

Quantum Theory in de-Sitter Space*

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Abstract. A general method for constructing fields in spaces with transitive group of transformations is presented. Quantum-theory of free fields with spin 0, 1/2, and the connection of spin and statistics in de-Sitter space of constant positive curvature are discussed.

1. Introduction

Usually Quantum-field-theory is formulated in Minkowski-space. To the group of motions, the Poincare-group, correspond the fundamental conservation laws of energy-momentum and angular momentum. We study quantum-theory in de-Sitter space of positive curvature $(4 + 1)$, whose group of motions has 10 parameters as the Poincare-group. Classical fields with spin $\neq 0$, especially spin 1/2 have been discussed by DIRAC [1] and later for inst. by LEE and GÜRSEY [2]. For their approach it is essential to embed de-Sitter space in a flat 5-dim. space, moreover for spin 1/2 the accidental existence of a 4-dim. representation of the de-Sitter group is used. The method of covariant derivatives gives equations for arbitrary spin, but as pointed out in [2], it remains unclear under what representation of the de-Sitter group these equations shall be invariant.

We propose a general method for constructing fields in spaces with transitive transformation group, using a field theoretic version of the so called "induced representations". We apply this formalism to quantum theory of free fields with spin 0, 1/2, restmass $m > 0$, in de-Sitter space. The connection of spin and statistics in de-Sitter space is discussed following W. PAULI [3] for the flat space.

We have proved the cited theorems rigorously. For brevity the proofs are only sketched in an appendix.

Notation * conj. complex.
+ hermitian conj.
T transposed.

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Indices: μ, ν take on values $0, \dots, 4$, $i, j: 0, \dots, 3$, $\alpha, \beta: 1, 2, 3$.
 Units: $\hbar = c = 1$, R : Radius of de-Sitter space.

2. Field Theory in Spaces with Transitive Group of Transformations

Let M be a topological space, G a Liegroup acting as continuous group of transformations on M :

$$\begin{aligned} G \times M &\rightarrow M \\ (g, x) &\rightarrow g(x) \quad \text{continuous, } g \in G, x \in M. \end{aligned} \quad (2.1)$$

Choose $x_0 \in M$, denote by K the stable subgroup of x_0 , K is closed.

$$K = \{g \in G, g(x_0) = x_0\}. \quad (2.2)$$

M can be identified with the space of left cosets G/K :

$$M = G/K = \{g \cdot K, g \in G\}. \quad (2.3)$$

Suppose that parameters, continuous together with their partial derivatives up to the third order, are introduced in a neighbourhood U of $e \in G$.

Theorem 1. *Under the cited assumptions G and M can be made to differentiable manifolds so that the mappings a), b), c) are differentiable:*

- a) $G \rightarrow G \quad g \rightarrow g \cdot g_1$,
- b) $G \rightarrow G \quad g \rightarrow g_1 \cdot g$,
- c) $G \times M \rightarrow M \quad (g, x) \rightarrow g(x)$.

We can find a covering $\Delta = \{H\}$ of M with open coordinate neighbourhoods H and differentiable maps:

$$H \rightarrow G \quad x \rightarrow \underset{H}{g_x} \quad x \in H \in \Delta, \quad \underset{H}{g_x} \in G \quad (2.4)$$

so that $x = \underset{H}{g_x} \cdot K$ for all $x \in H$.

The proof of this theorem is based on theorem 62 § 44 of [4].

Def. Let $k \rightarrow T(k)$ be a differentiable representation of the group K in the linear space V . A field ψ to the representation T of K is defined by a family of differentiable maps:

$$\begin{aligned} \psi_H : H &\rightarrow V \quad H \in \Delta \\ x &\rightarrow \psi_H(x) \quad x \in H, \psi_H(x) \in V \end{aligned}$$

so that for any $H, H_1 \in \Delta$, $x \in H \cap H_1$, the condition is satisfied:

$$\psi_{H_1}(x) = T\left(\underset{H_1}{g_x^{-1}} \cdot \underset{H}{g_x}\right) \psi_H(x). \quad (2.5)$$

By $F(T, M)$ we denote the linear space of these fields.

Theorem 2. For $g \in G$ we define a map $T_g : F(T) \rightarrow F(T)$

$$\begin{aligned} \psi &\rightarrow \psi' = T_g \cdot \psi \\ \psi'_H(x) &= T \left(g_x^{-1} g \cdot g_{x'} \right) \psi_{H'}(x') \\ x' &= g^{-1}(x), \quad x \in H, \quad x' \in H', \quad H, H' \in \Delta. \end{aligned} \tag{2.6}$$

We have: $g_x^{-1} g \cdot g_{x'} \in K$

$$x_0 \xrightarrow{g_{x'}} x' = g^{-1}(x) \xrightarrow{g} x \xrightarrow{g_x^{-1}} x_0.$$

T_g is a representation of G .

$$T g_1 \cdot T g_2 = T g_1 \cdot g_2. \tag{2.7}$$

Let $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^s)$ be differentiable parameters in a neighbourhood U of $e \in G$, so that

$$K \cap U = \{(0, y)\}. \tag{2.8}$$

Take $g \rightarrow (x, y) \in U$, g_1 arbitrary in G , then $g' = g_1 g g_1^{-1} \rightarrow (x', y')$ will be in U for g sufficiently near to e . $g \rightarrow g'$ is differentiable (theorem 1). We expand in a Taylor-series

$$\begin{aligned} x'^i &= l^i_j(g_1) x^j + p^{i\beta}(g_1) y^\beta + 2^{\text{nd}} \text{ order} \\ y'^\alpha &= q^\alpha_j(g_1) x^j + f^\alpha_\beta(g_1) y^\beta + 2^{\text{nd}} \text{ order} \end{aligned} \tag{2.9}$$

$$g_1 \rightarrow F(g_1) = \left\| \begin{array}{cc} l(g_1) & p(g_1) \\ q(g_1) & f(g_1) \end{array} \right\| \tag{2.10}$$

is the adjoint representation of G .

For $g_1 = k \in K$;

$$p(k) = 0, \quad k \rightarrow f(k): \text{adjoint representation of } K. \tag{2.11}$$

Assumption. The adjoint representation of G , restricted to K splits up into the parts $f(k)$, $l(k)$.

