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Some Probabilistic Techniques in Field Theory*

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Abstract. We study, in the context of the Markov hyerarchical fields ($d=2$, 3) the role of the Markov property, of formal renormalization and of formal positivity. We determine upper and lower bounds for the ground state energy and discuss their relation with the perturbation theory series.

Introduction and Motivation

The basic property which allows to prove the rigorous validity of the perturbation expansion in euclidean field theory of φ^4 type in $d=2$, 3 space-time dimensions, is the "ultraviolet stability". The ultraviolet stability is the existence of a lower bound to the minimum of the spectrum of the renormalized Hamiltonian. In this paper we propose a model and a method of analysis which allows, in our opinion, to clarify the statistical mechanical aspects of the ultraviolet stability theorem. To motivate this model, and to illustrate the reasons which make it essentially as difficult as the euclidean field theory, we proceed as follows.

The euclidean field on R^d is a gaussian field with covariance

$$C=(1-D)^{-1} \tag{1}$$

where D is the Laplace operator on R^d. The ultraviolet divergences, originate from the divergence of the kernel $C_{\xi,\eta}$ of the operator C, as operator on $L_2(R^d)$, as $|\xi-\eta|\to 0$, if $d\geqq 2$. This remark leads to the idea [1], of representing C as

$$C=\sum_{N=0}^{\infty}[(2^{2N}-D)^{-1}-(2^{2(N+1)}-D)^{-1}] \tag{2}$$

and, correspondingly, the field $\tilde{\varphi}$ as,

$$\tilde{\varphi}_\xi=\sum_{N=0}^{\infty}\tilde{\varphi}_\xi^{[N]} \tag{3}$$

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where $\tilde{\varphi}^{[N]}$ are independent fields (with respect to the integers index N). If $d=2, 3$, $\tilde{\varphi}^{[N]}$ has a bounded covariance:

$$\mathscr{E}((\tilde{\varphi}_\xi^{[N]})^2)=\gamma 2^{(d-2)N}, \quad \gamma<\infty \tag{4}$$

and it can be normalized by setting:

$$\tilde{z}_\xi^{(N)}=\frac{\tilde{\varphi}_\xi^{(N)}}{(2\gamma 2^{(d-2)N})^{1/2}}. \tag{5}$$

It is easy to check that the field $\tilde{z}_\xi^{(N)}$ is almost constant on a scale 2^{-N} and that its covariance decays exponentially fast on this same scale. We can imagine to construct a good model of the above field φ admitting a representation of the type Equations (3) and (5), where the random variables $\tilde{z}_\xi^{(N)}$ have the following properties:

i) they are "constant" over squares with scale 2^{-N}.

ii) they decay exponentially fast on a scale 2^{-N}.

A precise definition of such a field, is given in the next section and will be called a Markov hierarchical field. Using this field as a "free field", we shall then study the φ^4-interacting field. This problem, as it will turn out, presents exactly the same difficulties and divergences as the euclidean field presents.

1. The Model: Definitions and Notations

The free hierarchical Markov field over R^d is described in terms of a family of gaussian random variables indexed by the tesserae of a family $(Q_i)_{i=0}^\infty$ of compatible pavements of R^d. Each tesserae $\Delta\in Q_i$ is a cube with side size 2^{-i} $i=0,1,\ldots$. The random variable associated to Δ will be denoted by z_Δ and the variables z_Δ, $\Delta\in Q_i$, $z_{\Delta'}$, $\Delta'\in Q_j$ are assumed independent if $i\neq j$. Given $i\geqq 0$, the distribution of the z_Δ's for $\Delta\in Q_i$ is described by a gaussian Ising model with nearest neighbour interaction with formal density proportional to

$$\exp-\frac{\beta}{2}\left[\sum_{\Delta,\Delta'\in Q_i}^{N}{}^{*}(z_\Delta-z_{\Delta'})^2+\alpha^2\sum_{\Delta\in Q_i}z_\Delta^2\right] \tag{1.1}$$

where $\sum^*$ runs over the pairs of nearest neighbour tesserae $\Delta,\Delta'\in Q_i$ and β,α are positive parameters fixed so that the expectation of z_Δ^2 is $\frac{1}{2}$; α,β are fixed once for all.

The free hierarchical Markov field with ultraviolet cut-off of length 2^{-N} is defined as the gaussian field over R^d:

$$\varphi_\xi^{[\leqq N]}=\sum_{k=0}^{N}\sum_{\substack{\Delta\in Q_k\\ \xi\in\Delta}}\sqrt{2\gamma_k}\,z_\Delta \tag{1.2}$$

where

$$\gamma_k=2^{(d-2)k}, \quad k=0,1,\ldots. \tag{1.3}$$

We define the normalized field with cut-off 2^{-N}

$$x_\Delta^{(N)} = \frac{\varphi_\xi^{[\leqq N]}}{\sqrt{2\sum_0^N{}_i\,\gamma_i}}, \qquad \xi\in\Delta\in Q_N \tag{1.4}$$

which obeys the recursion relation

$$x_\Delta^{(N)} = \frac{z_\Delta + \sqrt{\Gamma_N}\,x_{\Delta'}^{(N-1)}}{\sqrt{1+\Gamma_N}} \tag{1.5}$$

where

$$\Gamma_N = \sum_0^{N-1}{}_k\,\gamma_k/\gamma_N:$$

We shall denote $\hat{\mathscr{E}}_N(\cdot)$ the expectation with respect to the probability distribution $\hat{P}_N$ of the field $(z_\Delta)_{\Delta\in Q_N}$. We shall define

$$P_N = \prod_{i=0}^{N} \hat{P}_i \tag{1.6}$$

The "interaction" is defined for $d=2,3$ as

$$V_{0,I}^{(N)} = -\lambda \int_I :(\varphi_\xi^{[\leqq N]})^4: d\xi, \qquad \lambda>0 \tag{1.7}$$

where I is a bounded set exactly paved by Q_0,

$$:(\varphi_\xi^{[\leqq N]})^n: = \left(\sqrt{2\sum_0^N{}_i\,\gamma_i}\right)^n H_n(x_\Delta), \qquad \xi\in\Delta\in Q_N \tag{1.8}$$

and H_n is the n-th Hermite polynomial ($H_0(x)=1$, $H_1(x)=x$, $H_2(x)=x^2-1/2$, $H_4(x)=x^4-3x^2+3/4$). The "renormalized interaction to order 3", will be defined

$$V_I^{(N)} = V_{0,I}^{(N)} - \frac{1}{2!}\langle V_{0,I}^{(N)2}\rangle_{(2)} - \frac{1}{2!}\langle V_{0,I}^{(N)2}\rangle_{(0)} - \frac{1}{3!}\langle V_{0,I}^{(N)3}\rangle_{(0)} \tag{1.9}$$

where

$$\frac{1}{2}\langle V_{0,I}^{(N)2}\rangle_{(2)} = \frac{\lambda^2}{2}4^2 3!\iint_{I\times I} d\xi\,d\eta\,(C_{\xi\eta}^{[\leqq N]})^3:(\varphi_\xi^{[\leqq N]})^2: \tag{1.10}$$

$$\frac{1}{2}\langle V_{0,I}^{(N)2}\rangle_{(0)} = \frac{\lambda^2}{2}4!\iint_{I\times I} d\xi\,d\eta\,(C_{\xi\eta}^{[\leqq N]})^4 \tag{1.11}$$

$$\frac{1}{3!}\langle V_{0,I}^{(N)3}\rangle_{(0)} = -\frac{\lambda^3}{3!}\binom{4}{2}^3(2!)^3\iiint_{I\times I\times I} d\xi\,d\eta\,d\zeta\,(C_{\xi\eta}^{[\leqq N]})^2(C_{\eta\zeta}^{[\leqq N]})^2(C_{\zeta\xi}^{[\leqq N]})^2 \tag{1.12}$$

and

$$C_{\xi\eta}^{[\leqq N]} \equiv \int \varphi_\xi^{[\leqq N]}\varphi_\eta^{[\leqq N]} P_N(dz). \tag{1.13}$$

The "ultraviolet problem" that we study in this paper, is the following: to prove the existence of $E_+(\lambda)$, $E_-(\lambda)$, such that, $\forall I, N$ ($|I|$ is the volume of I):

$$\text{i)}\quad \exp[-E_-(\lambda)|I|] \leqq \int \exp V_I^{(N)} P_N(dz) \leqq \exp[E_+(\lambda)|I|] \tag{1.14}$$

$$\text{ii)}\quad \lim_{\lambda\to 0} E_\pm(\lambda)/\lambda^3 = 0\,. \tag{1.15}$$

The technique we use would allow to treat more general problems and does not distinguish the cases $d=2$ and $d=3$ (see §6).

The results i), ii) are obtained in this paper by using a technique which is completely different from the one used in [2] and seems to simplify the classic approach, [1, 3] to perturbation theory, at least for the class of models considered here. In this paper we also give a complete derivation of an estimate for E_+, along the lines which were only summarily sketched in [2].

2. $E_-(\lambda)$: Structure of the Bound

The estimate for $E_-(\lambda)$ will be obtained by studying and bounding

$$\int \exp V_I^{(N)} \chi^{(N)} P_N(dz) \tag{2.1}$$

where $\chi^{(N)}$ is a suitably chosen characteristic function. To describe $\chi^{(N)}$ and the other characteristic functions which will appear in the following, we shall adopt the convection that χ ("something") is the characteristic function of the events for which the "something" is verified. We introduce the sequence

$$B_k = B(1+k)^4 \log\left(e + \frac{1}{\lambda}\right) \tag{2.2}$$

where $B>0$ will be chosen later, and if $\Delta \in Q_k$

$$\chi_\Delta = \chi(|x_\Delta^{(k)}| < B_k[1 + 2^k d(\Delta, I)]) \tag{2.3}$$

where $d(\Delta, I)$ is the distance between the sets Δ and I.

We than define

$$\chi^{(N)} = \prod_{i=0}^{N} \prod_{\Delta \in Q_i} \chi_\Delta \tag{2.4}$$

of course the sequence B_k has been chosen so that there exists a function $e(\lambda)$ such that (see Appendix A)

$$\text{i)}\quad \int \chi^{(N)} P_N(dz) \geqq \exp\{-e(\lambda)|I|\} \tag{2.5}$$

$$\text{ii)}\quad \lim_{\lambda\to 0} \lambda^{-k} e(\lambda) = 0\,, \qquad k = 0, 1, \ldots\,. \tag{2.6}$$

To describe the inductive procedure to find an expression for $E_-(\lambda)$ we need few more definitions.

