Commun. math. Phys. 58, 273—290 (1978)

Communications in Mathematical Physics

Passive States and KMS States for General Quantum Systems

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Abstract. We characterize equilibrium states of quantum systems by a condition of passivity suggested by the second principle of thermodynamics. Ground states and β-KMS states for all inverse temperatures $\beta \geqq 0$ are completely passive. We prove that these states are the only completely passive ones. For the special case of states describing pure phases, assuming the passivity we reproduce the results of Haag et al.

Introduction

The main aim of the equilibrium statistical physics is a description and investigation of equilibrium states for large physical systems. To this end we use infinite systems as good mathematical models.

The physical motivation for this paper is a question; how to describe the equilibrium states for a given infinite quantum system?

As it is well known, in the traditional approach we consider the finite systems for which the equilibrium states are better known and then we take the thermodynamical limit. If the limit exists, one assumes that it describes the equilibrium state of the infinite system.

Starting with the Gibbs canonical ensemble characterized by inverse temperature β for finite systems and keeping β constant, one can easily prove that the limit state is β-KMS state for the evolution group of the infinite system. In other words, if H is a generator of this group then the modular automorphism group associated with this state is given by $-\beta H$.

The KMS states are formal generalizations of the canonical Gibbs states for infinite systems and it is not obvious whether they possess properties attributing to equilibrium states.

Some number of papers are devoted to answer this question. For quantum lattice systems Araki proved that every KMS state is a limit of Gibbs states [1], moreover for translationally invariant states he showed that the KMS condition is equivalent to the known variational principle saying that for a given energy and density, the maximum entropy is achieved on the equilibrium state [2]. Quite recently it was proved to be equivalent with local thermodynamical stability [4].

In this paper we give another description without any reference to Gibbs states and the limiting procedure. We characterize the equilibrium states of general quantum systems by a condition saying that the systems are unable to perform mechanical work in cyclic processes. This condition called "passivity" is suggested by the second principle of thermodynamics.

For states describing pure phases (weakly clustering) we prove that passive states are either KMS-states with some non-negative inverse temperature β or ground states. In the general case, assuming a little stronger condition of the complete passivity we get the same results. On the other hand KMS-states (with positive inverse temperature) and ground states are completely passive.

In the special case, for passive states which are weakly clustering with respect to the time evolution group and not central we have some additional results.

To our knowledge the paper of Haag et al. [8] is the first paper devoted to this problem in the general setting. The other versions of this type are [6, 10]. They proved that "pure phase" states (some technical assumptions which are related to the clustering property with respect to the time evolution) stable under local perturbation of the dynamics are either KMS-states (with $\beta \in \mathbb{R}$) or ground states. This result corresponds to our last special case.

For finite systems it can be shown that the stability under local perturbation of dynamics means the functional dependence between the state and the hamiltonian, our condition of passivity gives a decreasing (not necessarily functional) dependence, both these conditions are weaker than KMS-condition asserting the functional dependence of the form

$$\varrho \sim e^{-\beta H}.$$

1. Passive States: Main Definitions and Results

We investigate a quantum system S (finite or not). To describe such a system we use C^*-algebra language. Let $(\mathfrak{A}, \alpha)$ be the C^*-dynamical system assigned to S. More precisely $\mathfrak{A}$ is the C^*-algebra of observables and $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ is the strongly continuous one-parameter group of automorphisms of $\mathfrak{A}$ describing the time evolution of the system. Assume the system is in a state ω at time $t=0$. Then the state ω_t of the system at time t is given by

$$\omega_t(A) = \omega(\alpha_t(A)) \quad A \in \mathfrak{A}. \tag{1.1}$$

The family of states (1.1) satisfies the following evolution equation

$$\frac{d\omega_t}{dt}(A) = \omega_t(\delta(A)) \quad A \in D(\delta), \tag{1.2}$$

where δ denotes the generator of the group α;

$$\delta(A)=\lim_{t\to 0}\frac{1}{t}(\alpha_t(A)-A)$$

and $D(\delta)$ is the set of all $A\in\mathfrak{A}$ for which the above limit exists.

All the formulae obtained so far are based on the assumption that the dynamics of the system is described by a one-parameter group of automorphisms of $\mathfrak{A}$. In the physical language this assumption means that our system is closed i.e.: thermally isolated and placed in the immutable external conditions.

The main idea of the present paper relies on the investigation, how the system reacts to a change of the external conditions. Such a change can be achieved for instance by switching on some external fields or by moving the walls confining the space region admissible for the system.

