axiom 1 is satisfied.

The intersection of two lines $I - P_1$ and $I - P_2$ is the point

$$P_{12} = \frac{P_1 \times P_2}{\mathrm{tr} P_1 \times P_2} \tag{4.8}$$

which, by the same arguments, can be shown to be unique. Hence, axiom 2 is satisfied. To prove axiom 3, consider the following four points

$$P_i = E_i \qquad i = 1, 2, 3$$

with P_4 being given by the general formula (2.14). The line l_K passing through points E_i and E_j is given by

$$\frac{E_i \times E_j}{\mathrm{tr} E_i \times E_j} = E_k = I - l_k ; i, j, k \qquad \text{in cyclic order} . \tag{4.9}$$

The points E_k and P_4 do not lay on the line l_k since

$$E_k \circ E_k \neq 0 ; \qquad E_k \circ P_4 \neq 0 .$$

Thus, there exist four points, no three of which lay on the same line. Such a projective plane constructed with Jordan matrices is called a Moufang plane.

Let us now show that this projective plane is effectively non-Desarguian. First let us recall that Desargues' theorem states that given two triangles ABC and $A'B'C'$ such that the lines AA', BB', and CC' intersect at a common point 0, then the intersection points of the lines AB and $A'B'$, BC and $B'C'$, AC and $A'C'$ all lay on a straight line.

We now give an explicit non-Desarguian configuration. For this consider the three lines

$$l_3 = I - E_3 ; \qquad l_2 = I - E_2 ; \qquad l_\theta = I - \theta_{23}$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} .$$

These three lines intersect at the point represented by the idempotent E_1 since

$$E_2 \times E_3 = \frac{1}{2} E_1$$

$$E_2 \times \theta_{23} = \frac{\cos^2\theta}{2} E_1$$

$$E_3 \times \theta_{23} = \frac{\sin^2\theta}{2} E_1 .$$

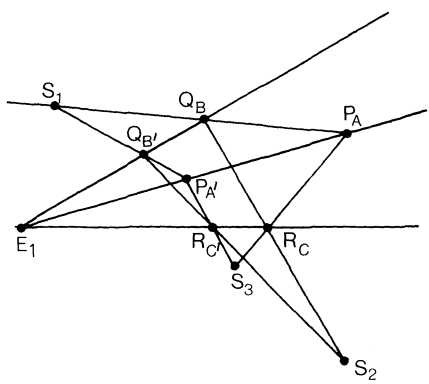
Now take two points on each line

$$P_A, P_{A'} \in l_3$$

$$Q_B, Q_{B'} \in l_2$$

$$R_C, R_{C'} \in l_\theta .$$

Graphically, we have



The line passing through points P_A and Q_B is represented by $I - L_{PQ}$; $L_{PQ} = P_A \times Q_B / \mathrm{tr}\, P_A \times Q_B$ and intersection point of the lines l_{PQ} and $l_{P'Q'}$ is the point (up to a trace factor)

$$S_1 = (P_A \times Q_B) \times (P_{A'} \times Q_{B'}) .$$

Similarly for the points B, C, B', C' and A, C, A', C' one finds

$$S_2 = (Q_B \times R_C) \times (Q_{B'} \times R_{C'})$$

$$S_3 = (P_A \times R_C) \times (P_{A'} \times R_{C'}) .$$

Now Desargues' theorem would require that the points S_1, S_2, and S_3 lay on a straight line, which in our notation is equivalent to the condition

$$(S_1 \times S_2, S_3) = [S_1, S_2, S_3] = 0 .$$

Let us now show that this condition does not in general hold for this Moufang plane. As the points in our plane consider the idempotents

$$P_A=\frac{1}{2}\begin{pmatrix}1 & -e_2 & 0\\ e_2 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad P_{A'}=\frac{1}{2}\begin{pmatrix}1 & -e_1 & 0\\ e_1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$Q_B=\frac{1}{2}\begin{pmatrix}1 & 0 & e_7\\ 0 & 0 & 0\\ -e_7 & 0 & 1\end{pmatrix} \quad Q_{B'}=\frac{1}{2}\begin{pmatrix}1 & 0 & e_3\\ 0 & 0 & 0\\ -e_3 & 0 & 1\end{pmatrix}$$

$$R_C=\frac{1}{2}\begin{pmatrix}1 & -\cos\theta e_6 & -\sin\theta e_6\\ \cos\theta e_6 & \cos^2\theta & \cos\theta\sin\theta\\ \sin\theta e_6 & \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}$$

$$R_{C'}=\frac{1}{2}\begin{pmatrix}1 & -\cos\theta e_5 & -\sin\theta e_5\\ \cos\theta e_5 & \cos^2\theta & \cos\theta\sin\theta\\ \sin\theta e_5 & \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}.$$