Then by a linear transformation of variables we can achieve:

$$F(k) = \left\| \begin{array}{c} l(k) \\ f(k) \end{array} \right\| \tag{2.12}$$

Covariant Differential Forms

Denote by L_x the space of differential forms at the point $x \in M$. Take $g \in G$, $g(x) = x'$. g induces a map $g^* : L_{x'} \rightarrow L_x$

$$L_{x'} \ni \alpha = a_j dx'^j \rightarrow g^* \alpha = a_j \frac{\partial g^j(x)}{\partial x^k} dx^k \in L_x. \tag{2.13}$$

We have:

$$(g_1 \cdot g_2)^* = g_2^* \cdot g_1^* \quad (\text{see f. inst. [5]}). \tag{2.14}$$

From (2.8) we see that we can regard (x^1, \dots, x^n) as parameters on M in a neighbourhood $U^* = \{u \cdot K, u \in U\}$ of x_0 .

$$\begin{aligned}\omega^i_0 &= \omega^i(x_0) = dx^i \\ \omega^i_H(x) &= (g_x^{-1})^* \omega^i_0 \quad x \in H \\ \omega_H(x) &= (\omega_H^1(x), \dots, \omega_H^n(x)).\end{aligned}\tag{2.15}$$

The following relations can be verified:

$$\text{If } x \in H_1 \cap H, \text{ then } \omega_{H_1}(x) = l(g_{H_1}^{-1} g_x) \omega_H(x)\tag{2.16}$$

$l(k)$ as in (2.12),

$$\begin{aligned}g \in G, x \in H, x' = g^{-1}(x) \in H' \\ g^* \omega_H(x) &= l(g_x^{-1} \cdot g \cdot g_{x'}) \omega_{H'}(x').\end{aligned}\tag{2.17}$$

We call $\omega_H(x)$ a covariant basis of differential forms. An analogous basis for higher forms is constructed by taking products of the elements of ω .

Covariant Derivative

We define:

$$\begin{aligned}(\partial_j \psi)_H(x) &= \frac{\partial}{\partial x^j} T_{g_x g'^{-1} g_x^{-1}} \psi_H(x) \Big|_{g' = (x', 0) = e} \quad (j = 1, \dots, u) \\ (Q_\alpha \psi)_H(x) &= \frac{\partial}{\partial y^\alpha} T_{g_x g'^{-1} g_x^{-1}} \psi_H(x) \Big|_{g' = (0, y) = e} \quad (\alpha = 1, \dots, s)\end{aligned}\tag{2.18}$$

∂_j are differential operators of first order.

Theorem 3. $d: \psi_H(x) \rightarrow u_H(x) = \|\partial_j \psi_H(x)\|$ defines a map $F(T) \rightarrow F(l^{-1\top} \otimes T)$. That is: $\{u_H(x)\}$ defines a field, transforming with the representation $l^{-1\top} \otimes T$ of K . $l(k)$ as in (2.12). d satisfies:

$$d \cdot T_g = T_g \cdot d\tag{2.19}$$

where T_g denotes the representations of G in $F(T)$ and $F(l^{-1\top} \otimes T)$ respectively. Analogously:

$$\begin{aligned}\psi_H \rightarrow v_H(x) &= \|Q_\alpha \psi_H(x)\| \\ F(T) &\rightarrow F(f^{-1\top} \otimes T).\end{aligned}$$

Let us introduce:

$$\partial_{j_n} = \sum \partial_{i_1}(x) \dots \partial_{i_n}(x)\tag{2.20}$$

Σ : complete symmetrization of i_1, \dots, i_n . For a matrix a^i_j ($i, j = 1, \dots, n$), $a^{j_j'}$ denote the corresponding matrix elements in the completely symmetric part of the Kronecker-product $[a]^n$.

We are interested in differential operators $D: F(T) \rightarrow F(T)$, defined by:

$$(D\psi)_H(x) = \sum_{m=0}^n C_H^{i_1, \dots, i_m} \frac{\partial^m}{\partial x^{i_1}, \dots, \partial x^{i_m}} \psi_H(x) \quad (2.21)$$

x : parameters on H , $C_H(x)$: operators over V , differentiable in x .

Theorem 4. *Any differential operator (2.21) can be uniquely represented as*

$$(D\psi)_H(x) = \sum_{m=0}^n C_H^{J_m}(x) \partial_{J_m} \psi_H(x). \quad (2.22)$$

If D is an invariant differential operator

$$T_g \cdot D = D \cdot T_g \quad \text{for all } g \in G \quad (2.23)$$

then the C^J are constant operators, independent of H , satisfying the conditions:

$$T(k) \cdot C^J = \iota^J_{J'}(k) C^{J'} \cdot T(k) \quad \text{for all } k \in K. \quad (2.24)$$

Conclusion. The structure of invariant equations for arbitrary $T(k)$ (T corresponds to the spin) is completely determined by the stable subgroup K , and depends only on K . Analogous assertions are true for local Lagrange-densities etc.

3. de-Sitter Space, de-Sitter Group

A model for de-Sitter space is the 4-dimensional hyperboloid (3.1) in the 5-dimensional flat space with the induced metric:

$$M: x^\mu x^\nu \eta_{\mu\nu} = (x^0)^2 - (x^1)^2 - \dots - (x^4)^2 = -R^2$$

$$\eta = \left\| \begin{array}{cccc} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \\ & & & & -1 \end{array} \right\| \quad x = \left\| \begin{array}{c} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{array} \right\| \quad (3.1)$$

$$\text{group of motions: } SO_0(4, 1) = \{g, g^T \eta g = \eta\} = G \quad (3.2)$$

$$g = \left\| \begin{array}{cccc} g^0_0 & \dots & g^0_4 \\ \vdots & \ddots & \vdots \\ g^4_0 & \dots & g^4_4 \end{array} \right\|$$

$$g \in \text{component of the unity: } g^0_0 > 0 \quad \left\| \begin{array}{ccc} g^1_1 & \dots & g^1_4 \\ \vdots & \ddots & \vdots \\ g^4_1 & \dots & g^4_4 \end{array} \right\| > 0$$