The first is the definition of "cumulants" (or truncated expectations) of a family of random variables $x_1 \dots x_s$ with respect to the probability measure P:

$$\mathscr{E}^T(x_1, \dots, x_s; k_1, \dots, k_s) = \left[\frac{\partial^{k_1 + \dots + k_s}}{\partial\theta_1^{k_1} \dots \partial\theta_s^{k_s}} \log \int \exp \sum_1^s{}_i\, \theta_i x_i dP\right]_{\theta_i = 0}, \qquad k_i = 0, 1, \dots . \tag{2.7}$$

which makes sense in an obvious way if $\int |x_i|^l P(dx) < \infty$, $l = 0, 1, \dots; i = 1, 2, \dots$. The second definition gives a meaning to the symbol $[p(\lambda)]_t$ for any polynomial $p(\lambda)$: if $p(\lambda) = \sum_{k \geqq 0} e_k \lambda^k$ we denote:

$$[p(\lambda)]_{(t)} = \sum_0^t{}_k\, e_k \lambda^k . \tag{2.8}$$

Finally we define the symbols $\tilde{V}_I^{(h)}$ inductively for $h = N, N-1, \dots, 1, 0, -1$:

$$\begin{aligned} \tilde{V}_I^{(N)} &= V_I^{(N)} \\ \tilde{V}_I^{(N-k)} &= \left[\hat{\mathscr{E}}_{N-k+1}(\tilde{V}_I^{(N-k+1)}) + \frac{1}{2}\hat{\mathscr{E}}^T_{N-k+1}(\tilde{V}_I^{(N-k+1)}; 2) \right. \\ &\quad \left. + \frac{1}{3!}\hat{\mathscr{E}}^T_{N-k+1}(\tilde{V}_I^{(N-k+1)}; 3)\right]_{(3)} . \end{aligned} \tag{2.9}$$

It is not difficult to realize that $\tilde{V}_I^{(-1)} = 0$. The bound is obtained recursively by proving that there exists G, ϱ, ϱ', ϱ'' such that

$$\int \chi^{(k)} \exp(\tilde{V}_I^{(k)}) \hat{P}_k(dz) = \chi^{(k-1)} \exp(\tilde{V}_I^{(k-1)}) \cdot \exp\{\tau G \varepsilon(k, \lambda)|I|\} \tag{2.10}$$

where

$$\varepsilon(k, \lambda) = [k^\varrho 2^{-(4-d)k} \lambda e^{\varrho' \lambda^{\varrho''}}]^{4-1/2} 2^{dk} \tag{2.11}$$

where τ is some function which takes values on $[-1, +1]$.

It will turn out from the proof that in (2.11) $4-1/2$ can be changed in $4-\varepsilon$ provided G is accordingly changed in some G_ε.

The above (2.10) implies that one can take

$$E_-(\lambda) = G(\lambda e^{\varrho' \lambda^{\varrho''}})^{4-1/2} \sum_0^\infty{}_k\, [k^\varrho 2^{-(4-d)k}]^{4-1/2} 2^{dk} . \tag{2.12}$$

The proof of (2.10) will be given in §5.

3. $E_+(\lambda)$

In this section we remove the field cut-off making explicit use of the positivity of $H_4(x)$ for large values of x.

The basic idea is to represent the integral as a sum of integrals in each of which the regions where the fields $x^{(N)}$, $x^{(N-1)}$ are small, are specified. We then treat the integral over these fields as in the determination of the lower bound. The integral over the remaining fields will be studied by using that either the field $x^{(N)}$ is large and therefore $V^{(N)} \ll 0$ or z_Δ is large and, hence, has very small probability. The possibility of a separate treatment of the field with support in complementary regions relies on the Markov property of the fields $\{z_\Delta\}_{\Delta \in Q_i}$ $i=0,1,\ldots$. To implement the above program we introduce the following characteristic functions:

$$\chi_\Delta = \chi(|x_\Delta^{(N)}| < B_N[1+2^N d(I,\Delta)]) \tag{3.1}$$

$$\chi_\Delta^c = 1 - \chi_\Delta \tag{3.2}$$

and we shall use the decomposition of unity:

$$1 = \sum_{D_N} \prod_{\substack{\Delta \subset D_N \\ \Delta \in Q_N}} \chi_\Delta^c \prod_{\substack{\Delta \subset I \setminus D_N \\ \Delta \in Q_N}} \chi_\Delta \equiv \sum_{D_N} \chi_{D_N}^c \chi_{D_N^c} \tag{3.3}$$

where the sum runs over the subsets of I which are exactly paved by Q_N and the abbreviations of the second equality are, selfexplanatory.

Starting from the identity:

$$\int \exp V_I^{(N)} P_N(dz) = \sum_{D_N} \int \chi_{D_N}^c \chi_{D_N^c} \exp V_I^{(N)} P_N(dz)$$

we shall first prove that there is a $k(\lambda)$ such that if $N \geqq k(\lambda)$

$$\int \exp V_I^{(N)} P_N(dz) \leqq \sum_{D_N} \int \chi_{D_N}^c \chi_{D_N^c} \exp V_{I \setminus D_N}^{(N)} P_N(dz) \tag{3.4}$$

where $V_{I\setminus D_N}^{(N)}$ is defined as in (1.9) by changing I into $I \setminus D_N$.

The second step, will be to prove that, for $k \geqq k(\lambda)$

$$\sum_{D_k} \int \chi_{D_k}^c \chi_{D_k^c} \exp \tilde{V}_{I\setminus D_k}^{(k)} P_k(dz)$$
$$\leqq \left(\sum_{D_{k-1}} \int \chi_{D_{k-1}}^c \chi_{D_{k-1}^c} \exp \tilde{V}_{I\setminus D_{k-1}}^{(k-1)} P_{k-1}(dz) \right) \cdot \exp \bar{G} \varepsilon(k,\lambda) |I| \tag{3.5}$$

where $\tilde{V}_{I\setminus D_k}^{(k)}$ is defined recursively as in (2.9) replacing I by $I \setminus D_k$, $\bar{G}$ is a suitably chosen constant and $\varepsilon(k,\lambda)$ is defined in (2.11). The above formula clearly imply that one can take for $E_+(\lambda)$

$$E_+(\lambda) = \bar{G} \sum_{k \geqq k(\lambda)} \varepsilon(k,\lambda) + \sup_{D_{k(\lambda)-1}} \max |\tilde{V}_{I\setminus D_{k(\lambda)-1}}^{k(\lambda)-1}| \chi_{D_{k(\lambda)-L}^c} \frac{1}{|I|} \tag{3.6}$$

furthermore the function $k(\lambda)$ can be taken identicaly zero for λ small enough (hence the second term is absent for small λ). We now prove (3.4) and (3.5). The proof is based on the following structural properties of $\tilde{V}_J^{(k)}$ with J exactly paved by Q_k (see Appendix B): there is a constant $b > 0$, such that if we write

$$\tilde{V}_J^{(k)} = V_{0,J}^{(k)} + \bar{V}_J^{(k)} \tag{3.7}$$

where $V_{0,J}^{(k)}$ is defined as in (1.7), the following relation holds:

$$V_{0,J}^{(k)} \leqq V_{0,J\setminus D}^{(k)} - 4\lambda(2^{-(4-d)k})(1+\Gamma_k)^2 H_4(F) 2^{dk} |D| \tag{3.8}$$

valid if $D \subset J$, D exactly paved by Q_k and if $|x_\Delta^{(k)}| > F \geqq 2$ for $\Delta \subset D$, $\Delta \in Q_k$. Furthermore if J is paved by Q_N,

$$\bar{V}_J^{(N)} \leqq \bar{V}_{J\setminus D}^{(N)} + b(\lambda^2 + \lambda^3)N(1+\Gamma_N)^4 2^{-2(4-d)N} 2^{dN}|D| \tag{3.9}$$

valid without exceptions, or, if J is paved by Q_k, $k < N$

$$|\bar{V}_J^{(k)} - \bar{V}_{J\setminus D}^{(k)}| \leqq b(\lambda^2 + \lambda^3)k(1+\Gamma_k)^4 2^{-2(4-d)k} 2^{dk}|D|F^8 \tag{3.10}$$

if $|x_\Delta^{(k)}| < F$ for all $\Delta \subset J$, $\Delta \in Q_k$.

We shall define the above mentioned function $k(\lambda)$ as the smallest integer k such that:

$$4\lambda 2^{-(4-d)k}(1+\Gamma_k)^2 H_4(\tilde{B}_k) - b(\lambda^2+\lambda^3)k(1+\Gamma_k)^4 2^{-2(4-d)k} B_k^8 > 0 \tag{3.11}$$

where $\tilde{B}_k < B_k$ is defined as

$$\begin{aligned} \tilde{B}_k &= \left[\sqrt{\Gamma_k} B_{k-1} - \frac{B_k}{8(1+k)^2}\right] \frac{1}{\sqrt{1+\Gamma_k}}, \qquad k \geqq 1 \\ \tilde{B}_0 &= B_0 . \end{aligned} \tag{3.12}$$

and to obtain $\tilde{B}_k \geqq 2$, we shall choose $B \geqq 12$ [see Equation (2.2)]. The reason for this choice of $k(\lambda)$ will become clear soon. The first statement [Equation (3.4)] follows immediately from Equations (3.3), (3.8), (3.9), (3.11).

To prove the second statement (Equation (3.5)) we introduce

$$\begin{aligned} \hat{\chi}_\Delta &= \chi\left(|z_\Delta| < \frac{B_k}{8(1+k)^2}(1+2^k d(\Delta, I))\right) \\ \hat{\chi}_\Delta^c &= 1 - \hat{\chi}_\Delta \\ \hat{\chi}_{R_k}^c &= \prod_{\substack{\Delta \subset R_k \\ \Delta \in Q_k}} \hat{\chi}_\Delta^c, \qquad \chi_{R_k^c} = \prod_{\substack{\Delta \cap R_k = \emptyset \\ \Delta \in Q_k}} \hat{\chi}_\Delta \end{aligned} \tag{3.13}$$

if R_k is paved by Q_k. Then,

$$\begin{aligned} \sum_{D_k} \int \chi_{D_k}^c \chi_{D_k^c} \exp \tilde{V}_{I\setminus D_k}^{(k)} P_k(dz) &= \sum_{D_k} \sum_{D_{k-1}} \sum_{R_k} \int \chi_{D_k}^c \chi_{D_k^c} \\ &\quad \cdot \chi_{D_{k-1}}^c \chi_{D_{k-1}^c} \hat{\chi}_{R_k}^c \hat{\chi}_{R_k^c} \exp \tilde{V}_{I\setminus D_k}^{(k)} P_k(dz) . \end{aligned} \tag{3.14}$$

Let

$$\hat{R}_k = \left\{\xi \mid \xi \in \Delta \in Q_{k-1}, \quad 2^k d(\Delta, R_k) \leqq \left(\frac{B_k}{8(1+k)^2}\right)^3\right\}. \tag{3.15}$$

Than (3.8), (3.9), (3.11), (3.12), imply, for $k \geqq k(\lambda)$

$$\begin{aligned} [(3.14)] &\leqq \sum_{D_k} \sum_{D_{k-1}} \sum_{R_k} \int \chi_{D_k}^c \chi_{D_k^c} \chi_{D_{k-1}}^c \chi_{D_{k-1}^c} \hat{\chi}_{R_k}^c \hat{\chi}_{R_k^c} \exp \tilde{V}_{I\setminus D_k \cup D_{k-1} \cup \hat{R}_k}^{(k)} P_k(dz) \cdot \\ &\quad \exp 2^{dk} |\hat{R}_k \cap I| [b(\lambda^2+\lambda^3)k(1+\Gamma_k)^4 \\ &\quad \cdot 2^{-(4-d)k} B_k^8 + 6\lambda(1+\Gamma_k)^2] 2^{-(4-d)k} . \end{aligned} \tag{3.16}$$