In the case of changing external conditions, the system is no longer closed (although it is still thermally isolated) and the evolution Equation (1.2) should be replaced by the following one:

$$\frac{d\omega_t}{dt}(A)=\omega_t(\delta_t(A)) \quad A\in D(\delta_t), \tag{1.3}$$

where $\{\delta_t\}_{t\in\mathbb{R}}$ is a family of derivations of $\mathfrak{A}$. Following the philosophy of [8] we assume that $D(\delta_t)=D(\delta)$ and

$$\delta_t(A)=\delta(A)+i[h_t,A] \quad A\in D(\delta), \tag{1.4}$$

where $\{h_t\}_{t\in\mathbb{R}}$ is a family of self-adjoint elements of $\mathfrak{A}$.

Clearly the family $\{h_t\}_{t\in\mathbb{R}}$ describes the way, the external conditions are changing. We shall always assume that $h_t=0$ for $t\leqq 0$ (up to this moment the system is closed), h_t is continuous on $[0, T]$ and differentiable (in the norm topology) inside this interval (smooth changes) and $h_t=\text{const}$ for $t\geqq T$ (the system is closed again although the external conditions may differ from the original ones).

From the physical point of view the assumption (1.4) restricts the class of external conditions. We are allowed to make only local changes, for instance switched on external fields should be of finite extent.

According to (1.3) and (1.4) the evolution of our system in the case of changing external conditions is governed by the equation

$$\frac{d\omega_t}{dt}(A)=\omega_t(\delta(A))+i\omega_t([h_t,A]) \quad A\in D(\delta). \tag{1.5}$$

The general solution of this equation can be written in the following form similar to (1.1):

$$\omega_t(A)=\omega(\alpha_t^h(A)) \quad A\in\mathfrak{A}, \tag{1.6}$$

where $\omega=\omega_0$ is the state of the system at time $t=0$ and $\{\alpha_t^h\}_{t\in\mathbb{R}}$ is a family of automorphisms of $\mathfrak{A}$ satisfying the following evolution equation;

$$\frac{d\alpha_t^h}{dt}(A)=\alpha_t^h(\delta(A))+i\alpha_t^h([h_t,A]) \quad A\in D(\delta) \tag{1.7a}$$

and the initial condition

$$\alpha_0^h(A)=A \quad A\in\mathfrak{A}. \tag{1.7b}$$

It can be shown that $\{\alpha_t^h\}_{t\in\mathbb{R}}$ exists and is uniquely determined by (1.7).

The physical meaning of the operators h_t follows easily from (1.4). This equation shows that h_t is the additional energy operator related to the considered change of the external conditions. It means that by changing external conditions we can transmitte the energy to the system (there is a work done by external forces!). The energy transmitted to the system S during the time interval $[0, T]$ is given by the formula

$$L^h(\omega)=\int_0^T \omega_t\left(\frac{dh_t}{dt}\right)dt=\int_0^T \omega\left(\alpha_t^h\left(\frac{dh_t}{dt}\right)\right)dt. \tag{1.8}$$

To justify this formula we divide $[0,T]$ into small intervals $[0,T]=\bigcup_{k=1}^{N}[t_{k-1},t_k]$ (where $0=t_0<t_1<\ldots<t_{N-1}<t_N=T$) such that ω_t is almost constant on each small interval. Then

$$L^h(\omega)\simeq\sum_{k=1}^{N}\omega_{t_k}(h_{t_k}-h_{t_{k-1}})$$

which is the correct expression (see also [11], pp. 20 and 21).

If $L^h(\omega)\geqq 0$ then the positive work is done by external forces and the energy of the system increases. In the other case $L^h(\omega)<0$, the positive work is done by the system and its energy decreases.

Both cases are possible for real systems in equilibrium. Consider for example a gas in a thermally isolated cylinder. Then $L^h(\omega)<0$ (resp. $L^h(\omega)>0$) if the gas is decompressed (resp. compressed).

There is however one important property of the equilibrium states which is related to the concept of the work. Namely for such states $L^h(\omega)\geqq 0$ provided the final external conditions coincide with the original ones: $h_T=0$. This fact is strongly related to the second principle of thermodynamics saying that systems in the equilibrium are unable to perform mechanical work in cyclic processes. We describe this property saying that the equilibrium states are passive.

Definition 1.1. Let ω be a state of C^*-dynamical system $(\mathfrak{A},\alpha)$. We say that ω is passive iff

$$L^h(\omega)\geqq 0 \tag{1.9}$$

for any differential family $\{h_t\}_{t\in\mathbb{R}}$ of self-adjoint elements of $\mathfrak{A}$ such that $h_t=0$ for $t\leqq 0$ and $t\geqq T$.