(see f. inst. [6])

$$g: x^\mu \rightarrow g^\mu_\nu x^\nu = g(x). \quad (3.3)$$

Theorem 5. *To every point $x \in M$ and to every normed tetrad ω^i in x ($ds^2 = \omega^i \omega^j \eta_{ij}$) there exists one and only one basis of the Liealgebra of G :*

$$M_{ij}(x), P_j(x) \quad i, j = 0, \dots, 3. \quad (3.4)$$

so that $M_{i,j}$ generate the stability group $L(x)$ of x (homogenous Lorentz-group) and P_j the infinitesimal translations in the directions ω^i

$$\begin{aligned} (I + i a^j P_j): \omega^j &\rightarrow \omega^j + a^j \\ \left(I + \frac{i}{2} a^{jk} M_{jk}\right): \omega^j &\rightarrow \omega^j + a^{jk} \omega_k \end{aligned} \quad (3.5)$$

$M_{i,j}, P_j$ satisfy the commutation relations:

$$\begin{aligned} [M_{i,j}, M_{k,l}] &= -i(\eta_{il} M_{jk} + \eta_{jk} M_{il} - \eta_{ik} M_{jl} - \eta_{jl} M_{ik}) \\ [M_{i,j}, P_k] &= i(\eta_{ik} P_j - \eta_{jk} P_i) \\ [P_j, P_k] &= -\frac{i}{R^2} M_{jk}. \end{aligned} \quad (3.6)$$

Under a homogeneous Lorentztransformation $\Lambda \in L(x)$ P_j transforms like a 4-vector:

$$\Lambda^{-1} P_j \Lambda = (\Lambda^{-1})^k_j P_k. \quad (3.7)$$

The Casimiroperator of 2nd order is given by:

$$I_2 = P_j P^j - \frac{1}{2R^2} M_{ij} M^{ij}. \quad (3.8)$$

Parameters on M

We choose $x_0 = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ -R \end{pmatrix}$ and introduce the following parameters

$$\begin{cases} x^0 = \frac{R}{2} (\lambda - \lambda^{-1}) - \frac{1}{2R} y^2 \lambda^{-1} & -\infty < \lambda < \infty, \lambda \neq 0 \\ \mathbf{x} = \lambda^{-1} \mathbf{y} & \mathbf{y} = (y^1, y^2, y^3) \\ x^4 = -\frac{R}{2} (\lambda + \lambda^{-1}) + \frac{1}{2R} y^2 \lambda^{-1} & -\infty < y^\alpha < \infty \quad \alpha = 1, 2, 3. \end{cases} \quad (3.9)$$

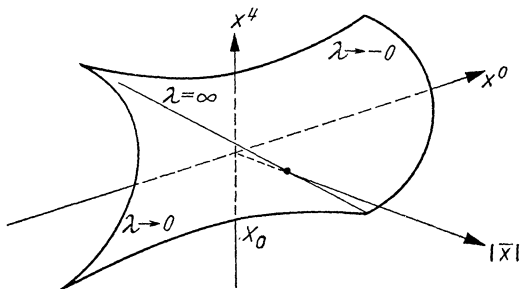


Fig. 1

Metric tensor:

$$g_{ij} = \left\| \begin{array}{ccc} \frac{R^2}{\lambda^2} & & \\ & -\frac{1}{\lambda^2} & \\ & & -\frac{1}{\lambda^2} \\ & & & -\frac{1}{\lambda^2} \end{array} \right\| \quad \sqrt{-g} = \frac{R}{\lambda^4} \quad (3.10)$$

λ, y are regular parameters in the open set $H: \lambda > 0$. A covering of M by open coordinate neighbourhoods is given by:

$$\Delta = \{g(H) \mid g \in G\}. \quad (3.11)$$

As parameters of $x \in g(H)$ we take $x' = g^{-1}(x) \rightarrow (\lambda, y) \in H$.

To apply our general theory we need elements g_x satisfying (2.4)

$$\text{for } H: \lambda > 0 \quad x \rightarrow (\lambda, y) \quad (3.12)$$

$$g_x = \left\| \begin{array}{ccc} 1 + \frac{1}{2} \frac{y^2}{R^2} & -\frac{y^\Gamma}{R} & \frac{y^2}{2R^2} \\ -\frac{y}{R} & I_3 & -\frac{y}{R} \\ -\frac{y^2}{R^2} & \frac{y^\Gamma}{R} & 1 - \frac{1}{2} \frac{y^2}{R^2} \end{array} \right\| \left\| \begin{array}{cc} \frac{\lambda^{-1} + \lambda}{2} & \frac{\lambda^{-1} - \lambda}{2} \\ & I_3 \\ \frac{\lambda^{-1} - \lambda}{2} & \frac{\lambda^{-1} + \lambda}{2} \end{array} \right\|$$

$$g_x: x_0 \rightarrow x.$$

The elements g_x form a subgroup of G .

For $H' = g(H)$, $x' \in H$, we set

$$g_{x'} = g \cdot g_x \quad x' = g(x) \quad x \in H. \quad (3.13)$$

Infinitesimal parameters on G in which the adjoint representation of G , restricted to $K = L(x_0)$ is completely reduced are:

$$a^{ij}, a^j \quad g = \left\| \begin{array}{cc} \delta^j + a^j & -\frac{a^i}{R} \\ \frac{a_j}{R} & 1 \end{array} \right\| \quad (3.14)$$

$$M_{ij} = \frac{1}{i} \frac{\partial g}{\partial a^{ij}}; \quad P_j = \frac{1}{i} \frac{\partial g}{\partial a^j} \quad (3.15)$$

This is the decomposition of the Liealgebra of G from theorem 5, corresponding to the point x_0 and the tetrad dx^j ($j = 0, \dots, 3$).