It is easily seen that $D_k \backslash D_{k-1} \subset R_k$, than we can do the sum on D_k and we get

$$[(3.14)] \leqq \sum_{D_{k-1}} \sum_{R_k} \int \chi^c_{D_{k-1}} \chi_{D^c_{k-1}} \hat{\chi}^c_{R_k} \hat{\chi}_{R^c_k} \exp \tilde{V}^{(k)}_{I \backslash D_{k-1} \cup \hat{R}_k} P_k(dz)$$
$$\cdot \exp 2^{dk} |\hat{R}_k \cap I| [b(\lambda^2 + \lambda^3) k (1+\Gamma_k)^4 2^{-(4-d)k} B^8_k + 6\lambda(1+\Gamma_k)^2]_2^{-(4-d)k} . \tag{3.17}$$

We can write:

$$\int \chi^c_{D_{k-1}} \chi_{D^c_{k-1}} \hat{\chi}^c_{R_k} \hat{\chi}_{R^c_k} \exp \tilde{V}^{(k)}_{I \backslash D_{k-1} \cup \hat{R}_k} P_k(dz)$$
$$= \int P_{k-1}(dz) \chi^c_{D_{k-1}} \int \hat{P}_k(dz_{(i)}) \hat{\chi}^c_{R_k}$$
$$\cdot \int \hat{P}_k(dz_{(e)} | z_{(i)}) \chi_{D^c_{k-1}} \hat{\chi}_{R^c_k} \exp \tilde{V}^{(k)}_{I \backslash D_{k-1} \cup \hat{R}_k} \tag{3.18}$$

where

$$z_{(i)} = \{z_\Delta | \Delta \subset (R_k \cup \partial^+ R_k)\}$$
$$z_{(e)} = \{z_\Delta | \Delta \not\subset (R_k \cup \partial^+ R_k)\} \tag{3.19}$$

$$\partial^+ R_k = \{\Delta \in Q_k | d(\Delta, R_k) = 0, \Delta \not\subset R_k\} \tag{3.20}$$

and $P_k(dz_{(i)})$ denotes the distribution of the $z_{(i)}$ variables with respect to the measure P_k and $P_k(dz_{(e)}|z_{(i)})$ is the distribution of the variables $z_{(e)}$ conditioned, in $\hat{P}_k$, to given values of $z_{(i)}$ (but it depends only on $z^+_{\partial R_k}$ because the Markov property of the field).

We now use the inequality, valid if $B > 8b^*$, where b^* is defined in § 5:

$$\int \hat{P}_k(dz_{(e)} | z_{(i)}) \chi_{D^c_{k-1}} \hat{\chi}_{R^c_k} \exp \tilde{V}^{(k)}_{I \backslash D_{k-1} \cup \hat{R}_k}$$
$$\leqq \chi_{D^c_{k-1}} \exp \tilde{V}^{(k-1)}_{I \backslash D_{k-1} \cup \hat{R}_k} \exp G\varepsilon(k, \lambda) |I| . \tag{3.21}$$

This inequality will be proven, together with the similar one (2.10), used in the theory of E_- in § 5.

We use next the inequality

$$\chi_{D^c_{k-1}} \exp \tilde{V}^{(k-1)}_{I \backslash D_{k-1} \cup \hat{R}_k} \leqq \chi_{D^c_{k-1}} \exp \tilde{V}^{(k-1)}_{I \backslash D_{k-1}}$$
$$\cdot \exp \{ |\hat{R}_k \cap I| 2^{dk} [b(\lambda^2 + \lambda^3) 2^{-(4-d)k} B^8_k$$
$$+ 4H_4(B_k) \lambda (1+\Gamma_k)^2] 2^{-(4-d)k} \}$$
$$\leqq \chi_{D_{k-1}} \exp \tilde{V}^{(k-1)}_{I \backslash D_{k-1}} \exp \left\{ |R_k \cap I| 2^{dk} \frac{\mu(k, \lambda)}{2} \right\} \tag{3.22}$$

where

$$\mu(k, \lambda) = \left[2 \left(\frac{B_k}{8(1+k)^2} \right)^3 \right]$$
$$\cdot [b(\lambda^2 + \lambda^3) k (1+\Gamma_k)^4 2^{-(4-d)k} B^8_k + 4\lambda H_k(B_k)(1+\Gamma_k)^2] 2^{-(4-d)k} \tag{3.23}$$

and the first factor of $\mu(k,\lambda)$ takes into account the replacement of $\hat{R}_k$ by R_k in the last step, hence:

$$[(3.14)] \leqq \left[\sum_{D_{k-1}} \sum_{R_k} e^{\mu(k,\lambda)2^{dk}|R_k \cap I|} \int \chi^c_{D_{k-1}} \chi_{D^c_{k-1}} \hat{\chi}^c_{R_k} \cdot \exp \tilde{V}^{(k-1)}_{I \setminus D_{k-1}} P_k(dz)\right] \exp G|I|\varepsilon(k,\lambda). \tag{3.24}$$

To perform the sum over R_k, we use the inequality

$$\sum_{R_k} e^{\mu(k,\lambda)2^{dk}|R_k \cap I|} \int \hat{\chi}^c_{R_k} \hat{P}_k(dz) \leqq \exp\left[c|I|2^{dk} e^{\mu(k,\lambda) - \frac{B_k^2}{64(1+k)^4}\frac{\alpha^2\beta}{2}}\right] \tag{3.25}$$

where $c>0$ is a suitable constant. Inequality (3.25) is a property of the free field $\hat{P}_k$ and it is an immediate consequence of Lemma 1 in Appendix C.

4. The Structure of $\tilde{V}^{(k)}_J$

To find what has to be proven to obtain the basic inequalities (2.11) and (3.25) we have to use explicitely the structure of $\tilde{V}^{(k)}_J$. This structure can be studied by computing explicitely the function $\tilde{V}^{(k)}_J$. The calculation straightforward, but lengthy, and the definitive result is in the Appendix B; here we describe only some of the main features:

$$\tilde{V}^{(k)}_J = \sum_{1p}^{3} \sum_{\substack{\Delta_i \in Q_k i:1,\ldots,p \\ \Delta_i \subset J i:1,\ldots,p}} \sum_{\substack{n_1 \ldots n_p \\ n_i > 0 \\ \sum_1^p n_i \leqq 8}} \bar{A}^{n_1 \ldots n_p}_{\Delta_1 \ldots \Delta_p}(N,k,J) \cdot e^{-\varkappa 2^k d(\Delta_1,\ldots,\Delta_p)} x^{(k)n_1}_{\Delta_1} \ldots x^{(k)n_p}_{\Delta_p} + \sum_{\substack{\Delta \in Q_k \\ \Delta \subset J}} \bar{C}_\Delta(N,k,J) \tag{4.1}$$

where $d(\Delta_1,\ldots,\Delta_p)$ is the length of the smallest path connecting $\Delta_1 \ldots \Delta_p$, $\varkappa$ is a positive constant and $\bar{A}^{n_1 \ldots n_p}_{\Delta_1 \ldots \Delta_p}(N,k,J)$, $\bar{C}_\Delta(N,k,J)$, are suitable coefficients which verify the estimates

$$\sup_{\substack{J,N,p \\ n_1,\ldots,n_p \\ \Delta_1,\ldots,\Delta_p}} |\bar{A}^{n_1 \ldots n_p}_{\Delta_1 \ldots \Delta_p}(N,k,J)| \leqq \bar{A} 2^{-(4-d)k}(1+\Gamma_k)^2 \tag{4.2}$$

$$\sup_{J,N,\Delta} |\bar{C}_\Delta(N,k,J)| \leqq \bar{A} 2^{-(4-d)k}(1+\Gamma_k)^2 \tag{4.2a}$$

for a suitable chosen constant $\bar{A}$. We shall now consider only the case in which $x^{(k)}$ can be written as $x^{(k)}_\Delta = (z_\Delta + \sqrt{\Gamma_k} x^{(k-1)}_\Delta)/\sqrt{1+\Gamma_k}$ with $|x^{(k-1)}_\Delta| \leqq B_{k-1}$, and we shall regard $\tilde{V}^{(k)}_J$ as a function of the $(z_\Delta)_{\Delta \in Q_k}$ which will take the following form

$$\tilde{V}^{(k)}_J = \sum_{1}^{3}{}_p \sum_{\substack{\Delta_i \in Q_k i:1\ldots p \\ \Delta_i \subset J i:1\ldots p}} \sum_{\substack{n_1 \ldots n_p \\ n_i > 0 \\ \sum n_i \leqq 8}} A^{n_1 \ldots n_p}_{\Delta_1 \ldots \Delta_p}(N,k,J) \cdot e^{-\frac{\varkappa}{2} 2^k d(\Delta_1 \ldots \Delta_p)} z^{n_1}_{\Delta_1} \ldots z^{n_p}_{\Delta_p} + \sum_{\substack{\Delta \in Q_k \\ \Delta \subset J}} e_\Delta(N,k,J) \tag{4.3}$$

and the coefficients verify an estimate of the type (4.2) above with $\bar{A}$ replaced by

$$A=\bar{A}B_{k-1}^{8}r \tag{4.4}$$

where $r>0$ is a constant. It is easy to realize that the estimates that we are seeking can be deduced from the following general lemma on the theory of gaussian processes. Let, for J exactly paved by Q_0:

$$H_J=\sum_{1}^{s}{}_p \sum_{\substack{\Delta_i\in Q_0\\ \Delta_i\subset J\\ i:1\dots p}} \sum_{\substack{n_i>0\\ \sum n_i\leqq D\\ i:1\dots p}} A_{\Delta_1\dots\Delta_p}^{n_1\dots n_p} e^{-\frac{\varkappa}{2}d(\Delta_1\dots\Delta_p)} z_{\Delta_1}^{n_1}\dots z_{\Delta_p}^{n_p} \tag{4.5}$$

and call $A\equiv\sup|A_{\Delta_1\dots\Delta_p}^{n_1\dots n_p}|$.

We shall consider the z_Δ's, $\Delta\in Q_0$, as random variables with the conditional distribution $\hat{P}_0(dz|(\bar{z}_\Delta)_{\Delta\in C})$, (c.f.r. Eq. 3.18), hereafter abriged as $\bar{P}(dz)$, where C is a region paved by Q_0 at distance b^3 from J.