In our example if we decompress the gas and next compress it to the previous volume then the total work we have done is positive (it is zero if the process is done in the quasi-static way).

Now we are able to state the main results of the paper:

Theorem 1.1. *Let ω be a passive state of a C^*-dynamical system $(\mathfrak{A},\alpha)$. Then ω is α-invariant i.e. $\omega(\alpha_t(A))=\omega(A)$ for all $t\in\mathbb{R}$ and $A\in\mathfrak{A}$.*

This result is not surprising. Equilibrium states should be stationary.

Theorem 1.2. *Let $(\mathfrak{A},\alpha)$ be a C^*-dynamical system. Assume that a state ω is either a ground state or a* KMS-*state with some positive inverse temperature $\beta \geqq 0$ for $(\mathfrak{A},\alpha)$. Then ω is passive.*

The converse statement, although very desired is not true in general. The very reason for this is the linear nature of the passivity condition: it is obvious that any mixture of passive states is passive. On the other hand a nontrivial mixture of KMS-states with different temperatures is neither KMS nor ground state.

The simplest way to exclude such mixtures is to assume that our state describes a pure thermodynamical phase. As the mathematical expression of this fact we adopt the weak clustering property.

Definition 1.2. Let G be a locally compact amenable group of automorphisms of $\mathfrak{A}$. The action of G on $\mathfrak{A}$ will be denoted by $\varkappa$. We always assume that the mappings $G \ni x \mapsto \varkappa_x(A) \in \mathfrak{A}$ are continuous. An invariant mean on G will be denoted by η. A state ω of $\mathfrak{A}$ is called G-weakly clustering iff

$$\eta_x(\omega(A\varkappa_x(B))) = \omega(A)\omega(B) \quad A, B \in \mathfrak{A}.$$

In the physical applications G is the group of space translations or just the time evolution group.

For states describing pure thermodynamical phases we have the converse of Theorem 1.2.

Theorem 1.3. *Let $(\mathfrak{A},\alpha)$ ba a C^*-dynamical system and G be a locally compact amenable group of automorphisms of $\mathfrak{A}$ commuting with α. Assume that ω is a G-weakly clustering and passive state of $(\mathfrak{A},\alpha)$. Then either*

1) *ω is a* KMS-*state of $(\mathfrak{A},\alpha)$ with some non-negative inverse temperature $\beta \geqq 0$ or*
2) *ω is a ground state of $(\mathfrak{A},\alpha)$.*

In the general case, to get the conclusion of Theorem 1.3, we have to assume a stronger version of the passivity condition. It is called the complete passivity and seems to be as natural as the previous one. To introduce it we consider the system S^N consisting of N-copies of the system S. The complete passivity means, that if all these copies are in the same state ω and are uncorrelated then the resulting state of S^N is passive and this fact should hold for all $N = 1, 2, \ldots$.

To make this definition precise, for a given C^*-dynamical system $(\mathfrak{A},\alpha)$ we consider the system $(\mathfrak{A},\alpha)^N = \left(\bigotimes^N \mathfrak{A}, \bigotimes^N \alpha\right)$.

Definition 1.3. Let ω be a state of a C^*-dynamical system $(\mathfrak{A},\alpha)$. ω is called completely passive iff for all natural N, $\bigotimes^N \omega$ is a passive state of $(\mathfrak{A},\alpha)^N$.

We have

Theorem 1.4. *Assume that ω is a completely passive state of a C^*-dynamical system $(\mathfrak{A},\alpha)$. Then either*

1) *ω is a* KMS-state *of $(\mathfrak{A},\alpha)$ with some non-negative inverse temperature $\beta \geqq 0$, or*
2) *ω is a ground state of $(\mathfrak{A},\alpha)$.*

Remark. It follows immediately from Theorem 1.2 that KMS-states (with $\beta \geqq 0$) and ground states are completely passive.

All the results mentioned above will be proved in the next sections. In the last section we also prove some deeper statements concerning the spectrum of the hamiltonian operator and the type of the factors generated by GNS-representations related to passive states weakly clustering with respect to the time translation group.

2. The Mathematical Sense of the Passivity

The main result of this section is contained in the following theorem.

Theorem 2.1. *Let $(\mathfrak{A}, \alpha)$ be a C^*-dynamical system, δ be the generator of α and ω be a state of $\mathfrak{A}$. Then ω is passive iff*

$$\omega\left(U^* \frac{\delta}{i}(U)\right) \geqq 0 \tag{2.1}$$

for any $U \in \mathcal{U}_0(\mathfrak{A}) \cap D(\delta)$, where $\mathcal{U}_0(\mathfrak{A})$ denotes the connected component of the unity of the group $\mathcal{U}(\mathfrak{A})$ of all unitary elements of $\mathfrak{A}$ with the uniform topology.