After some long and tedious calculation, one finds that

$$(S_1\times S_2, S_3)=\frac{1}{2^{19}}\cos^2\theta\sin^2\theta(3+\sin\theta-\cos\theta+\sin\theta\cos\theta).$$

This shows the existence of non-Desargian configurations in the Moufang plane.

5. Orthocomplementation of the Moufang Plane

A projective plane is orthocomplemented if on it there exists an involution on the linear varieties which reverses the order of inclusion and maps each point to a line outside this point. Existence of such involutions over the Moufang plane has long been established [26] and thoroughly studied [27]. The simplest such involution is to associate with each projector P the projector

$$P'=I-P. \tag{5.1}$$

If P is one dimensional ($\mathrm{tr}\,P=1$), then $I-P$ is of dimension 2 and represents a line. This involution reverses the order of inclusion since if $P\circ l=P$ then

$$P'\circ l'=(I-P)\circ(I-l)=I-l-P+P=l'.$$

A linear variety is by definition orthogonal to another if the orthocomplement of one contains the other. Two points P_1 and P_2 are orthogonal if and only if $P_1\circ P_2=0$

$$P_1\circ P_2=0\Leftrightarrow P_1\perp P_2. \tag{5.2}$$

In this case

$$P_1\times P_2=\frac{1}{2}(I-P_1-P_2) \tag{5.3}$$

and

$$l_{12}=P_1+P_2. \tag{5.4}$$

The third point orthogonal to P_1 and P_2 is just $(I-P_1-P_2)$, which is the orthocomplement of the line l_{12}. We remark that for any orthogonal linear varieties l_1 and l_2 we have the simple formula for their union

$$l_1 \cup l_2 = l_1 + l_2. \tag{5.5}$$

Having now at our disposal an orthocomplemented projective geometry, we can construct a quantum mechanical system of propositions, satisfying all axioms, as shown in Ref. [15]. As in [15], two propositions l_1 and l_2 are compatible if and only if $(l_1 \cap l_2) \cup (l_1 \cap l_2') = l_1$ i.e. if l_1 can be written as the sum of two projectors, one in l_2 and one in l_2'

$$\begin{aligned} l_1 = P_1 + P_2 \quad \text{with} \quad & P_1 \circ l_2 = P_1 \\ & P_2 \circ l_2 = 0. \end{aligned} \tag{5.6}$$

In §6 we show the existence of a unique probability function and in §7 the theory of measurements and observables is discussed.

The automorphism group of the orthocomplemented Moufang plane is again F_4 [3]. Indeed, since F_4 preserves the Jordan product and the Freudenthal product, F_4 transforms points into points, lines into lines and orthogonal linear varieties into orthogonal linear varieties.

6. The Probability Function

We now show that given any one-dimensional projector P, there exists a unique probability function

$$W_p(l) = \mathrm{tr}(P \circ l) \qquad l \in G \tag{6.1}$$

satisfying Gleason's axioms [28]

$$1) \quad 0 \leqq W_p(l) \leqq 1 \tag{6.2}$$

$$2) \quad W_p(P) = 1 \tag{6.3}$$

$$\begin{aligned} 3) \quad & W_p(l_1 \vee l_2) = W_p(l_1) + W_p(l_2) \\ & l_1 \perp l_2 \in G. \end{aligned} \tag{6.4}$$

To prove (6.2), we first choose for l a point Q. Since $\mathrm{tr} P \circ Q$ is invariant under F_4, we consider P in the form E_1 and Q general given by (2.14). Then $\mathrm{tr} P \circ Q = |a|^2 \leqq 1$. Next, for $l = I - Q$, $\mathrm{tr} P \circ l = 1 - |a|^2 \geqq 0$.