Lemma. *Given an integer $t\geqq 0$ and $b>b^*$, let $J\subseteqq I$ and C be regions exactly paved by Q_0. There exist constants S, ϱ_1, ϱ_2, ϱ_3, ϱ_4 depending only on t, D, d, $\varkappa$ such that:*

$$\int\prod_{\Delta}\hat{\chi}_\Delta \exp H_J\bar{P}(dz)\leqq\exp\left\{\left[\sum_{1}^{t}{}_k\frac{\hat{\mathscr{E}}_0^T(H_J;k)}{k!}\right]+|I|S((Ab^{\varrho_1}e^{\varrho_2 Ab^{\varrho_3}})^{t+1}+e^{-\varrho_3 b^{3/2}}e^{\varrho_4 Ab^{\varrho_3}})\right\} \tag{4.6}$$

and if $C=\emptyset$

$$\int\prod_{\Delta}\hat{\chi}_\Delta \exp H_J\hat{P}_0(dz)\geqq\exp\left\{\left[\sum_{1}^{t}{}_k\frac{\hat{\mathscr{E}}_0^T(H_J;k)}{k!}\right]-|I|S\cdot((Ab^{\varrho_1}e^{\varrho_2 Ab^{\varrho_3}})^{t+1}+e^{-\varrho_3 b^{3/2}}e^{\varrho_4 Ab^{\varrho_3}})\right\} \tag{4.7}$$

where

$$\hat{\chi}_\Delta=\chi(|z_\Delta|\leqq b(1+d(I,\Delta))).$$

Remark 1. The truncated expectations are to be computed with respect to the unconditional $\hat{P}_0$ measure.

Remark 2. In the applications we shall identify Q_0 with Q_k and identify J with $I\backslash D_k\cup\hat{R}_k$ and choose $b=\frac{B_k}{8}(1+k)^{-2}$. In this way we make use of the scale invariance of the z-components of the free field.

Remark 3. The term in square brackets in (4.6) will be, in the application, a polynomial in λ of degree 9 (since $t=3$) which we replace by $\left[\sum_{1}^{3}{}_k\frac{\hat{\mathscr{E}}_0^T}{k!}(H_J;k)\right]_{(3)}$ [see (2.9)]. As it is implied by the structure of $\tilde{V}^{(k)}$ this replacement produces an error which is of the same form of the one in (4.6).

Remark 4. The above lemma is very weak from the point of view of statistical mechanics and becomes interesting only in the limit $A\to 0$ $b\to\infty$ so that $Ab^{\varrho}\to 0$ for all $\varrho>0$.

5. Proof of the Basic Lemma

The proof of this lemma is quite simple but it is burdened by many technical details. To help the reader, we give first a short sketch.

The goal is to evaluate (disregarding the characteristic functions which are the origin of many technical difficulties) the integral $\int \exp H_J \bar{P}(dz)$ by the cumulant formula. The a priori error would be however:

$$\text{cost} \exp(\max |H_J|)^{t+1} \tag{5.1}$$

which is, of course, too large ("wrong $|I|$ dependence").

If the z_Δ's were independent variables (rather than almost such) and if H_J were "strictly local", i.e. $H_J = \sum_\Delta H_\Delta$ with H_Δ depending only on z_Δ (rather than almost such) we could write the integral as

$$\prod_\Delta \left(\int \exp H_\Delta \bar{P}(dz)\right) \tag{5.2}$$

and then apply to each factor the cumulant formula with an error:

$$\exp \sum_\Delta (\max |H_\Delta|)^{t+1} \tag{5.3}$$

which is much better than (5.1) and is precisely what we want.

The fact that $\bar{P}$ does not factorize will be cured by collecting many Δ's into large boxes $\square$, still very small compared to I. Then we shall fix the values of the z_Δ variables for the Δ's near the boundaries of $\square$ and call them $\bar{z}$. The measure $\bar{P}$, conditioned to the fixed values $\bar{z}$ will then factorize "over the boxes $\square$" because of the Markov property of $\bar{P}$. If the boxes are large the non locality of H_J will be negligeable and we shall perform the conditional integral by the cumulant formula to order t making an error of the type (5.3) with $\sum_\Delta$ replaced by $\sum_\square$. The result will unfortunately depend on the conditions $\bar{z}$. It will in fact have the form of a linear combinations of terms of the form:

$$\mathscr{E}_{\bar{z}}^T(z_{\Delta_1}, \ldots, z_{\Delta_p}; n_1, \ldots, n_p) \tag{5.4}$$

where $\mathscr{E}_{\bar{z}}^T$ denotes the truncated expectation with respect to the conditioned measure. Such expectations are polynomials in the $\bar{z}$ and differ very little from the ones we want (i.e. the unconditional ones) if $\Delta_1, \ldots, \Delta_p$ are far from the region $\bigcup_\square (\partial \square)$ because the covariance of the z_Δ's decays exponentially and, far from $\bigcup_\square (\partial \square)$, coincides with the unconditional covariance. This remark shows that the above procedure has reduced the problem of proving the lemma to the special case in which J is replaced by $J \cap \Gamma_1$ where Γ_1 is a region around $\bigcup_\square (\partial \square)$ with width of the order of the maximum between the correlation length of the z_Δ-covariance and the range $\varkappa^{-1}$ of the "hamiltonian" H_J.

The location in space of the $\square$'s was however arbitrary. Hence we can apply the same argument of $H_{J \cap \Gamma_1}$ by choosing the $\square$'s out of a pavement with boxes of the same size of the former ones but shifted in location.

In this way the initial problem is reduced to the case in which J is replaced by $(J \cap \Gamma_1) \cap \Gamma_2$ where Γ_2 are the new corridors. After $(d+1)$ steps one can obviously manage by suitably choosing the successive displaced pavements so that $(J \cap \Gamma_1) \cap \Gamma_2 \cap \ldots \cap \Gamma_{d+1} = \emptyset$ thereby reducing the proof of the lemma to the trivial case $H_J = 0$.

The proof that we give here is different from the analogous result of [2] and closer in spirit to the general methods of Statistical Mechanics [4, 5].

The technique discussed seems close to the one used in the theory of the critical point of the almost gaussian Ising model [6].

Proof of the Lemma. Throughout the proof C, I, and J are fixed. Let R be a region paved by Q_0 and let

$$H_R = \sum_{1}^{S}{}_p \sum_{\substack{\Delta_i \subset Q_0 \\ \Delta_i \subset R \\ i: 1 \ldots p}} \sum_{\substack{n_i > 0 \\ \sum_i n_i \leq D \\ i: 1 \ldots p}} A^{n_1 \ldots n_p}_{\Delta_1 \ldots \Delta_p} e^{-\frac{\varkappa}{2} d(\Delta_1 \ldots \Delta_p)} z^{n_1}_{\Delta_1} \ldots z^{n_p}_{\Delta_p} \tag{5.5}$$

where the definitions of the A's is extended so that

$$A^{n_1 \ldots n_p}_{\Delta_1 \ldots \Delta_p} = 0 \text{ if some } \Delta_i \not\subset J, \quad i = 1, \ldots, p.$$

Given two different regions R and S (paved by Q_0) we define the interaction between R and S as

$$H_{R,S} = H_{R \cup S} - H_R - H_S. \tag{5.6}$$

We consider tesserae $\square$ paved by Q_0 of side b^2 (for simplicity we assume $b^{1/2}/4$ integer, the modifications needed in the general case are trivial and will not be considered). Let Q^b be the corresponding pavement made up by the tesserae $\square$. For any $\square \in Q^b$ we put

$$\square = \square' \cup \Gamma_2(\square) \cup \Gamma_1(\square) \tag{5.7}$$

where $\Gamma_2(\square)$ and $\Gamma_1(\square)$ are corridors of width $b^{3/2}$, $\Gamma_1(\square)$ is adjacent to the boundary of $\square$ and $\Gamma_2(\square)$ is adjacent to the internal boundary of $\Gamma_1(\square)$ (see Fig. 1).

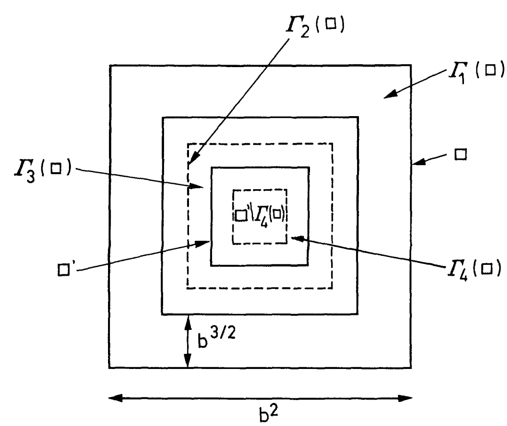

Fig. 1

We define

$$\Gamma_1 = \bigcup_{\square \in Q^b} \Gamma_1(\square) \tag{5.8}$$

and we have

$$H_J = H_{\Gamma_1} + \sum_{\substack{\square \\ \square \cap J \neq \emptyset}} \Psi_\square + H^{(l)} \equiv \hat{H}_J + H^{(l)} \tag{5.9}$$

$$\Psi_\square = H_{\square' \cup \Gamma_2(\square)} + H_{\Gamma_2(\square), \Gamma_1(\square)} \tag{5.10}$$

where $H^{(l)} = H_J - \hat{H}_J$ can be bounded as [see (4.5)]

$$\left| H^{(l)} \prod_\Delta \hat{\chi}_\Delta \right| \leqq s_1 A b^D e^{-\frac{\varkappa}{4} b^{3/2}} |I| . \tag{5.11}$$

Therefore the main point is to estimate

$$\int \bar{P}(dz) \prod_\Delta \hat{\chi}_\Delta \exp \hat{H}_J . \tag{5.12}$$

Let us first prove a lower bound for (5.12); in this case $C = \emptyset$. We have

$$\begin{aligned} [(5.12)] = \int \bar{P}(dz_{\Gamma_1}) \prod_{\Delta \in \Gamma_1} \hat{\chi}_\Delta \exp H_{\Gamma_1} \left(\prod_{\square \cap J \neq \emptyset} \int \bar{P}(dz_\square | z_{\Gamma_1}) \prod_{\Delta \subset \square} \hat{\chi}_\Delta \exp \Psi_\square \right) \\ \cdot \left(\prod_{\square \cap J = \emptyset} \int \bar{P}(dz_\square | z_{\Gamma_1}) \prod_{\Delta \subset \square} \hat{\chi}_\Delta \right) \end{aligned} \tag{5.13}$$

where $\bar{P}(dz_{\Gamma_1})$ denotes the probability distribution of the r.v. $(z_\Delta)_{\Delta \subset \Gamma_1} \equiv z_{\Gamma_1}$, $z_\square$ are the r.v. $(z_\Delta)_{\Delta \in \square}$, $\bar{P}(dz_\square | z_{\Gamma_1})$ is their conditional probability for fixed z_{Γ_1} and the Markov property of $\bar{P}$ has been used.