To prove this result we need a formula giving a deeper insight into the structure of the perturbed dynamics α_t^h. To get this formula we consider the following differential equation:

$$\frac{dU_t}{dt} = -i\alpha_t(h_t)U_t \tag{2.2a}$$

with the initial condition:

$$U_0 = I. \tag{2.2b}$$

One can easily prove that there exists one and only one solution $\{U_t\}_{t\in\mathbb{R}}$ of the problem (2.2). Moreover the selfadjointness of h_t implies that all operators U_t are unitary.

The relation between $\{U_t\}_{t\in\mathbb{R}}$ and the perturbed dynamics $\{\alpha_t^h\}_{t\in\mathbb{R}}$ is given by

$$\alpha_t^h(A) = U_t^* \alpha_t(A) U_t \qquad A \in \mathfrak{A}. \tag{2.3}$$

To prove this formula one has to check that α_t^h introduced by it is the solution of (1.7). This verification can be done by the direct computation and will be omitted.

Assume now that $h_t \in D(\delta)$ and that the mapping $t \mapsto \delta(h_t)$ is continuous. Then $U_t \in D(\delta)$, $\delta(U_t)$ is differentiable with respect to t and $\frac{d}{dt}\delta(U_t) = \delta\left(\frac{d}{dt}U_t\right)$. Since $U_0 = I$ and $\delta(I) = 0$ we have $\omega\left(U_0^* \frac{\delta}{i}(U_0)\right) = 0$. Therefore

$$\begin{aligned}\omega\left(U_T^* \frac{\delta}{i}(U_T)\right) &= \int_0^T \frac{d}{dt}\omega\left(U_t^* \frac{\delta}{i}(U_t)\right) dt \\ &= \int_0^T \omega\left(\left(\frac{dU_t}{dt}\right)^* \frac{\delta}{i}(U_t) + U_t^* \frac{\delta}{i}\left(\frac{dU_t}{dt}\right)\right) dt.\end{aligned}$$

Now, using (2.2a) and remembering that δ is a derivation commuting with α we get

$$\omega\left(U_T^*\frac{\delta}{i}(U_T)\right)=\int_0^T \omega(U_t^*\alpha_t(h_t)\delta(U_t)-U_t^*\delta(\alpha_t(h_t)U_t))dt$$
$$=-\int_0^T \omega(U_t^*\alpha_t(\delta(h_t))U_t)dt=-\int_0^T \omega(\alpha_t^h(\delta(h_t)))dt.$$

On the other hand, remembering that $h_0=0=h_T$ and using (1.7a) we have

$$L^h(\omega)=\int_0^T \omega\left(\alpha_t^h\left(\frac{dh_t}{dt}\right)\right)dt=\omega(\alpha_t^h(h_t))|_0^T-\int_0^T \omega\left(\frac{d\alpha_t^h}{dt}(h_t)\right)dt$$
$$=-\int_0^T \omega(\alpha_t^h(\delta(h_t)))dt-i\int_0^T \omega(\alpha_t^h([h_t,h_t]))dt$$
$$=-\int_0^T \omega(\alpha_t^h(\delta(h_t)))dt.$$

Therefore

$$L^h(\omega)=\omega\left(U_T^*\frac{\delta}{i}(U_T)\right). \tag{2.4}$$

Now the proof of Theorem 2.1 is very simple. Assume that (2.1) holds and that $h=\{h_t\}_{t\in\mathbb{R}}$ satisfies the conditions formulated in Definition 1.1. If $h_t\in D(\delta)$ and $t\mapsto\delta(h_t)$ is continuous then (1.9) follows immediately from (2.1) and (2.4) (Note that U_T belongs to $\mathscr{U}_0(\mathfrak{A})$). If $\{h_t\}_{t\in\mathbb{R}}$ is not contained in $D(\delta)$ then we put

$$h_t^n=\int_{-\infty}^{+\infty} n\varphi(ns)\alpha_s(h_t)ds,$$

where n is integer and φ is a positive C^1-class function with compact support on the real line such that $\int_{-\infty}^{+\infty}\varphi(s)ds=1$. One can easily check that $h^n=\{h_t^n\}_{t\in\mathbb{R}}$ satisfies all the conditions used in the derivation of (2.4). Therefore $L^{h^n}(\omega)\geqq 0$ for every n. On the other hand, it is rather obvious that $h_t^n\to h_t$ and $\frac{dh_t^n}{dt}\to\frac{dh_t}{dt}$ uniformly for $t\in\mathbb{R}$ as $n\to\infty$. Then, using the simple continuity property of $L^h(\omega)$ we have $L^{h^n}(\omega)\to L^h(\omega)$ and (1.9) follows. This way we proved that (2.1) implies passivity.