The covariant basis of 1-forms on M (2.15) is given by

$$\begin{aligned} & \text{for } H: \lambda > 0 \\ \omega^0(x) &= \frac{R}{\lambda} d\lambda \\ \omega^\alpha(x) &= \frac{1}{\lambda} dy^\alpha \quad \alpha = 1, 2, 3. \end{aligned} \quad (3.16)$$

Let $\mathcal{A} \rightarrow T(\mathcal{A})$ be a repr. of the homogenous Lorentzgroup with the generators: $\frac{\partial T(\mathcal{A})}{\partial a^{ij}} = L_{ij}$. The covariant derivative (2.18) in $F(T)$ is given by:

$$\begin{aligned} & \text{for } H: \lambda > 0 \\ \partial_0 &= \frac{\lambda}{R} \frac{\partial}{\partial \lambda}; \quad \partial_\alpha = \lambda \frac{\partial}{\partial y^\alpha} + \frac{1}{R} L_{0\alpha}. \end{aligned} \quad (3.17)$$

4. Scalar Particles in de-Sitter Space

a) Classical Solution

$$\text{Representation: } \varphi(x) \text{ real functions on } M \quad (4.1)$$

$$T_g \varphi(x) = \varphi(g^{-1}(x))$$

$$\text{Lagrange-density } \mathcal{L} = \frac{1}{2} (\partial_i \varphi \partial^i \varphi - m^2 \varphi^2) \quad (4.2)$$

$$\text{Action-function } \mathcal{S} = \int \mathcal{L} [\omega^0 \omega^1 \omega^2 \omega^3]$$

$$\text{field equations: } \delta \mathcal{S} = 0 \quad (4.3)$$

$$\partial_i \partial^i \varphi + m^2 \varphi = 0$$

$$\left(\frac{\lambda^2}{R^2} \frac{\partial^2}{\partial \lambda^2} - \frac{2\lambda}{R^2} \frac{\partial}{\partial \lambda} - \lambda^2 \Delta + m^2 \right) \varphi = 0; \quad \Delta = \frac{\partial^2}{\partial y^\alpha \partial y^\alpha}. \quad (4.3)$$

Solution for

$$\begin{aligned} m > \frac{3}{2R}; \quad \varrho = +\sqrt{R^2 m^2 - 9/4}; \quad \omega^2 = \mathbf{k}^2 + (\varrho/R)^2 \\ & \text{for } H: \lambda > 0 \end{aligned} \quad (4.4)$$

$$\varphi(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} (K_+ a^*(\mathbf{k}) + K_- a(\mathbf{k}))$$

$$\varphi^\pm(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} K_\pm(x; \mathbf{k}) \begin{pmatrix} a^*(\mathbf{k}) \\ a(\mathbf{k}) \end{pmatrix}$$

$$\begin{aligned} K_+(x; \mathbf{k}) &= e^{i\mathbf{k} \cdot \mathbf{y}} \lambda^{3/2} \left(\frac{|\mathbf{k}|}{\omega - m} \right)^{i\varrho} e^{-i\omega R} e^{i\pi/4} \times \\ & \times \sqrt{\frac{R\pi\omega}{2}} e^{-\varrho\pi/2} H_{i\varrho}^1(|k| \lambda R) \end{aligned}$$

$$K_- = K_+^*; \quad H^{1,2}: \text{Hankelfunctions.}$$

Every solution of (4.3) which is sufficiently regular can be represented for $x \in H$ by (4.4). Namely, let $\varphi(x)$ be such a solution, then

$$f(\lambda, \mathbf{k}) = \int e^{i\mathbf{k} \cdot \mathbf{y}} \varphi(\lambda, \mathbf{y}) \frac{d^3 y}{(2\pi)^{3/2}}$$

will satisfy an ordinary differential equation of second order in λ , for which (4.4) gives a fundamental system.

Lemma 1. *If $g \in G$, $x, g(x) \in H$, then*

$$\varphi^-(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} K_-(g(x); k) T_g a(k) \quad (4.5)$$

where T_g on the right side denotes a unitary representation of G in the space of square integrable functions $a(k)$. The infinitesimal operators of this

representation are given below, (4.13). Analogous assertions hold for φ^+ , φ .

For $H' = g^{-1}(H)$ we set:

$$\varphi_H^-(x) = \int K_-(g(x); \mathbf{k}) T_g a(\mathbf{k}) \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \quad (4.6)$$

analogously for φ^+ , φ .

Using lemma 1 and the invariance of the field equations it can be verified that φ satisfies the field equations everywhere and that condition (2.5) is fulfilled.

Asymptotically for $R \rightarrow \infty$, $x \rightarrow x_0$, we have:

$$K_{\pm} \rightarrow e^{\pm i(\omega x^0 + \mathbf{k} \cdot \mathbf{y})}. \quad (4.7)$$

From (4.6) it follows that φ^{\pm} are prop. to $e^{\pm i\omega t}$ for arbitrary x (t : local proper time). Therefore φ^{\pm} must be regarded as analogues of the pos. and neg. frequency part. (We follow [7] in denoting pos. and neg. frequency part.)

Theorem 6. φ^{\pm} respectively a^* , a transform with equivalent repr. of the de-Sitter group. This is in contrast to the Poincare group, where φ^{\pm} transform with repr. distinguished by $\text{sgn } P_0$.

Normalization.

$$\begin{aligned} N(\varphi, \psi) &= i \int (\varphi \partial^j \psi - \partial^j \varphi \psi) \sigma_j \\ &= i \int_{\lambda = \text{const}} (\varphi \partial^0 \psi - \partial^0 \varphi \cdot \psi) [\omega^1 \omega^2 \omega^3] \quad (4.8) \end{aligned}$$

$$\sigma_0 = [\omega^1 \omega^2 \omega^3] \text{ etc.}$$

$$N(\varphi^+, \varphi^-) = \int d^3 k a^*(k) a(k). \quad (4.9)$$

Noether-theorems.

$$I_\nu \varphi(x) = \frac{\partial}{\partial g^\nu} T_g \varphi(x)|_{g=e} \quad \text{infin. operators} \quad (4.10)$$

$$g^\nu: \text{inf. parameters on } G \quad (g^\nu) = (a^{ij}, a^j)$$

$$\omega^i(g(x))|_{g=e} = \xi_\nu^i(x) dg^\nu \quad \text{Killing vectors}. \quad (4.11)$$

$$\bar{P}_j = - \int \left(\frac{\partial \mathcal{L}}{\partial \partial_i \varphi} I_\nu \varphi + \mathcal{L} \xi_\nu^i \right) \sigma_j = - \frac{1}{2} N(\varphi, P_j \varphi) \quad (4.12)$$

analogously for \bar{M}_{ij} .