We choose $0 < \gamma < 1$ as in Lemma 2 of Appendix C $\left(\gamma = \frac{1}{2} \left(\frac{2d}{\alpha^2 + 2d} \right)^2 \right)$, and we define

$$\begin{aligned} \chi_{\gamma b}^{\Gamma_1} &= \chi(|z_\Delta| \leqq \gamma b (1 + d(\Delta, I)), \forall \Delta \in \Gamma_1) \\ \chi_b^\square &= \chi(|z_\Delta| \leqq b(1 + d(\Delta, I)), \forall \Delta \in \square' \cup \Gamma_2(\square)) . \end{aligned} \tag{5.14}$$

Then

$$\begin{aligned} [(5.12)] &\geqq \int \bar{P}(dz_{\Gamma_1}) \chi_{\gamma b}^{\Gamma_1} \exp H_{\Gamma_1} \left(\prod_{\square \cap J = \emptyset} \int \bar{P}(dz_\square | z_{\Gamma_1}) \chi_b^\square \right) \\ &\quad \cdot \left(\prod_{\square \cap J \neq \emptyset} \int \bar{P}(dz_\square | z_{\Gamma_1}) \chi_b^\square \exp \Psi_\square \chi_b^\square \right) \\ &\geqq \int \bar{P}(dz_{\Gamma_1}) \chi_{\gamma b}^{\Gamma_1} \exp H_{\Gamma_1} \left(\prod_{\square \cap J = \emptyset} \int \bar{P}(dz_\square | z_{\Gamma_1}) \chi_b^\square \right) \prod_{\square \cap J \neq \emptyset} \\ &\quad \cdot \left\{ \int \bar{P}(dz_\square | z_{\Gamma_1}) e^{\Psi_\square \chi_b^\square} \exp \left[-3 b^{2d} e^{-\frac{b^2}{4}} e^{2 S_2 A b^{D+2d}} \right] \right\} \end{aligned} \tag{5.15}$$

where the ch.f. in the second integral have been eliminated as follows:

i) There is a constant s_2 such that $|\Psi_\square \chi_b^\square| \leqq s_2 A b^{D+2d}$ as it follows from the assumed structure of H_J.

ii) we write

$$-\log\left[\frac{\int \bar{P}(dz_\square|z_{\Gamma_1})\chi^b_\square e^{\Psi_\square \chi^b_\square}}{\int \bar{P}(dz_\square|z_{\Gamma_1})e^{\Psi_\square \chi^b_\square}}\right] = \int_1^\infty du \frac{d}{du}(\log \int \bar{P}(dz_\square|z_{\Gamma_1})\chi^{bu}_\square e^{\Psi_\square \chi^b_\square})$$

$$= \int_1^\infty du \frac{\int \bar{P}(dz_\square|z_{\Gamma_1})\left(\frac{d}{du}\chi^{bu}_\square\right)e^{\Psi_\square \chi^b_\square}}{\int \bar{P}(dz_\square|z_{\Gamma_1})\chi^{bu}_\square e^{\Psi_\square \chi^b_\square}}. \tag{5.16}$$

Since $\frac{d}{du}\chi^{bu}_\square$ is a combination with non negative coefficients of δ-functions, and therefore is positive we can extract the factors $\exp \Psi_\square \chi^b_\square$ and perform again the integral:

$$[(5.16)] \leqq \exp 2s_2 A b^{D+2d}(-\log \int \bar{P}(dz_\square|z_{\Gamma_1})\chi^b_\square). \tag{5.17}$$

Hence

$$\int \bar{P}(dz_\square|z_{\Gamma_1})\chi^b_\square \exp \Psi_\square \chi^b_\square \geqq (\int \bar{P}(dz_\square|z_{\Gamma_1})\exp \Psi_\square \chi^b_\square)$$
$$\cdot (\int \bar{P}(dz_\square|z_{\Gamma_1})\chi^b_\square)^{\exp 2s_2 A b^{D+2d}}. \tag{5.18}$$

Finally we use the lemma on the free field (see Appendix C)

$$\int \chi^b_\square \bar{P}(dz_\square|z_{\Gamma_1}) \geqq \exp\left(-3b^{2d}e^{-\frac{b^2}{4}}\right) \tag{5.19}$$

to derive the bound used in (5.15)

$$\int \bar{P}(dz_\square|z_{\Gamma_1})\chi^b_\square \exp \Psi_\square \chi^b_\square \geqq |\int \bar{P}(dz_\square|z_{\Gamma_1})\exp \Psi_\square \chi^b_\square$$
$$\cdot \exp\left(-3b^{2d}e^{-\frac{b^2}{4}}e^{2s_2 A b^{D+2d}}\right). \tag{5.20}$$

We now try to compute the second integral by a cumulant expansion [see (2.7)] to order t:

$$\log \int \bar{P}(dz_\square|z_{\Gamma_1})\exp \Psi_\square \chi^b_\square = \sum_1^t k \frac{\bar{\mathscr{E}}^T_{z_{\Gamma_1}}(\Psi_\square \chi^b_\square; k)}{k!}$$
$$+ \frac{2^{(t+1)^2}(t+1)!}{(t+1)!}\tau(s_2 b^{D+2d}A)^{t+1}\exp 2(s_2 A b^{D+2d}) \tag{5.21}$$

where $\bar{\mathscr{E}}_{z_{\Gamma_1}}$ denotes the expectation with respect to $\bar{P}(dz_\square|z_{\Gamma_1})$ and τ is a function with values in $[-1, 1]$.

Combining (5.21), (5.20) and (5.15)

$$[(5.12)] \geqq \int \bar{P}(dz_{\Gamma_1})\chi^{\Gamma_1}_{\gamma b}\exp H_{\Gamma_1}$$
$$\cdot \left(\prod_{\square \cap J = \emptyset} \int \bar{P}(dz_\square|z_{\Gamma_1})\chi^\square_b\right) \prod_{\square \cap J \neq \emptyset}\left\{\exp \sum_1^t {}_k \frac{1}{k!}\bar{\mathscr{E}}^T_{z_{\Gamma_1}}(\Psi_\square \chi^b_\square; k)\right.$$
$$\left.\cdot \exp - \left[\frac{2^{(t+1)^2}(t+1)!}{(t+1)!}(s_2 b^{D+2d}A)^{t+1}e^{2s_2 A b^{D+2d}} + 3b^{2d}e^{-\frac{b^2}{4}}e^{2s_2 A b^{D+2d}}\right]\right\}. \tag{5.22}$$

The next step consists of trying to replace the conditional expectation $\mathscr{E}_{z\Gamma_1}$ by the unconditioned one $\hat{\mathscr{E}}_0$.

Consider $\mathscr{E}^T_{z\Gamma_1}(\Psi_\Box \chi_b^\Box; k)$ and decompose $\Psi_\Box$ as

$$\Psi_\Box = (H_{\Box'} + H_{\Box', \Gamma_3(\Box)}) + (H_{\Gamma_1(\Box), \Gamma_2(\Box)} + H_{\Gamma_2(\Box)}) + H'^{(l)} \tag{5.23}$$

where $\Gamma_3(\Box)$ is a corridor adjacent to $\Box'$ with width $\frac{1}{2}b^{3/2}$ [hence $\Gamma_3(\Box) \subset \Gamma_2(\Box)$] (see Fig. 1) and $H'^{(l)}$ can be bounded by

$$|H'^{(l)}\chi_b^\Box| \leqq s_1 A b^{D+2d} e^{-\frac{\varkappa}{8}b^{3/2}}. \tag{5.24}$$

We denote the three addends in (5.23) Ψ_1, Ψ_2, Ψ_3 respectively and $\chi_b^\Box$ by χ then

$$\begin{aligned}
&\mathscr{E}^T_{z\Gamma_1}((\Psi_1 + \Psi_2 + \Psi_3)\chi; k) \\
&= \sum_{\substack{k_1, k_2, k_3 \\ (k_1+k_2+k_3)=k}} \frac{k!}{k_1!k_2!k_3!} \mathscr{E}^T_{z\Gamma_1}(\Psi_1\chi, \Psi_2\chi, \Psi_3\chi; k_1, k_2, k_3) \\
&= \mathscr{E}^T_{z\Gamma_1}(\Psi_1\chi; k) + \sum_{k_2>0} \mathscr{E}^T_{z\Gamma_1}(\Psi_1\chi, \Psi_2\chi, k_1, k_2)\frac{k!}{k_1!k_2!} \\
&\quad + \sum_{k_3>0} \mathscr{E}^T_{z\Gamma_1}(\Psi_1\chi, \Psi_2\chi, \Psi_3\chi; k_1, k_2, k_3)\frac{k!}{k_1!k_2!k_3!}
\end{aligned} \tag{5.25}$$

and we have used the elementary summation properties of the truncated functions following from their definitions ("Leibnitz formula").

We bound the third term by using that $\mathscr{E}^T_{z\Gamma_1}$ is a sum of at most s_3 products of powers of $\Psi_1\chi$, $\Psi_2\chi$, $\Psi_3\chi$ and $\Psi_3\chi$ appears in at least one of the factors.

Then (5.24) implies:

$$\left|\sum_{k_3>0} \mathscr{E}^T_{z\Gamma_1}(\Psi_1\chi, \Psi_2\chi, \Psi_3\chi; k_1, k_2, k_3)\frac{k!}{k_1!k_2!k_3!}\right| \leqq s_3(s_1 A b^{D+2d})^k e^{-\frac{\varkappa}{8}b^{3/2}}. \tag{5.26}$$

To treat the second term we define $\Gamma_4(\Box)$ as the corridor adjacent to the boundary of $\Box'$ contained inside $\Box'$ and with width $\frac{b^{3/2}}{2}$ and decompose Ψ_1 as $\Psi_1' + \Psi_1''$ where

$$\Psi_1' = H_{\Gamma_4(\Box)} + H_{\Gamma_4(\Box), \Gamma_3(\Box)}. \tag{5.27}$$

The feature of Ψ_1'' to be retained is that in its expression as a polynomial in the z_Δ's each monomials contains at least one z_Δ with $\Delta \subset \Box' \backslash \Gamma_4(\Box)$. Then the second term in (5.25) can be written

$$\sum_{k_2>0} \frac{k!}{k_1!k_2!} \mathscr{E}^T_{z\Gamma}(\Psi_1'\chi, \Psi_2\chi; k_1, k_2) + \sum_{\substack{k_2>0 \\ h_1, h_2 \\ h_2>0 \\ h_1+h_2=k-k_2}} \mathscr{E}^T_{z\Gamma_1}(\Psi_1'\chi, \Psi_1''\chi, \Psi_2\chi; h_1, h_2, k_2) \tag{5.28}$$

and the properties of the free field allow to bound the second term of this sum as

$$s_4\left((s_2 b^{D+2d}A)^k e^{-\frac{b^2}{4}} + (s_2 b^{D+2d}A)^k e^{-\tilde{\varkappa} b^{3/2}}\right) \tag{5.29}$$

where the first term comes from the replacement of the χ's by 1, the second from the properties of the conditioned measure and from the Wick theorem (see Appendix D) and s_k and $\tilde{\varkappa}$ are positive constants.