Assume conversely that ω is a passive state and that $U\in\mathscr{U}_0(\mathfrak{A})\cap D(\delta)$. Since U belongs to the connected component of the unity of the group of all unitary elements of $\mathfrak{A}$, one can find a finite sequence $\{A_k\}_{k=1,2\ldots,N}$ of self-adjoint elements of $\mathfrak{A}$ such that $\|A_k\|\leqq\pi$ and

$$U=e^{iA_1}e^{iA_2}\ldots e^{iA_N}.$$

Remembering that $D(\delta)$ is dense in $\mathfrak{A}$ in the norm topology, one may assume that $A_k\in D(\delta)$ for $k=1,2,\ldots,N-1$. Then $A_N\in D(\delta)$ because $U\in D(\delta)$.

Let f be a smooth function on $\mathbb{R}$ such that $f(s)=0$ for $s\leqq 0$ and $f(s)=1$ for $s\geqq 1$. We set

$$U_t=\begin{cases} I & \text{for} \quad t\leqq 0 \\ e^{iA_1}\ldots e^{iA_k}\cdot e^{if(t-k)A_{k+1}} & \text{for} \quad t\in[k,k+1] \quad k=0,\ldots,N-1 \\ U & \text{for} \quad t\geqq N \end{cases} \tag{2.5}$$

and

$$h_t = i\alpha_{-t}\left(\frac{dU_t}{dt}U_t^*\right). \tag{2.6}$$

One can easily check that $\{h_t\}_{t\in\mathbb{R}}$ satisfies all the assumptions used in the derivation of (2.4) and (2.1) follows immediately from (1.9) (note that $U_N = U$). This ends the proof of Theorem 2.1.

To understand better the content of the condition (2.1) we have to rewrite it in the Hilbert space language.

Let $(\mathfrak{A}, \alpha)$ be a C^*-dynamical system and ω be a α-invariant state. By using the GNS-method we construct $(\mathscr{H}, \pi, \Omega, T)$, where $\mathscr{H}$ is a Hilbert space, π is a cyclic representation of $\mathfrak{A}$ acting on $\mathscr{H}$, $\Omega\in\mathscr{H}$ is the cyclic vector and $T=\{T_t\}_{t\in\mathbb{R}}$ is a one-parameter, strongly continuous group of unitary operators acting on $\mathscr{H}$ such that

$$(\Omega|\pi(A)\Omega) = \omega(A), \tag{2.7}$$

$$T_t\Omega = \Omega, \tag{2.8}$$

$$\pi(\alpha_t(A)) = T_t\pi(A)T_t^* \tag{2.9}$$

for all $t\in\mathbb{R}$ and $A\in\mathfrak{A}$. In particular

$$\pi(\alpha_t(A))\Omega = T_t\pi(A)\Omega. \tag{2.10}$$

Let H denotes the self-adjoint generator of $T: T_t = e^{itH}$. Operator H is closely related to the derivation δ. One can easily check [using (2.10)] that $\pi(A)\Omega\in D(H)$ and

$$\pi\left(\frac{\delta}{i}(A)\right)\Omega = H\pi(A)\Omega$$

for all $A\in D(\delta)$. To present the simple application of the above formulae we transform the LHS of (2.1):

$$\begin{aligned}\omega\left(U^*\frac{\delta}{i}(U)\right) &= \left(\Omega|\pi\left(U^*\frac{\delta}{i}(U)\right)\Omega\right) \\ &= \left(\pi(U)\Omega|\pi\left(\frac{\delta}{i}(U)\right)\Omega\right) = (\pi(U)\Omega|H\pi(U)\Omega).\end{aligned} \tag{2.11}$$

Now we are able to prove Theorem 1.2. If ω is a ground state, then $H\geqq 0$ and the passivity follows immediately from (2.11) and Theorem 2.1. If ω is a KMS-state with the inverse temperature $\beta>0$, then Ω is a cyclic and separating vector for the von Neumann algebra $\mathscr{A}$ generated by $\pi(\mathfrak{A})$ and $e^{-\beta H}$ coincides with the modular operator constructed for $(\mathscr{A}, \Omega)$. Using the well known properties of the modular operator we have:

$$(e^{-\beta/2H}\pi(U)\Omega|e^{-\beta/2H}\pi(U)\Omega) = (\pi(U^*)\Omega|\pi(U^*)\Omega) = 1.$$

Now, the passivity of ω follows from the obvious inequality: $H\geqq \frac{1}{\beta}(I-e^{-\beta H})$.