In momentum space: (assuming commutativity of a^* , a)

$$\bar{P}_0 = \int d^3 k a^*(\mathbf{k}) \left(\omega - \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) a(\mathbf{k}) \quad (4.13)$$

$$\begin{aligned} \bar{P}_\alpha = \int d^3 k a^*(\mathbf{k}) &\left(k_\alpha \left(1 - \frac{i}{2R\omega} \right) + \left(\frac{i\omega}{R} + \frac{3}{2R^2} \right) \frac{\partial}{\partial k^\alpha} + \right. \\ &\left. + \frac{k^\beta}{R^2} \frac{\partial^2}{\partial k^\beta \partial k^\alpha} + \frac{k_\alpha}{2R^2} \frac{\partial^2}{\partial k^\beta \partial k^\beta} \right) a(\mathbf{k}) \end{aligned}$$

$$\bar{M}_{\alpha\beta} = \int d^3 k a^*(\mathbf{k}) \frac{1}{i} \left(k_\alpha \frac{\partial}{\partial k^\beta} - k_\beta \frac{\partial}{\partial k^\alpha} \right) a(\mathbf{k})$$

$$\begin{aligned} \bar{M}_{0\alpha} = \int d^3 k a^*(\mathbf{k}) &\left(\frac{ik_\alpha}{2\omega} - \left(i\omega + \frac{3}{2R} \right) \frac{\partial}{\partial k^\alpha} - \right. \\ &\left. - \frac{k^\beta}{R} \frac{\partial^2}{\partial k^\beta \partial k^\alpha} - \frac{k_\alpha}{2R} \frac{\partial^2}{\partial k^\beta \partial k^\beta} \right) a(\mathbf{k}). \end{aligned}$$

b) Quantization

Assumption. Let the space of physical states be a Hilbert-space \mathcal{H} with unitary representation of the de-Sitter group: T_g , $\varphi(x)$ operator distributions over \mathcal{H} so that:

$$\begin{aligned} \langle | \varphi(x) | \rangle &\in F(T) \quad \text{for a dense set of states } | \rangle & (4.14) \\ \langle | T_g^{-1} \varphi(x) T_g | \rangle &= T_g \langle | \varphi(x) | \rangle. \end{aligned}$$

The existence of such a space is assured by the Fock-space in momentum repr.

Assumption. $[a(k), a(k')]_{\pm} = [a^+(k), a^+(k')]_{\pm} = 0$ that is: either commutativity or anticommutativity is assumed. $[a(k), a^+(k')]_{\pm} = c(k, k')$ scalar distribution.

We require invariance under the transformations (of the component of the unity) of the de-Sitter group. Then, in view of the irreducibility of the representation of the de-Sitter group in the 1-particle subspace $c(k, k')$ must be a multiple of the unit operator.

$$[a(\mathbf{k}), a(\mathbf{k}')]_{\pm} = \delta^3(\mathbf{k} - \mathbf{k}'). \quad (4.15)$$

In coordinate space:

For $-$, equal λ

$$\begin{aligned} [\varphi(\lambda, y), \varphi(\lambda, y')]_{-} &= 0 & (4.16) \\ [\varphi(\lambda, y), \partial_0 \varphi(\lambda, y')]_{-} &= i\lambda^3 \delta^3(y - y'). \end{aligned}$$

The δ function is normalized correctly, since:

$$\int_{\lambda = \text{const}} \lambda^3 \delta^3(y - y') \sigma_0 = 1.$$

For $+$, equal λ : f. inst. $\lambda = 1, y, y' \rightarrow 0$

$$[\varphi(\lambda, y), \varphi(\lambda, y')]_{+} \cong iD_1(y - y', 0) \cong ie^{-m|y-y'|}.$$

Conclusion. For a scalar field locality is sufficient to derive the correct quantization prescription. This is in complete analogy to the flat space (see [3]). Let us define the dynamical variables \bar{P}_j, \bar{M}_{ij} in the usual way as normal products, then the transition to Quantum-theory is made by simply replacing $a^*(k)$ by $a^+(k)$ in (4.12).

It would be tentative to interpret $\bar{P}_0, \bar{P}_\alpha, \bar{M}_{ij}$ as operators of energy, momentum and angular momentum of an observer at the point x_0 . However, this leads to difficulties, at least for \bar{P}_0 .

Take

$$g = \left\| \begin{array}{cccc} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \\ & & & & -1 \end{array} \right\| \in SO_0(4, 1) \quad (4.17)$$

$g \rightarrow T_g$ is a unitary transformation in \mathcal{H} . The following relations can be verified:

$$\begin{aligned} T_g^{-1} P_0 T_g &= -P_0 & T_g^{-1} M_{0\alpha} T_g &= -M_{0\alpha} \\ T_g^{-1} P_\alpha T_g &= P_\alpha & T_g^{-1} M_{\alpha\beta} T_g &= M_{\alpha\beta} . \end{aligned} \quad (4.18)$$

P_0 cannot have a positive spectrum. It is easily seen that P_0 has a purely continuous spectrum $(-\infty, \infty)$ in the 1-particle subspace. If we make the transition to flat space by letting R tend to ∞ in the expressions for the infinitesimal operators of the de-Sitter group, we get the infinitesimal operator of the representation of Poincare-group with $P_0 > 0$, starting from (4.13), applying first T_g (4.18) and letting then tend $R \rightarrow \infty$ we get the representation with $P_0 < 0$.

Conclusion. The same unitary representation of de-Sitter group can be contracted in the limit $R \rightarrow \infty$ to inequivalent representations of Poincare-group.

The behaviour of P_0 can be explained by observing that the corresponding Killing vector ξ_0 points in positive time direction for $x^4 < 0$, in negative time direction for $x^4 > 0$.

5. Dirac Particles in de-Sitter space

a) Classical Solution

In this section G denotes the covering group of $SO_0(4,1)$.