We can summarize the above calculations in the following relation:

$$[(5.12)] \geqq \exp(\text{error}) \int \bar{P}(dz_{\Gamma_1}) \chi_{\gamma b}^{\Gamma_1} \exp H_{\Gamma_1} \Big(\prod_{\square \cap J = \emptyset} \int \bar{P}(dz_\square | z_{\Gamma_1}) \chi_b^\square \Big)$$
$$\prod_{\square \cap J \neq \emptyset} \Bigg\{ \exp \Bigg[\sum_1^t \frac{1}{k!} \mathscr{E}_{z_{\Gamma_1}}^T (\chi_b^\square (H_{\square'} + H_{\square', \Gamma_3(\square)}); k) + \sum_1^t {}_k \sum_{\substack{k_1, k_2 \\ k_1 + k_2 = k \\ k_2 > 0}} \frac{1}{k_1! k_2!}$$
$$\cdot \mathscr{E}_{z_{\Gamma_1}}^T (\chi_b^\square (H_{\Gamma_4(\square)} + H_{\Gamma_4(\square), \Gamma_3(\square)}), \chi_b^\square (H_{\Gamma_2(\square)} + H_{\Gamma_1(\square), \Gamma_2(\square)}); k_1, k_2) \Bigg] \Bigg\}. \tag{5.30}$$

The error has the same form as the one appearing in the text of the lemma [Equation (4.6)] with different values of the constants (which can be desumed from the above calculation).

We now write

$$\mathscr{E}_{z_{\Gamma_1}}^T (\chi_\square^b (H_{\square'} + H_{\square', \Gamma_3(\square)}); k)$$
$$= \mathscr{E}_{z_{\Gamma_1}}^T (\chi_\square^b (H_{\Gamma_4(\square)} + H_{\Gamma_4(\square), \Gamma_3(\square)}); k)$$
$$+ \sum_{\substack{k_1 + k_2 = k \\ k_1 > 0}} \frac{k!}{k_1! k_2!} \mathscr{E}_{z_{\Gamma_1}}^T (\chi_\square^b (H_{\square'} + H_{\square', \Gamma_3(\square)} - H_{\Gamma_4(\square)} - H_{\Gamma_4, (\square), \Gamma_3(\square)}),$$
$$\chi_\square^b (H_{\Gamma_4(\square)} + H_{\Gamma_4(\square), \Gamma_3(\square)}); k_1, k_2)$$
$$= \mathscr{E}_{z_{\Gamma_1}}^T (\chi_\square^b (H_{\Gamma_4(\square)} + H_{\Gamma_4(\square), \Gamma_3(\square)}); k)$$
$$+ \sum_{\substack{k_1 + k_2 = k \\ k_1 > 0}} \frac{k!}{k_1! k_2!} \hat{\mathscr{E}}_0^T ((H_{\square'} + H_{\square, \Gamma_3(\square)}) - H_{\Gamma_4(\square)} - H_{\Gamma_4(\square), \Gamma_3(\square)},$$
$$H_{\Gamma_4(\square)} + H_{\Gamma_4(\square), \Gamma_3(\square)}; k_1, k_2) + (\text{error}) \tag{5.31}$$

and the error can be studied along the same lines of the argument leading to the bound (5.30) and has the same form of (5.29) with new constants.

Hence

$$[(5.12)] \geqq \exp(\text{error}) \int \bar{P}(dz_{\Gamma_1}) \chi_{\gamma b}^{\Gamma_1} \Big(\prod_{\square \cap J = \emptyset} \int \bar{P}(dz_\square | z_{\Gamma_1}) \chi_b^\square \Big)$$
$$\cdot \exp \Bigg[\sum_1^t {}_k \sum_{\substack{k_1 + k_2 = k \\ k_1 > 0}} \frac{1}{k_1! k_2!} \sum_\square \hat{\mathscr{E}}_0^T (\Psi_1'', \Psi_1'; k_1, k_2) \Bigg]$$
$$\cdot \exp H_{\Gamma_1} \prod_{\square \cap J \neq \emptyset} \Bigg\{ \exp \Bigg[\sum_1^t {}_k \sum_{k_1 + k_2 = k} \frac{1}{k_1! k_2!} \mathscr{E}_{z_{\Gamma_1}}^T (\chi_\square^b (H_{\Gamma_4(\square)}$$
$$+ H_{\Gamma_4(\square), \Gamma_3(\square)}), \chi_\square^b (H_{\Gamma_2(\square)} + H_{\Gamma_1(\square), \Gamma_2(\square)}); k_1, k_2) \Bigg] \Bigg\}. \tag{5.32}$$

Each factor in the product sign can be reexpressed using the cumulant formula backwards as

$$\int \bar{P}(dz_{\square}|z_{\Gamma_1})\chi^b_{\square}\exp(H_{\Gamma_2(\square)\cup\Gamma_4(\square)}+H_{\Gamma_2(\square),\Gamma_1(\square)}-H_{\Gamma_4(\square),\Gamma_2(\square)\setminus\Gamma_3(\square)})$$

$$\cdot\exp\left\{\tau\left[\frac{2^{(t+1)^2}(t+1)!}{(t+1)!}(s_2 b^{D+2d}A)^{t+1}\exp 2(s_2 A b^{D+2d})\right]\right\} \tag{5.33}$$

where as usual τ is a function with values in $[-1,1]$.

We now bound $H_{\Gamma_4(\square),\Gamma_2(\square)\setminus\Gamma_3(\square)}$ as usual [see (5.24)] use the Markov property of $\bar{P}$ and remark that if $\bar{\Gamma}_1=\bigcup_{\square\in Q^b}[\Gamma_1(\square)\cup\Gamma_2(\square)\cup\Gamma_4(\square)]$

$$H_{\Gamma_1}+\sum_{\square\in Q^b}(H_{\Gamma_2(\square)\cup\Gamma_4(\square)}+H_{\Gamma_2(\square),\Gamma_1(\square)})=H_{\bar{\Gamma}_1}+\tau|I|\left(s_1 A b^D e^{-\frac{\varkappa}{8}b^{3/2}}\right) \tag{5.34}$$

and obtain

$$[(5.12)]\geqq\left(\int\bar{P}(dz)\prod_{\square\in Q^b}\chi^{\square}_{\gamma b}\exp H_{\bar{\Gamma}_1}\right)$$

$$\cdot\exp\left\{\sum_{\substack{k_1+k_2\leqq t\\ k_1>0}}\frac{1}{k_1!\,k_2!}\sum_{\square}\hat{\mathscr{E}}^T_0(\Psi''_1,\Psi'_1;k_1,k_2)\right\}\exp(\text{error}) \tag{5.35}$$

where the error can be described as after (5.31); Now we study the integral and remark that it has the same structure as (5.12) with J replaced by $\bar{\Gamma}_1$ and b by γb. Therefore we can proceed as before choosing a new pavement Q'^b displaced by $b^2/_2$ with respect to Q^b obtaining a result similar to (5.31) with $\bar{\Gamma}_1$ replaced by a new set $\bar{\Gamma}'_1$ (much smaller than $\bar{\Gamma}_1$). Again one proceeds as before by choosing a third pavement Q''^b displaced by $b^2/_4$ with respect to Q'^b and, if $d=3$, a fourth one displaced by $b^2/_{16}$ w.r.t. Q''^b. Collecting all the errors made in this process (4.7) is proven.

We now study the second estimate (4.6) which is obtained by simple modifications of the first one (4.7). We start from (5.13), then assuming $|z_\Delta|\leqq b(1+d(\Delta,I)),\ \forall\Delta\in C$

$$[(5.12)]\leqq\int\bar{P}(dz_{\Gamma_1})\chi^{\Gamma_1}_b\exp H_{\Gamma_1}\left(\prod_{\square\cap J=\emptyset}\int\bar{P}(dz_{\square}|z_{\Gamma_1})\chi^{\square}_b\right)$$

$$\cdot\left(\prod_{\square\cap J\neq\emptyset}\int\bar{P}(dz_{\square}|z_{\Gamma_1})\exp\Psi_{\square}\chi^{\square}_b\right). \tag{5.36}$$

By beeing careful in attributing to the various τ the value opposite to the one chosen in the estimate (4.7) we obtain that (5.12) can be bounded above by the r.h.s. of (5.35) with b replaced by $\gamma^{-1}b$.

It follows that b^* can be chosen $=\max\{10^4,\gamma^{-3}\bar{b}\}$ where $\bar{b}$ and γ are defined in Appendix A and C respectively.

6. Concluding Remarks

1) The interest of the above approach to the ultraviolet stability is the clear distinction of the role plaid by formal perturbation theory.

The ultraviolet stability follows by standard arguments in the statistical mechanics of Markov processes (§5). The applicability of such arguments is guaranteed by formal renormalization theory [Sec. 4 and (3.21), (3.22), (3.23)].

The scale invariance of the free field reduces the statistical mechanics part of the argument to a lemma on a standard Ising model on a unit lattice.

2) The technique used can be clearly extended to prove "perturbation theory" in the following sense.

Define

$$V_I'^{(N)} = \left(V_{0,I}^{(N)} - \frac{\langle (V_{0,I}^{(N)})^2 \rangle_{(2)}}{2!} \right) \tag{6.1}$$

and, if $\mathscr{E}$ denotes the expectation with respect to P_N [c.f.r. (1.6)]

$$V_I^{(N,t)} = \left[V_I'^{(N)} - \sum_{k=1}^{t} \frac{\mathscr{E}^T((V_I'^{(N)});k)}{k!} \right]_{(t)} . \tag{6.2}$$

Then there exists $E_\pm(\lambda, t)$ such that for $t \geqq 3$

$$\text{i)} \quad \exp - E_-(\lambda, t)|I| \leqq \int \exp V_I^{(N,t)} dP_N \leqq \exp E_+(\lambda, t)|I| \tag{6.3}$$

$$\text{ii)} \quad \lim_{\lambda \to 0} E_\pm(\lambda, t)\lambda^{-t} = 0 . \tag{6.4}$$

Notice that the renormalization theory implies that all the terms in the sum in (6.2) with $k \geqq 4$ are finite.

3) If $d=2$ the condition $t \geqq 3$ can be dropped and also $V_I'^{(N)}$ could be replaced by $V_{0,I}^{(N)}$.

4) The above techniques allow to study the integrals:

$$\frac{\int (\exp z_\Delta \omega) \exp V_I^{(N)} dP_N}{\int \exp V_I^{(N)} dP_N} \tag{6.5}$$

and to derive for the moments of the z_Δ's bounds of the type of the ones found by Feldman [3].