Indeed

$$\omega\left(U^*\frac{\delta}{i}(U)\right)=(\pi(U)\Omega|H\pi(U)\Omega)$$
$$\geqq\frac{1}{\beta}[(\pi(U)\Omega|\pi(U)\Omega)-(e^{-\beta/2H}\pi(U)\Omega|e^{-\beta/2H}\pi(U)\Omega)]=0\,.$$

If ω is a trace state (a KMS state with $\beta=0$) the previous argument does not apply but the passivity can be proven directly. For the proof in this case we are indebted to Professor Araki.

By the α invariance of ω for $A=A^*\in D(\delta)$ we have

$$\omega(A^m\delta(A)A^n)=\frac{1}{m+n+1}\,\omega(\delta(A^{m+n+1}))=0$$

and hence $\omega(U^*\delta(U))=0$ for $U=e^{i\varepsilon A}$. If unitary operators U_1 and U_2 satisfy $\omega(U_j^*\delta(U_j))=0$ then $U=U_1U_2$ also satisfy the same equation:

$$\omega(U^*\delta(U))=\omega(U_2^*U_1^*\delta(U_1)U_2)+\omega(U_2^*U_1^*U_1\delta(U_2))$$
$$=\omega(U_1^*\delta(U_1))+\omega(U_2^*\delta(U_2))=0$$

due to the trace property of ω.

These two arguments are sufficient to prove

$$\omega(U^*\delta(U))=0$$

for $U\in D(\delta)$ in the connected component of the identity.

This ends the proof of Theorem 1.2.

Remark. The reader familiar with the Araki notion of the relative entropy (cf. [3]) certainly noticed that in the KMS-case considered above

$$L^h(\omega)=S(\omega/\varphi)\,,$$

where φ is the final state of our process:

$$\varphi(A)=\omega(\alpha_T^h(A))\qquad A\in\mathfrak{A}\,.$$

We would like to present one nice application of Theorem 1.2 and formula (2.4). Let us consider a C^*-dynamical system $(\mathfrak{A},\alpha)$ consisting of two non-interacting subsystems $(\mathfrak{A}_1,\alpha^1)$ and $(\mathfrak{A}_2,\alpha^2)$:

$$\begin{aligned}&\mathfrak{A}=\mathfrak{A}_1\otimes\mathfrak{A}_2\\&\alpha_t=\alpha_t^1\otimes\alpha_t^2\,.\end{aligned}\tag{2.12}$$

Assume that the state of our system is given by $\omega=\omega^1\otimes\omega^2$, where ω^1 (resp. ω^2) is a KMS-state of $(\mathfrak{A}_1,\alpha^1)$ [resp. $(\mathfrak{A}_2,\alpha^2)$] with the inverse temperature β_1 (resp. β_2). Suppose that

$$0<\beta_1<\beta_2\,.$$

Let $\{h_t\}_{t\in\mathbb{R}}$ satisfy all the assumptions used in the derivation of (2.4). In virtue of (2.12) $\delta=\delta_1\otimes\mathrm{id}_2+\mathrm{id}_1\otimes\delta_2$ (where δ_1 and δ_2 are generators of α^1 and α^2

respectively) and

$$L^h(\omega)=\omega\left(U_T^*\frac{\delta}{i}(U_T)\right)=L_1^h(\omega)+L_2^h(\omega),$$

where

$$L_1^h(\omega)=\omega\left(U_T^*\frac{\delta_1\otimes \mathrm{id}_2}{i}(U_T)\right),$$

$$L_2^h(\omega)=\omega\left(U_T^*\frac{\mathrm{id}_1\otimes \delta_2}{i}(U_T)\right).$$

It turns out that the quantities introduced above have clear physical meaning: $L_1^h(\omega)$ and $L_2^h(\omega)$ are the amounts of the energy transmitted to the first and the second subsystem respectively. One can check this statement, either by making a little deeper analysis than that given in Section 1, or by considering confinite systems which dynamics are described by hamiltonian operators.