Representation of the stability group of x_0 (hom. Lorentz-group) Dirac-representation $D(1/2, 0) + D(0, 1/2)$; $\Lambda \rightarrow T(\Lambda)$

$$\gamma^0 = \left\| \begin{array}{c} I \\ -I \end{array} \right\| \quad \gamma^\alpha = \left\| \begin{array}{c} \sigma^\alpha \\ -\sigma^\alpha \end{array} \right\| . \quad (5.1)$$

Covariant derivatives:

$$\partial_0 = \frac{\lambda}{R} \frac{\partial}{\partial \lambda} ; \quad \partial_\alpha = \lambda \cdot \frac{\partial}{\partial y^\alpha} + \frac{1}{2R} \gamma_0 \gamma_\alpha \quad (\lambda > 0) \quad (5.2)$$

$$\text{Lagrange-density: } \mathcal{L} = \frac{i}{2} (\bar{\psi} \gamma^j \partial_j \psi - \partial_j \bar{\psi} \gamma^j \psi) - m \bar{\psi} \psi \quad (5.3)$$

$$\text{Action function: } \mathcal{S} = \int \mathcal{L} [\omega^0 \omega^1 \omega^2 \omega^3] \quad (5.4)$$

$$\text{Field equations: } \delta \mathcal{S} = 0 \quad (5.5)$$

$$i \gamma^j \partial_j \psi - m \psi = 0 .$$

Solution for H : $\lambda > 0$; $\varrho = R \cdot m$; $\omega^2 = \mathbf{k}^2 + m^2$; $m > 0$

$$\psi(\lambda, y) = \int \frac{d^3 k}{(2\pi)^{3/2}} K(x, \mathbf{k}) \cdot \begin{pmatrix} a(-\mathbf{k}) \\ b^*(\mathbf{k}) \end{pmatrix} \quad (5.6)$$

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad b = \begin{pmatrix} b_3 \\ b_4 \end{pmatrix}$$

$$K(x; \mathbf{k}) = e^{i k_\alpha y^\alpha} \lambda^2 \left\| \begin{array}{l} -i(kR)^{-i\varrho+1/2} e^{\varrho\pi/2} \frac{\sqrt{\pi}}{2} ((\omega - m)R)^{i\varrho} e^{i\omega R} H_{i\varrho+1/2}^3(k\lambda R) \\ -\frac{\sigma^\alpha k_\alpha}{|k|} (kR)^{-i\varrho+1/2} e^{\varrho\pi/2} \frac{\sqrt{\pi}}{2} ((\omega - m)R)^{i\varrho} e^{i\omega R} H_{i\varrho-1/2}^3(k\lambda R) \\ (k \cdot R)^{i\varrho+1/2} \frac{i\sigma^\alpha k_\alpha}{|k|} e^{-\varrho\pi/2} \frac{\sqrt{\pi}}{2} ((\omega - m)R)^{-i\varrho} e^{-i\omega R} H_{i\varrho+1/2}^1(k\lambda R) \\ (kR)^{i\varrho+1/2} e^{-\varrho\pi/2} \frac{\sqrt{\pi}}{2} ((\omega - m)R)^{-i\varrho} e^{-i\omega R} H_{i\varrho-1/2}^1(k\lambda R) \end{array} \right\|$$

$$K^+ \cdot K = \lambda^3 I$$

$$K \xrightarrow[x \rightarrow x_0]{R \rightarrow \infty} e^{i\mathbf{k}y} \left\| \begin{array}{ll} \frac{k}{\sqrt{2\omega(\omega-m)}} e^{-i\omega x^0} & \frac{\sigma^\alpha k_\alpha}{|k|} \sqrt{\frac{\omega-m}{2\omega}} e^{i\omega x^0} \\ -\frac{\sigma^\alpha k_\alpha}{|k|} \sqrt{\frac{\omega-m}{2\omega}} e^{-i\omega x^0} & \frac{k}{\sqrt{2\omega(\omega-m)}} e^{i\omega x^0} \end{array} \right\|.$$

The assertion corresponding to lemma 1 is:

Lemma 2. For $g \in G$, $x, x' = g(x) \in H$

$$\psi_H(x) = \int T(g_x^{-1} g^{-1} g_x) K(x'; \mathbf{k}) \left(\begin{array}{c} T_g a(-\mathbf{k}) \\ T_g b^*(\mathbf{k}) \end{array} \right) \frac{d^3 k}{(2\pi)^{3/2}}. \quad (5.7)$$

Again $T_g a, T_g b^*$ denote (eq.) unitary representations of G .

Using this lemma we can verify that (5.8) satisfies (2.5)

$$H' = g^{-1}(H); \quad x' \in H' \quad x = g(x') \in H \quad (5.8)$$

$$\psi_{H'}(x') = \int \frac{d^3 k}{(2\pi)^{3/2}} T \left(\begin{array}{c} g_x^{-1} \\ H' \end{array} \right) T \left(\begin{array}{c} g^{-1} \\ H \end{array} \right) g_x K(x; \mathbf{k}) \left(\begin{array}{c} T_g a(-\mathbf{k}) \\ T_g b^*(\mathbf{k}) \end{array} \right).$$

Normalization.

$$\begin{aligned} \psi &= \int \frac{d^3 k}{(2\pi)^{3/2}} e^{i k_\alpha y^\alpha} K \left(\begin{array}{c} a(-\mathbf{k}) \\ b^*(\mathbf{k}) \end{array} \right) \\ \varphi &= \int \frac{d^3 k}{(2\pi)^{3/2}} e^{i k_\alpha y^\alpha} K \left(\begin{array}{c} a'(-\mathbf{k}) \\ b^{*\prime}(\mathbf{k}) \end{array} \right) \end{aligned} \quad (5.9)$$

$$\begin{aligned} N(\psi, \varphi) &= -i \int_{\lambda = \text{const}} \left(\frac{\partial \mathcal{L}}{\partial \partial_j \psi} \varphi - \bar{\psi} \frac{\partial \mathcal{L}}{\partial \partial_j \bar{\varphi}} \right) \sigma_j \\ &= \int d^3 k \left(\sum_{\nu=1,2} a_\nu^{*\prime}(\mathbf{k}) a_\nu(\mathbf{k}) + \sum_{\nu=3,4} b'_\nu(\mathbf{k}) b_\nu^*(\mathbf{k}) \right). \end{aligned}$$

Conservation laws.

$\mathcal{L} = 0$ because the field equations are satisfied.

$$\bar{P}_j = - \int \left(\frac{\partial \mathcal{L}}{\partial \partial_1 \psi} i P_j \psi + \overline{(i P_j \psi)} \frac{\partial \mathcal{L}}{\partial \partial_1 \bar{\psi}} \right) \sigma_i = -N(\psi, P_j \psi) \quad (5.10)$$

analogously for M_{ij} .

Charge $Q = N(\psi, \psi)$.