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Appendix A

We define, for $\Delta \in Q_0$, $\hat{\chi}_\Delta^b = \chi(|z_\Delta| \leqq b(1 + d(\Delta, I)) \equiv b_\Delta)$ and prove the following lemma:

Lemma. *Let* $\hat{P}_0(dz)$ *be defined as before* [Eq. (1.6)]. *Then there exist* $\bar{b}$, k_1, k_2, *such that, for* $b > \bar{b}$

$$F_b \equiv \int \hat{P}_0(dz) \prod_{\Delta \in Q_0} \hat{\chi}_\Delta^b \geqq \exp[-|I| e_b] \tag{A.1}$$

$$e_b = k_1 \exp(-k_2 b^2). \tag{A.2}$$

Proof.

$$\log F_b = -\int_1^\infty \frac{d}{du} \log F_{bu} du = -\sum_\Delta b_\Delta \int_1^\infty du \frac{\mathscr{E}_{\Delta,u}(\delta(|z_\Delta| - ub_\Delta))}{\mathscr{E}_{\Delta,u}(\hat{\chi}_\Delta^{u,b})} \tag{A.3}$$

where $\mathscr{E}_{\Delta,u}$ denotes the expectation w.r.t. the distribution of the variables z_Δ in the random field with formal density

$$\text{const} \exp\left\{-\frac{\beta}{2}\left[\sum_{\Delta,\Delta'}^* (z_\Delta - z_{\Delta'})^2 + \alpha^2 \sum_{\Delta'} z_{\Delta'}^2\right]\right\} \prod_{\Delta \neq \Delta'} \hat{\chi}_{\Delta'}^{bu}. \tag{A.4}$$

We notice that this random field is superstable with superstability parameters (see [5] for the notations) $A = \frac{\beta\alpha^2}{2}$, $B = 0$ and free density 1 for the variable z_Δ and $\hat{\chi}_{\Delta'}^{ub}$, for the variables $z_{\Delta'}$, $\Delta' \neq \Delta$.

Supposing that $\bar{b} \geqq 1$, we remark that the parameter $\lambda = \int_0^1 d$ (free measure) $= 1$; hence from Ref. [5] it follows that the variable z_Δ in the field (A.4) has density bounded by

$$\exp\left\{-\frac{\beta\alpha^2}{4} z_\Delta^2 + \delta(\alpha,\beta)\right\} \tag{A.5}$$

where $\delta(\alpha,\beta) = 0$. By inserting this expression in (A.3) the lemma follows immediately and we can take for instance:

$$\bar{b} = \max\left\{1, \frac{8}{\beta\alpha^2} \log\left(\frac{e^\delta 4\sqrt{2\pi}}{\alpha\sqrt{\beta}}\right)\right\}$$

$$k_1 = \frac{4}{\alpha}\sqrt{\frac{2\pi}{\beta}} e^\delta \sum_{u=0}^{\infty} 2^{du} e^{-\beta\frac{\alpha^2 u^2}{8}} \tag{A.6}$$

$$k_2 = \frac{\beta\alpha^2}{8}.$$

Since $\{|x_\Delta^{(k)}| \leqq B_k(1 + 2^k d(I,\Delta)),\ \Delta \in Q_k,\ k: 0, 1, \ldots N\}$ implies [see (1.5)] that

$$|z_\Delta| \leqq (\sqrt{1+\Gamma_k} B_k - \sqrt{\Gamma_k} B_{k-1})(1 + 2^k d(\Delta, I)), \quad \Delta \in Q_k;\ k = 0, 1, \ldots, N \tag{A.7}$$

it is clear that the above lemma implies (2.5) and one can take

$$e(\lambda) = k_1 \sum_{k=0}^{\infty} e^{-k_2(\sqrt{1+\Gamma_k} B_k - \sqrt{\Gamma_k} B_{k-1})^2}. \tag{A.8}$$

Appendix B

The following expressions are easily proven by induction combined with simple algebra on Wick polynomials (see for example [7]).

One finds, if $1 \leqq k \leqq N$:

$$\begin{aligned}\tilde{V}_I^{(N-k)} &= V_I^{(N-k)} + \hat{\mathscr{E}}_{N-k+1}\hat{\mathscr{E}}_{N-k+2}\dots\hat{\mathscr{E}}_N\left\{\sum_{N-k}^{N-1}{}_i\,(W_i+F_i+H_i)\right.\\ &\quad\left.+\sum_{N-k+1}^{N-1}{}_i\,[W_i(V_{0,I}^{(i)}-V_{0,I}^{(N-k)})-\mathscr{E}(W_i(V_{0,I}^{(i)}-V_{0,I}^{(N-k)}))]\right\}\\ &\quad-\mathscr{E}\left(V_{0,I}^{(N-k)}\sum_{N-k}^{N-1}{}_i\,W_i\right)\\ &\equiv V_I^{(N-k)}+W_I^{(N-k)}+F_I^{(N-k)}+H_I^{(N-k)}+G_I^{(N-k)}+C_I^{(N-k)}\end{aligned} \tag{B.1}$$

where $(C_N(\xi,\eta)\equiv C_{\xi\eta}^{[\leqq N]},\ \varphi_N(\xi)\equiv\varphi_\xi^{[\leqq N]})$,

$$\begin{aligned}W_i &= \frac{\lambda^2}{2}\sum_1^3{}_l\binom{4}{l}^2(4-l)!\iint_{I\times I}d\xi d\eta[C_{i+1}^{4-l}(\xi,\eta)-C_i^{4-l}(\xi,\eta)]\\ &\quad\cdot[:\varphi_i^l(\xi)\varphi_i^l(\eta):-\delta_{l1}:\varphi_i^{2l}(\xi):],\end{aligned} \tag{B.2}$$

$$\begin{aligned}F_i &= -\lambda^3\sum_{r_1r_2r_3}d_{r_1r_2r_3}\iiint_{I\times I\times I}d\xi d\eta d\zeta\{C_{i+1}^{s_{12}}(\xi,\eta)C_{i+1}^{s_{23}}(\eta\zeta)C_{i+1}^{s_{31}}(\zeta\xi)\\ &\quad-3C_{i+1}^{s_{12}}(\xi,\eta)C_i^{s_{23}}(\eta\zeta)C_i^{s_{31}}(\zeta\xi)+2C_i^{s_{12}}(\xi,\eta)C_i^{s_{23}}(\eta,\zeta)C_i^{s_{31}}(\zeta,\xi)\}\\ &\quad\cdot[:\varphi_i^{r_1}(\xi)\varphi_i^{r_2}(\eta)\varphi_i^{r_3}(\zeta):],\end{aligned} \tag{B.3}$$

$$d_{r_1r_2r_3}=\begin{cases}\dfrac{1}{3!}\dbinom{4}{r_1}\dbinom{4}{r_2}\dbinom{4}{r_3}\dbinom{4-r_1}{s_{12}}\dbinom{4-r_2}{s_{23}}\dbinom{4-r_3}{s_{31}}s_{12}!s_{23}!s_{31}!\\ \text{if}\quad r_1+r_2+r_3=2r;\quad 1\leqq r\leqq 4;\quad r_i\leqq 3,\quad i=1,2,3\\ 0\quad\text{otherwise},\end{cases} \tag{B.4}$$

$$\begin{cases}S_{12}+S_{23}=4-r_2\\ S_{23}+S_{31}=4-r_3\\ S_{31}+S_{12}=4-r_1,\end{cases} \tag{B.5}$$

$$\begin{aligned}H_i &= \frac{\lambda^3}{2}\sum_0^1{}_p 4^2 3!\binom{2}{p}\binom{4}{p}(2-p)!\iiint_{I\times I\times I}d\xi d\eta d\zeta[C_{i+1}^{2-p}(\xi,\eta)\\ &\quad-C_i^{2-p}(\xi,\eta)]C_{i+1}^3(\eta,\zeta):\varphi_i^{p+2}(\xi)\varphi_i^p(\eta):,\end{aligned} \tag{B.6}$$

$$\begin{aligned}W_I^{(N-k)} &= \frac{\lambda^2}{2}\sum_1^3{}_l\binom{4}{l}^2(4-l)!\sum_{N-k+1}^{N}{}_i\iint_{I\times I}d\xi d\eta[C_i^{4-l}(\xi,\eta)-C_{i-1}^{4-l}(\xi,\eta)]\\ &\quad\cdot[:\varphi_{N-k}^l(\xi)\varphi_{N-k}^l(\eta):-\delta_{l,1}:\varphi_{N-k}^{2l}(\xi):],\end{aligned} \tag{B.7}$$

$$\begin{aligned}F_I^{(N-k)} &= -\lambda^3\sum_{r_1r_2r_3}d_{r_1r_2r_3}\sum_{N-k+1}^{N}{}_i\iiint_{I\times I\times I}d\xi d\eta d\zeta\{C_i^{s_{12}}(\xi,\eta)C_i^{s_{23}}(\eta,\zeta)\\ &\quad\cdot C_i^{s_{31}}(\zeta,\xi)-3C_i^{s_{12}}(\xi,\eta)C_{i-1}^{s_{23}}(\eta\zeta)C_{i-1}^{s_{31}}(\zeta,\xi)+2C_{i-1}^{s_{12}}(\xi,\eta)\\ &\quad\cdot C_{i-1}^{s_{23}}(\eta,\zeta)C_{i-1}^{s_{31}}(\zeta,\xi)\}:\varphi_{N-k}^{r_1}(\xi)\varphi_{N-k}^{r_2}(\eta)\varphi_{N-k}^{r_3}(\zeta):,\end{aligned} \tag{B.8}$$

$$H_I^{(N-k)}=\frac{\lambda^3}{2}\sum_{0}^{1}{}_p\, 4^2 3!\binom{2}{p}\binom{4}{p}(2-p)!\sum_{N-k+1}^{N}{}_i \iiint_{I\times I\times I} d\xi d\eta d\zeta$$
$$\cdot[C_i^{2-p}(\xi,\eta)-C_{i-1}^{2-p}(\xi,\eta)]C_i^3(\eta,\zeta):\varphi_{N-k}^{p+2}(\xi)\varphi_{N-k}^p(\eta):, \tag{B.9}$$

$$G_I^{(N-k)}=-3\lambda^3\sum_{r_1r_2r_3} d_{r_1r_2r_3}\sum_{N-k+1}^{N-1}{}_i \iiint_{I\times I\times I} d\xi d\eta d\zeta[C_{i+1}^{s_{12}}(\xi,\eta)$$
$$-C_i^{s_{12}}(\xi,\eta)]\{[C_i^{s_{23}}(\eta,\zeta)C_i^{s_{31}}(\zeta,\xi)-C_{N-k}^{s_{23}}(\eta,\zeta)C_{N-k}^{s_{31}}(\zeta,\xi)]$$
$$\cdot[:\varphi_{N-k}^{r_1}(\xi)\varphi_{N-k}^{r_2}(\eta)\varphi_{N-k}^{r_3}(\zeta):]-\delta_{3,s_{12}}[C_i^{s_{23}+s_{31}}(\eta,\zeta)$$
$$-C_{N-k}^{s_{23}+s_{31}}(\eta,\zeta)][:\varphi_{N-k}^{r_1+r_2}(\eta)\varphi_{N-k}^{r_3}(\zeta):]\}, \tag{B.10}$$

$$C_I^{(N-k)}=\frac{\lambda^3}{2}\binom{4}{2}^3 2^3\sum_{N-k+1}^{N}{}_i \iiint_{I\times I\times I} d\xi d\eta d\zeta[C_i^2(\xi,\eta)-C_{i-1}^2(\xi,\eta)]$$
$$\cdot[C_{N-k}^2(\eta,\zeta)C_{N-k}^2(\zeta,\xi)]. \tag{B.11}$$

Since the fields appearing in the Wick products are constant on the tesserae $\Delta\in Q_{N-k}$, it is clear that $\tilde{V}_I^{(N-k)}$ can be written as in §4.