Since $\beta_1<\beta_2$, our system is not in equilibrium and it may happen that

$$L_1^h(\omega)<0. \tag{2.13}$$

However we always have

$$\beta_1 L_1^h(\omega)+\beta_2 L_2^h(\omega)\geqq 0. \tag{2.14}$$

Indeed one can easily verify that ω is a KMS-state of the C^*-dynamical system $(\mathfrak{A},\tilde{\alpha})$, where $\tilde{\alpha}_t=\alpha^1_{\beta_1 t}\otimes\alpha^2_{\beta_2 t}$ and that $\tilde{\delta}=\beta_1\delta_1\otimes\mathrm{id}_2+\beta_2\mathrm{id}_1\otimes\delta_2$ is the generator of $\tilde{\alpha}$. Therefore the LHS of (2.14) coincides with $\omega\left(U_T^*\frac{\tilde{\delta}}{i}(U_T)\right)$ and is positive in virtue of Theorem 2.1 and Theorem 1.2.

If (2.13) holds then $L_1^h(\omega)<0$ (the energy is taken from the first subsystem and (2.14) can be written in the following equivalent form

$$\frac{-L^h(\omega)}{-L_1^h(\omega)}\leqq\frac{T_1-T_2}{T_1},$$

where $T_1+\frac{1}{k\beta_1}$ and $T_2=\frac{1}{k\beta_2}$. This way we got the famous Carnot formula saying that the efficiency of any heat motor is limited by $\frac{T_1-T_2}{T_1}$, where T_1 and T_2 are temperatures of the heat source and the heat sink respectively.

3. Spectral Properties of Passive States

In this section $(\mathfrak{A},\alpha)$ is a C^*-dynamical system and ω is a passive state of $(\mathfrak{A},\alpha)$. According to Theorem 2.1 we have

$$\omega\left(U^*\frac{\delta}{i}(U)\right)\geqq 0 \tag{3.1}$$

for all $U \in \mathscr{U}_0(\mathfrak{A}) \cap D(\delta)$. This condition, although very powerful is not convenient in applications, for the structure of $\mathscr{U}_0(\mathfrak{A}) \cap D(\delta)$ may be very complicated. It turns out that (3.1) implies a much simpler relation which is sufficient for our purposes. To derive this relation we take $A = A^* \in D(\delta)$. Then for any $\varepsilon \in \mathbb{R}$, $U_\varepsilon = e^{i\varepsilon A} \in \mathscr{U}_0(\mathfrak{A}) \cap D(\delta)$ and in virtue of (3.1) we have

$$\omega\left(e^{-i\varepsilon A} \frac{\delta}{i}(e^{i\varepsilon A})\right) \geqq 0 .$$

Expanding this expression up to the second power of ε we get

$$\omega(\delta(A)) \cdot \varepsilon + \omega([\delta(A), A]) \frac{i\varepsilon^2}{2} + \ldots \geqq 0 .$$

Therefore

$$\omega(\delta(A)) = 0 \tag{3.2}$$

and

$$i\omega([\delta(A), A]) \geqq 0 \tag{3.3}$$

for any self-adjoint $A \in D(\delta)$. Clearly (3.2) holds for any $A \in D(\delta)$. Now, the statement of Theorem 1.1 follows immediately. Indeed, for any $A \in D(\delta)$ we have

$$\frac{d}{dt} \omega(\alpha_t(A)) = \omega(\delta(\alpha_t(A))) = 0$$

and

$$\omega(\alpha_t(A)) = \text{const} = \omega(A) .$$

Let us note that

$$[\delta(A), A] = \delta(A)A - A\delta(A) = \delta(A^2) - 2A\delta(A)$$

and in virtue of (3.2) and (3.3) we get

$$\omega\left(A \frac{\delta}{i}(A)\right) \geqq 0 \qquad A = A^* \in D(\delta) . \tag{3.4}$$

This relation, although very similar to (3.1) is much more manageable for the structure of the set $\{A \in D(\delta) : A = A^*\}$ is essentially simpler then the structure of $\mathscr{U}_0(\mathfrak{A}) \cap D(\delta)$.

Let now $B \in D(\delta)$. Then $\frac{1}{2}(B + B^*)$ and $\frac{1}{2i}(B - B^*)$ are self-adjoint elements of $D(\delta)$ and in virtue of (3.4) we have

$$\omega\left(\frac{B+B^*}{2} \frac{\delta}{i}\left(\frac{B+B^*}{2}\right)\right) + \omega\left(\frac{B-B^*}{2i} \frac{\delta}{i}\left(\frac{B-B^*}{2i}\right)\right) \geqq 0 \qquad B \in D(\delta)$$

and, performing simple calculations, we get

$$\omega\left(B^* \frac{\delta}{i}(B)\right) + \omega\left(B \frac{\delta}{i}(B^*)\right) \geqq 0 \qquad B \in D(\delta) . \tag{3.5}$$

We shall use the Hilbert space language introduced in the previous section. Let $(\mathscr{H}, \pi, \Omega, T)$ be the result of the GNS-construction and H be the self-adjoint generator of T. Then (3.5) is equivalent to

$$(\pi(B)\Omega | H\pi(B)\Omega) + (\pi(B)^*\Omega | H\pi(B)^*\Omega) \geqq 0 \tag{3.6}$$

for any $B \in D(\delta)$.