The operator \bar{P}_0 is given by:

$$\begin{aligned} \bar{P}_0 &= \int d^3 k \left(\sum_{\nu=1,2} a_\nu^*(\mathbf{k}) \left(\omega - \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) a_\nu(\mathbf{k}) - \right. \\ &\quad \left. - \sum_{\nu=3,4} b_\nu(\mathbf{k}) \left(\omega + \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) b_\nu^*(\mathbf{k}) \right). \end{aligned} \quad (5.11)$$

The negative sign in front of the second term is *not* invariant under unitary transformations:

$$b(\mathbf{k}) = ((\omega - m)R)^{-2i\varrho} (Rk)^{2i\varrho} e^{-2i\omega R} b'(\mathbf{k}) \quad (5.12)$$

$$\begin{aligned} \bar{P}_0 = \int d^3k \left(\sum_{\nu=1,2} a_{\nu}^*(\mathbf{k}) \left(\omega - \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) a_{\nu}(\mathbf{k}) + \right. \\ \left. + \sum_{\nu=3,4} b'_{\nu}(\mathbf{k}) \left(\omega - \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) b'_{\nu}{}^*(\mathbf{k}) \right). \end{aligned} \quad (5.13)$$

b) Quantum-Theory

Assumption.

$$\begin{aligned} [a_{\mu}(k), a_{\nu}(k')]_{\pm} &= [a_{\mu}^+(k), a_{\nu}^+(k')]_{\pm} \\ &= [b_{\mu}(k), b_{\nu}(k')]_{\pm} = [b_{\mu}^+(k), b_{\nu}^+(k')]_{\pm} = 0 \\ [a_{\mu}(k), a_{\nu}^+(k')]_{\pm} &= c_{\mu\nu}(k, k') \\ [b_{\mu}(k), b_{\nu}^+(k')]_{\pm} &= c'_{\mu\nu}(k, k') \\ [a_{\mu}(k), b_{\nu}^+(k')]_{\pm} &= c''_{\mu\nu}(k, k'). \end{aligned} \quad (5.14)$$

Requiring invariance of the commutation relations under the transformations of the de-Sitter group and the gauge transformation:

$$\psi(x) \rightarrow e^{i\epsilon} \psi(x) \quad (5.15)$$

leads to the following commutation relations:

$$\begin{aligned} [a_{\mu}(k), a_{\nu}^+(k')]_{\pm} &= c_1 \delta_{\mu\nu} \delta^3(k - k') \\ [b_{\mu}(k), b_{\nu}^+(k')]_{\pm} &= c_2 \delta_{\mu\nu} \delta^3(k - k') \\ [a_{\mu}(k), b_{\nu}^+(k')]_{\pm} &= 0. \end{aligned} \quad (5.16)$$

In coordinate space:

$$\begin{aligned} [\psi_{\mu}(\lambda, y), \psi_{\nu}(\lambda, y')]_{\pm} &= [\psi_{\mu}^+(\lambda, y), \psi_{\nu}^+(\lambda, y')] = 0 \\ [\psi_{\mu}(\lambda, y), \psi_{\mu'}^+(\lambda, y')]_{\pm} &= \int \frac{d^3k}{(2\pi)^3} e^{ik_{\alpha}(y^{\alpha} - y'^{\alpha})} \times \\ &\times \left(\sum_{\nu=1,2} K_{\mu\nu} K_{\mu'\nu}^* c_1 \pm \sum_{\nu=3,4} K_{\mu\nu} K_{\mu'\nu}^* c_2 \right). \end{aligned} \quad (5.17)$$

Locality leads to $c_2 = \pm c_1$. For then we have $(K^+ K = \lambda^3 I)$

$$[\psi_{\mu}(\lambda, y), \psi_{\mu'}^+(\lambda, y')]_{\pm} = c_1 \delta_{\mu\mu'} \delta^3(y - y'). \quad (5.18)$$

For +

$$\begin{aligned} [a_{\mu}(k), a_{\nu}^+(k')]_{+} &= c_1 \delta^3(k - k') \\ [b_{\mu}(k), b_{\nu}^+(k')]_{+} &= c_1 \delta^3(k - k') \quad c_1 = c_1^*. \end{aligned} \quad (5.19)$$

For -

$$\begin{aligned} [a_{\mu}(k), a_{\nu}^+(k')] &= c_1 \delta^3(k - k') \\ [b_{\mu}(k), b_{\nu}^+(k')] &= -c_1 \delta^3(k - k') \quad c_1 = c_1^*. \end{aligned} \quad (5.20)$$

Now we cannot demand $\bar{P}_0 \geq 0$, for there are no unitary representations of the de-Sitter group fulfilling this. As substitute we require that

for states $|\rangle$ concentrated in the neighbourhood of x_0 the expectation value $\langle \bar{P}_0 \rangle$ shall be positive.

For +

Assumption. There exists a vacuum state $|0\rangle$, $a_\mu|0\rangle = b_\mu|0\rangle = 0$

$$\begin{aligned} |1\rangle &= \int d^3k \sum_{\mu=3,4} f_\mu(\mathbf{k}) b_\mu^+(\mathbf{k}) |0\rangle \\ \langle 1|1\rangle &= c_1 \int d^3k \sum_{\mu=3,4} f_\mu^*(\mathbf{k}) f_\mu(\mathbf{k}) \geq 0 \end{aligned} \quad (5.21)$$

it follows $c_1 > 0$ if the classical case $[b, b^+] = 0$ is excluded.

$$\langle 1| \bar{P}_0 |1\rangle = \int d^3k \sum_{\mu=3,4} f_\mu^*(\mathbf{k}) \left(\omega + \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) f_\mu(\mathbf{k}) + E_0 \quad (5.22)$$

$$E_0 = - \int d^3k' f_\nu^*(k') f_\nu(k') \cdot \int d^3k \left\langle 0 \left| b_\mu(k) \left(\omega + \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) b_\mu^+(k) \right| 0 \right\rangle$$

E_0 is no well defined quantity. We set as usual:

$$\langle 1| :P_0: |1\rangle = \langle 1| P_0 |1\rangle - E_0 .$$

As typical wave packet we choose:

$$f(k) = N e^{-(k-k_0)^2/h^2} \quad N: \text{normalization factor} \quad (5.23)$$

$$\langle 0| \psi^+(\lambda, \mathbf{y}) |1\rangle \cong N' e^{i\mathbf{k}_0 \cdot \mathbf{y}} e^{-y^2/4h^2} \quad (\text{für } \lambda = 1, \mathbf{y} \rightarrow 0)$$

$$\langle 1| : \bar{P}_0 : |1\rangle \cong \omega_0 = +\sqrt{k_0^2 + m^2} \quad (5.24)$$

the second term prop. i/R gives no contribution.