To prove (6.4), we remark that according to (5.5), $W_p(l_1 \cup l_2) = \mathrm{tr}(P \circ (l_1 + l_2)) = \mathrm{tr}(P l_1) + \mathrm{tr}(P l_2) = W_p(l_1) + W_p(l_2)$. On the other hand, such a function is unique i.e. if a real function on G satisfies (6.2)–(6.4) it is identical with the one given by the trace. To prove this, let us first make two remarks

1) Due to the condition (6.4) the function $W_p(l)$ is completely defined by its values on the points.

2) The restriction of W_p to the real subplane defined by the real 3×3 projection operators is a state on this plane and by Gleason theorem it is unique and given by the trace. This is indeed the case since the orthogonality condition $P\circ Q=0$ (Jordan product) restricted to real 3×3 projectors is identical with $PQ=0$ (ordinary matrix product).

Suppose there exists a point Q such that $W_p(Q)$ is different from $\mathrm{tr}(PQ)$. Lemma 3 of §3 shows that by an automorphism of F_4, we can choose P and Q real. Then by the second remark, we will have a contradiction and by the first remark the uniqueness is proven.

7. Measurements and Observables

We consider here ideal measurements of the first kind [15] which are simply called measurements in usual text-books. We show that the result of two successive, compatible experiments does not depend on the order in which they are performed, in spite of the non-associativity of octonions.

We first consider a physical system in the state P (one-dimensional projection operator) and we measure the proposition l (yes-no experiment). Suppose the result of the experiment is the answer yes (the value one). What is then the final state after the measurement? The problem is only non-trivial if l is a line. According to reference [29], the final state is the point

$$P_f=(P\cup l')\cap l, \tag{7.1}$$

where l', the orthocomplement of l, is a point. $P\cup l'$ is the line

$$\hat{l}=I-\frac{P\times l'}{\mathrm{tr}P\times l'}=I-\hat{L}. \tag{7.2}$$

Then the intersection of the lines l and $\hat{l}$ is the point $l'\times\hat{L}$. Normalizing, we get the final state

$$P_f=\frac{l'\times\hat{L}}{\mathrm{tr}(l'\times\hat{L})}=\frac{l'\times(P\times l')}{\mathrm{tr}(l'\times(P\times l'))}. \tag{7.3}$$

Using Equation (2.16) and the fact that the free Jordan algebra with two generators (here P and l') is special, i.e. equivalent to a matrix algebra over an associative field with Jordan product [4], one finds

$$\mathrm{tr}((P\circ l')\circ l')=\mathrm{tr}P\circ l', \tag{7.4}$$

$$\mathrm{tr}((P\times l')\times l')=\frac{1}{4}(1-\mathrm{tr}P\circ l'), \tag{7.5}$$

$$(P\times l')\times l'=(P\circ l')(l'-I)+\frac{1}{4}(P-l'\,\mathrm{tr}P\circ l'). \tag{7.6}$$

The probability to get the answer yes if one measures the proposition l on a system in state P is

$$W(l)=\mathrm{tr}P\circ l. \tag{7.7}$$

Suppose now we measure l_1 on the initial state P_1 and get the state P_{l_1}. Then we measure the compatible [see Eq. (5.6)] observable l_2. The probability to get the answer yes for l_1 and yes for l_2 is $\mathrm{tr}(P_{l_1} \circ l_2)\,\mathrm{tr}(P \circ l_1)$. Therefore, we must have

$$\mathrm{tr}(P_{l_1} \circ l_2)\,\mathrm{tr}(P \circ l_1) = \mathrm{tr}(P_{l_2} \circ l_1)\,\mathrm{tr}(P \circ l_2). \tag{7.8}$$

From Equation (7.3) we have

$$P_{l_1} = 4(1 - \mathrm{tr}\, P l_1')^{-1}(P \times l_1') \times l_1',$$

where $l_1' = I - l_1$.