We now give an example of the method to get the bounds of §3, 4. by explicitely estimating the contribution to $\bar{A}_{\Delta\Delta'}^{11}(N,k,I)$ coming from $W_I^{(N-k)}$, for $d=3$. We need to estimate:

$$W_{\Delta\Delta'}\equiv\lambda^2 2\gamma_{N-k}(1+\Gamma_{N-k})\sum_{N-k+1}^{N}{}_i \iint_{\Delta\times\Delta'} d\xi d\eta[C_i^3(\xi,\eta)-C_{i-1}^3(\xi,\eta)]$$
$$\leqq 4\lambda^2 2^{N-k}\sum_{N-k+1}^{N}{}_i \iint_{\Delta\times\Delta'} d\xi d\eta 3C_i^2(\xi,\eta)[C_i(\xi,\eta)-C_{i-1}(\xi,\eta)]. \tag{B.12}$$

By (C.2), (C.4), (1.2), (1.13), if $\Delta\neq\Delta'$ we have:

$$|W_{\Delta\Delta'}|\leqq 3\cdot 2^4\lambda^2 2^{N-k}\sum_{N-k+1}^{N}{}_i\, 2^{2i}\iint_{\Delta\times\Delta'} d\xi d\eta[C_i(\xi,\eta)-C_{i-1}(\xi,\eta)]$$
$$\leqq 3\cdot 2^4\lambda^2 2^{N-k}\sum_{N-k+1}^{N}{}_i\, 2^{2i}\sum_{\substack{\bar{\Delta}\subset\Delta,\bar{\Delta}'\subset\Delta'\\ \bar{\Delta},\bar{\Delta}'\in Q_i}}\iint_{\bar{\Delta}\times\bar{\Delta}'} d\xi d\eta[C_i(\xi,\eta)-C_{i-1}(\xi,\eta)]$$
$$\leqq 3\cdot 2^4\lambda^2 2^{N-k}2\|C\|\sum_{N-k+1}^{N}{}_i\, 2^{-3i}\sum_{\substack{\bar{\Delta}\subset\Delta,\bar{\Delta}'\subset\Delta'\\ \bar{\Delta},\bar{\Delta}'\in Q_i}} e^{-\varkappa' 2^i d(\bar{\Delta},\bar{\Delta}')}$$
$$\leqq 3\cdot 2^4 b_1 b_2(\lambda 2^{-(N-k)})^2 e^{-\frac{\varkappa}{2}2^{N-k}d(\Delta,\Delta')}, \tag{B.13}$$

where

$$b_1=2\|C\|\sum_{\Delta'\in Q_0} e^{-\frac{\varkappa'}{4}d(\Delta,\Delta')},\quad \Delta\in Q_0$$
$$b_2=\sum_{0}^{\infty}{}_n e^{-n\frac{k'}{4}}. \tag{B.14}$$

If $\Delta=\Delta'$, the best estimate of $W_{\Delta\Delta'}$ would be of order $k2^{-2(N-k)}$ which is bad. However (thank to the mass renormalization) the contribution of $W_{\Delta\Delta}$ in (B.7) is missing.

Appendix C

In this appendix we give some properties of gaussian fields related to the gaussian field $(z_\Delta)_{\Delta\in Q_0}$ with formal density

$$\text{const}\exp\left\{-\frac{\beta}{2}\left[\sum_{\Delta,\Delta'}^{*}(z_\Delta-z_{\Delta'})^2+\sum_{\Delta}\alpha^2 z_\Delta^2\right]\right\}. \tag{C.1}$$

1) The covariance $C_{\Delta,\Delta'}$ is such that, if ξ_Δ is the center of Δ

$$0\leqq C_{\Delta\Delta'}=\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\frac{e^{ik(\xi_\Delta-\xi_{\Delta'})}dk}{\alpha^2\beta+2\beta\sum_{1}^{d}{}_i(1-\cos k_i)}\leqq\frac{1}{\beta\alpha^2}\left(1+\frac{\alpha^2}{2d}\right)^{-d(\Delta,\Delta')}$$

$$\equiv\|c\|e^{-\varkappa' d(\Delta,\Delta')}. \tag{C.2}$$

If $\pi_n(\Delta,\Delta')$ is the number of walks on the lattice of the centers of the tesserae divided by (2d)

$$C_{\Delta\Delta'}=\frac{1}{\beta(\alpha^2+2d)}\sum_{0}^{\infty}{}_n\left(\frac{2d}{2d+\alpha^2}\right)^n\pi_n(\Delta,\Delta'), \tag{C.3}$$

$$0\leqq C_{\Delta\Delta'}<1/2, \tag{C.4}$$

$$\|C\|\equiv\sum_{\Delta'}C_{\Delta\Delta'}=\frac{1}{\alpha^2\beta}. \tag{C.5}$$

2) Let Γ be a region paved by Q_0 and let $\hat{P}_0(dz|z_\Gamma)$ denotes the above probability measure conditioned to fixed values of the z_Δ's, $\Delta\in\Gamma$. Then the conditioned variables $(z_\Delta)_{\Delta\notin\Gamma}$ are a non centered gaussian field with covariance $C^\Gamma_{\Delta\Delta'}$ such that

$$0\leqq C^\Gamma_{\Delta\Delta'}\leqq C_{\Delta\Delta'} \tag{C.6}$$

and center

$$u_\Delta=\beta\sum_{\Delta'\subset\Gamma}\left(\sum_{\substack{\Delta''\not\subset\Gamma\\ \Delta''\text{n.n.to}\Delta'}}C^\Gamma_{\Delta\Delta''}\right)z_{\Delta'}. \tag{C.7}$$

$C^\Gamma_{\Delta,\Delta'}$ is the covariance with "Ditrichelet boundary condition" on Γ. Hence using (C.6) (C.3), if $|z_\Delta|<b(1+d(\Delta,I))$

$$|u_\Delta|\leqq\left(\frac{\alpha^2+2d}{\alpha^2}\right)^2 b(1+d(\Delta,I)). \tag{C.8}$$

The above properties are well-known [7].

The following lemmas hold

Lemma 1. *Given n gaussian variables* $\{z_i\}_{i=1}^n$ *with covariance* $C_{ij}>0$, $i,j=1,\ldots,n$ *and given* $b_1,\ldots,b_n>0$

$$\int \prod_{i=1}^{n} \chi(|z_i|>b_i)dP \leqq \exp\left[-\sum_1^n{}_i\left(\frac{b_i^2}{2\|c\|}-\log 2\right)\right] \tag{C.9}$$

where $\|C\|=\sup_i \sum_1^n{}_j C_{ij}$

Proof. If $\gamma>0$:

$$e^{\gamma\sum_1^n{}_i b_i^2}\int\left[\prod_{i=1}^{k}\chi(z_i>b_i)\right]\left[\prod_{i=k+1}^{n}\chi(z_i<-b_i)\right]dP$$

$$\leqq \int \prod_{i=1}^{k}\chi(z_i>b_i)\prod_{i=k+1}^{n}\chi(z_i<-b_i)\exp\left(\sum_1^k{}_i \gamma b_i z_i - \sum_{k+1}^n{}_i \gamma b_i z_i\right)dP$$

$$\leqq \int \exp\left(\sum_1^k{}_i \gamma b_i z_i - \sum_{k+1}^n{}_i \gamma b_i z_i\right)dP \leqq \exp\left[\tfrac{1}{2}\sum_{i,j}\gamma^2 b_i b_j C_{ij}\right].$$

Hence

$$\int \prod_{i=1}^{k}\chi(z_i>b_i)\prod_{i=k+1}^{n}\chi(z_i<-b_i)dP \leqq \exp\left[-\gamma\left(1-\tfrac{\gamma}{2}\|c\|\right)\sum_1^n{}_i b_i^2\right]$$

and the lemma follows by choosing $\gamma=1/\|C\|$.

Lemma 2. *Let* Γ, I, $\square$ *be regions paved by* Q_0, *let* $\bar{z}_\Gamma\equiv(\bar{z}_\Delta)_{\Delta\in\Gamma}$ *be a given family of numbers such that* $|\bar{z}_\Delta|\leqq\gamma b(1+d(I,\Delta))$ *and let* $P(dz/\bar{z}_\Gamma)$ *be the measure* (C.1) *conditioned to the values* $\bar{z}_\Gamma$, then:

$$\int \prod_{\Delta\subset\square}\chi(|z_\Delta|<b(1+d(\Delta,I)))P(dz|\bar{z}_\Gamma)\geqq\left(1-2|\square|e^{-\frac{b^2}{4}}\right)$$

if

$$\gamma\leqq\tfrac{1}{2}\left(\frac{2d}{\alpha^2+2d}\right)^2.$$

Proof. Since the covariance with Dirichelet boundary conditions is bounded by (C.6) and since the center of the measure $P(dz_\square\backslash\bar{z}_\Gamma)$ is bounded as in (C.8) the above lemma is an immediate consequence of Lemma 1.

Appendix D

The replacement of the χ_s' by 1 is a trivial consequence of point 2) and Lemma 1 of Appendix C.

In order to complete the proof we need to evaluate

$$\mathscr{E}_0^T(\tilde{\Psi}'_1,\tilde{\Psi}''_1,\tilde{\Psi}_2;h_1,h_2,k_2).$$

Let us remark first the following points:

i) given the structure of the $\tilde{\Psi}'_s$ [see Equation (5.23)] $\mathscr{E}_0^T(\tilde{\Psi}\ldots)$ can be expressed as a sum of s_5 terms of the form

$$\mathscr{E}_0^T(z_{\Delta_{i_1}}\ldots z_{\Delta_{i_n}}, z_{\Delta_{j_1}}\ldots z_{\Delta_{j_s}}, z_{\Delta_{e_1}}\ldots z_{\Delta_{e_r}}, 1, 1, 1,)$$
$$\cdot e^{-\frac{\varkappa}{2}[d(\Delta_{i_1})+\ldots+d(\Delta_{i_r}))+d(\Delta_{j_1})+\ldots+d(\Delta_{j_s})+d(\Delta_{e_1})+\ldots+d(\Delta_{e_t})]}$$

with obvious shortened notations, where s_5 is a constant depending on h_1, h_2, k_2.

ii) $\mathscr{E}_0^T(z^{\Delta_1}, z^{\Delta_2}, z^{\Delta_3}; 1, 1, 1)$ is by definition an algebraic sum of products of expectations values. As it is well known, once these expectations are expressed via the Wick theorem as sum of products of 2-point functions, the only terms that survive are the so called "connected diagrams" where at least one z_Δ in each set z^{Δ_1}, z^{Δ_2}, z^{Δ_3} is connected to another z_Δ belonging to a different set.

If we recall now that at least one z_Δ in each term of $\tilde{\Psi}''_1$ belong to $\square'\backslash\Gamma_4(\square)$ the exponential factors arising from i) and ii) give rise to an overall dumping factor that is at least $\exp -\frac{\varkappa}{4}b^{3/2}$ where $b^{3/2}/2$ is the length of the smallest path connecting a set of points constructed in such a way that one point belongs by sure to $\square'/\Gamma_4(\square)$ and at least another one belongs to Γ_2.

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