For -

Requiring $a_\mu|0\rangle = b_\mu|0\rangle = 0$ leads to $c_1 = 0$, classical case.

$$|1'\rangle = \int f_\mu(k) a_\mu^+(k) |0\rangle \quad (5.25)$$

$$\langle 1'|1'\rangle = c_1 \int f_\mu^*(k) f_\mu(k) d^3k \geq 0 \rightarrow c_1 \geq 0$$

$$|1''\rangle = \int f_\mu(k) b_\mu^+(k) |0\rangle \quad (5.26)$$

$$\langle 1''|1''\rangle = -c_1 \int f_\mu^*(k) f_\mu(k) d^3k \geq 0 \rightarrow c_1 \leq 0 .$$

Therefore we should require $a_\mu|0\rangle = b_\mu^+|0\rangle = 0$. But then the wave packet (5.23) leads to $\langle 1| \bar{P}_0 |1\rangle \cong \pm \omega_0$ for

$$|1\rangle = \int d^3k f(k) a^+(k) |0\rangle \quad \text{resp.}$$

$$|1\rangle = \int d^3k f(k) b(k) |0\rangle .$$

Conclusion. Locality and the requirement $\langle | : \bar{P}_0(x) : | \rangle \geq 0$ for states $|\rangle$ localized in the neighbourhood of x , lead to the correct quantization prescription for spin 1/2. ($P_0(x)$: the basis element of the Lie-algebra of G , of Theorem 5, corresponding to the point x). The final expression for P_0 is:

$$\begin{aligned} : \bar{P}_0 : &= \int d^3k \left(\sum_{\nu=1,2} a_\nu^*(k) \left(\omega - \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) a_\nu(k) + \right. \\ &\quad \left. + \sum_{\nu=3,4} b_\nu^*(k) \left(\omega - \frac{i}{R} \left(\frac{3}{2} + k \frac{\partial}{\partial k} \right) \right) b_\nu(k) \right) . \end{aligned} \quad (5.27)$$

6. Summary

Let M be a space with transitive transformation group G , stable subgroup K of $x_0 \in M$.

Fields on M are classified by representations of K . (6.1)

Invariant local quantities as differential operators, local Lagrange-densities etc. are only dependent on K . The structure of invariant equations of two spaces is the same, if the stability groups are the same. (6.2)

Quantum theory of free fields can be readily formulated in de-Sitter space. It is dubious if a physical interpretation of the infinitesimal operators is possible. Especially no analogue of energy has been found. This seems plausible physically, since integral dynamical quantities would correspond to motions of the whole universe which cannot again take place in the universe. Only if a part of the system can be regarded as isolated, we can expect physically relevant integral dynamical quantities. (6.3)

The correct quantization prescription and thus the connection of spin and statistics can be derived for scalar particles from locality alone, for Dirac particles we add the further requirement $\langle | : \bar{P}_0(x) : | \rangle \geq 0$ for states $| \rangle$ localized in the neighbourhood of x . (6.4)

The analogues of positive and negative frequency part in de-Sitter space transform with the same unitary irreducible representation of the de-Sitter group. In the limit $R \rightarrow \infty$ the same irreducible representation of the de-Sitter group, written in different bases, contracts to inequivalent representations of Poincare group with $P_0 > 0$ or $P_0 < 0$. (6.5)

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7. Appendix

In this appendix we sketch the proofs of the cited theorems.

Theorem 1. Theorem 62 § 44 of [4] asserts, that in a neighbourhood V of $e \in G$ there exists a canonical coordinate system of the second kind $(x^1, \dots, x^n, y^1, \dots, y^s)$ so that $K \cap V = \{(0, y)\}$. Now we take a neighbourhood U of e , $U = U^{-1}$, $U^2 \subset V$. $U^* = \{u \cdot K, u \in U\}$. x can be regarded as parameters on U^* . The homeomorphisms $g: U \rightarrow g \cdot U$ and $U^* \rightarrow g(U^*)$ make G resp. M to differentiable manifolds satisfying

a), b), c). $\mathcal{A} = \{g(U^*), g \in G\}$. The fundamental map (2.4) is given by: $H = U^* \rightarrow G: x \rightarrow (x, 0) = g_x$; and analogously for $g(U^*)$.

Theorems 2, 3. are a matter of straightforward calculations.

Theorem 4 is proved by induction, using the fact that the coefficients of the terms in ∂_j containing first derivatives are c-numbers and not operators in V .

Theorem 5. It is sufficient to show the decomposition for the point $x_0 = (0, \dots, 0, -R)$ and the tetrad dx^j ($j = 0, \dots, 3$). One decomposition is given in (3.15). Let M'_{ij}, P'_j be another basis satisfying:

$$\left(I + i a^j P'_j + \frac{i}{2} a^{ij} M'_{ij} \right) (x) = x^i + a^i + \frac{1}{2} a^{ij} y_j + 2^{\text{nd}} \text{ order} .$$

by differentiating with respect to x^j it follows:

$$M_{ij} = M'_{ij}, \quad P_j = P'_j .$$

Lemma 1. By a direct calculation we verify (I_ν inf. op.)

$$I_\nu \varphi^-(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} K_-(x, k) I_\nu a(k) .$$

Take a one parameter subgroup $g(t)$,

$$I_\nu = \left. \frac{\partial g(t)}{\partial t} \right|_{t=0} .$$

Then lemma 1 follows by observing that

$$\begin{aligned} \varphi^-(t, x) &= \varphi(g^{-1}(t)(x)); \quad \text{and} \\ \varphi'(t, x) &= \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} K_-(x, k) T_{g(t)} a(k) \end{aligned}$$

satisfy the same differential equation with the same initial value for $t = 0$.

Theorem 6. The unitary transformation establishing the equivalence is (5.12).

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