Then

$$\begin{aligned}\mathrm{tr}\, P_{l_1} \circ l_2 &= 4(1 - \mathrm{tr}\, P l_1')^{-1}\,\mathrm{tr}[((P \times l_1') \times l_1') \circ l_2] \\ &= 1 - 4(1 - \mathrm{tr}\, P l_1')^{-1}\,\mathrm{tr}[(P \times l_1') \circ (l_1' \times l_2')] \\ &= 1 - 4(1 - \mathrm{tr}\, P l_1')^{-1}\,\mathrm{tr}[P \circ (l_1' \times (l_1' \times l_2'))].\end{aligned}$$

The compatibility of l_1 and l_2 implies $l_1' \circ l_2' = 0$, hence

$$l_1' \times (l_1' \times l_2') = \frac{1}{4} l_2'$$

$$\mathrm{tr}\, P \circ (l_1' \times (l_1' \times l_2')) = \frac{1}{4}\mathrm{tr}\, P \circ l_2'.$$

Thus, we find

$$\begin{aligned}\mathrm{tr}\, P_{l_1} \circ l_2 &= 1 - (1 - \mathrm{tr}\, P \circ l_1')^{-1}\,\mathrm{tr}\, P \circ l_2' \\ &= (1 - \mathrm{tr}\, P \circ l_1')^{-1}(1 - \mathrm{tr}\, P \circ l_1' - \mathrm{tr}\, P \circ l_2').\end{aligned} \tag{7.9}$$

Since $\mathrm{tr}\,(P l_1) = \mathrm{tr}\, P \circ (I - l_1') = 1 - \mathrm{tr}\, P l_1'$, we get the result

$$\mathrm{tr}\,(P_{l_1} \circ l_2)\,\mathrm{tr}\,(P \circ l_1) = 1 - \mathrm{tr}\, P \circ l_1' - \mathrm{tr}\, P \circ l_2' = 2\,\mathrm{tr}\, P \circ (l_1' \times l_2'). \tag{7.10}$$

Since the right-hand-side is symmetric in l_1' and l_2', we have

$$\mathrm{tr}\,(P_{l_1} \circ l_2)\,\mathrm{tr}\,(P \circ l_1) = \mathrm{tr}\,(P_{l_2} \circ l_2)\,\mathrm{tr}\,(P \circ l_2) = 2\,\mathrm{tr}\, P \circ (l_1' \times l_2'). \tag{7.11}$$

Note that the point $2l_1' \times l_2'$ is just the intersection of the lines l_1 and l_2, as expected.

What (7.11) implies is that the probability of successive, compatible experiments is independent of the order of the measurements. This is a somewhat surprizing result in view of the nonassociative nature of the octonion product.

We now come to the question of observables [30]. First, each projection operator defines an observable. In general, in the Moufang plane, an observable is defined by three one-dimensional projections. They can always be brought by an F_4-transformation to the form E_1, E_2, and E_3 (Lemma 2). The question arises if any Jordan matrix can be written as a linear superposition of three mutually orthogonal one-dimensional projection operators and interpreted as an observable. The answer is yes, since any element of J_3^8 can be diagonalized by an F_4-transformation (Lemma 5).

In order to get a Schrödinger equation, we need a time evolution operator. This should transform orthogonal states into orthogonal states and hence belong

to the automorphism group of the orthocomplemented Moufang plane, which is F_4. Some generator in the Lie algebra of F_4 will play the role of the Hamiltonian.

The subalgebra of complex Jordan matrices can be used to construct a Desarguian projective subgeometry of the Moufang geometry. The corresponding Quantum Mechanics can be realized in a three-dimensional Hilbert space. The subgroup of F_4 which leaves the complex subgeometry invariant is $SU(3)^c$. Jordan matrices whose elements comprise the six remaining octonionic units transform as 3 and $\bar{3}$ under $SU(3)^c$. More precisely, under the maximal subgroup $SU(3)^c \times SU(3)$ the representation $\underline{26}$ of F_4 reduces as

$$\underline{26} = (3^c, 3) \oplus (\bar{3}^c, \bar{3}) \oplus (1^c, 8). \tag{7.12}$$

Hence, it is tempting to identify $SU(3)^c$ with the color group, complex Jordan matrices with lepton states [18, 31], and the other with quark and antiquark states (note that a projection operator belongs to the reducible representation $27 = 26 \oplus 1$). Since a non-Desarguian projective plane cannot be embedded in a irreducible projective geometry of higher dimension, this would mean that quarks have no space properties, these latter requiring an infinite dimensional geometry. On the other hand, the lepton subgeometry can hopefully be embedded in a consistent way in such an infinite geometry. However, the definition of a consistent tensor product of quark states is still an open problem.

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Communicated by R. Haag

Received December 20, 1977