To procede the investigation of this relation we have to introduce a convenient notation. Let f be a function which is the Fourier transform of a finite complex measure:

$$f(\varepsilon) = \int_{-\infty}^{+\infty} e^{i\varepsilon t} d\mu_f(t).$$

For any such functions we put

$$f\left(\frac{\delta}{i}\right) = \int_{-\infty}^{+\infty} \alpha_t d\mu_f(t).$$

Using (2.10) one can easily check that

$$\pi\left(f\left(\frac{\delta}{i}\right)A\right)\Omega = f(H)\pi(A)\Omega \qquad A \in \mathfrak{A}. \tag{3.7}$$

Let $B \in \mathfrak{A}$. Then clearly $e^{+\delta^2/2}B = e^{-\frac{1}{2}\left(\frac{\delta}{i}\right)^2} B \in D(\delta)$ and using (3.6) and (3.7) we get

$$(\pi(B)\Omega | He^{-H^2}\pi(B)\Omega) + (\pi(B)^*\Omega | He^{-H^2}\pi(B)^*\Omega) \geqq 0.$$

As in the previous section we denote by $\mathscr{A}$ the von-Neumann algebra generated by $\pi(\mathfrak{A}): \mathscr{A} = \pi(\mathfrak{A})''$. According to the Kaplanski density theorem, for any $Q \in \mathscr{A}$ one can find a net $\{B_\alpha\}$ of elements of $\mathfrak{A}$ such that $\pi(B_\alpha) \to Q$ and $\pi(B_\alpha^*) \to Q^*$ strongly as $\alpha \to \infty$. Therefore (He^{-H^2} is a bounded operator!)

$$(Q\Omega | He^{-H^2}Q\Omega) + (Q^*\Omega | He^{-H^2}Q^*\Omega) \geqq 0 \tag{3.8}$$

for any $Q \in \mathscr{A}$.

There is no reason to expect that Ω is a separating vector for $\mathscr{A}$. In general, it is separating only for $E\mathscr{A}E$, where $E \in \mathscr{A}$ is the orthogonal projection onto the closure of $\mathscr{A}'\Omega$ (as usual $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$). We denote by Δ_E (resp. J_E) the modular operator (resp. the Tomita-Takesaki anti-unitary involution) related to the algebra $E\mathscr{A}E$ and vector Ω. Clearly Δ_E and J_E act on $E\mathscr{H}$. We extend Δ_E onto $\mathscr{H}$ by setting

$$\Delta = \Delta_E E. \tag{3.9}$$

Then Δ is a non-negative self-adjoint operator acting on $\mathscr{H}$ and $\mathscr{A}\Omega = \{A\Omega : A \in \mathscr{A}\}$ is a core of $\Delta^{1/2}$ (because $E\mathscr{A}\Omega = E\mathscr{A}E\Omega$ is a core of $\Delta_E^{1/2}$). Moreover, using the fundamental property of the modular operator in the Tomita-Takesaki theory and remembering that $E\Omega = \Omega$ we get

$$\begin{aligned}(\Delta^{1/2}R\Omega | \Delta^{1/2}Q\Omega) &= (\Delta_E^{1/2}ERE\Omega | \Delta_E^{1/2}EQE\Omega) \\ &= (EQ^*E\Omega | ER^*E\Omega) = (EQ^*\Omega | R^*\Omega)\end{aligned} \tag{3.10}$$

for any $R, Q \in \mathscr{A}$.

We know that $T_t \mathscr{A} T_t^* = \mathscr{A}$ [cf. formula (2.9)] and $T_t \Omega = \Omega$. Therefore $T_t \Delta T_t^* = \Delta$, because Δ is canonically related to $\mathscr{A}$ and Ω. It means that Δ and H strongly commute, so they have the common spectral decomposition. We write this decomposition in the following way:

$$H = \int_{\Lambda} \varepsilon dP(\varepsilon, \lambda), \tag{3.11}$$

$$\Delta = \int_{\Lambda} e^{\lambda} dP(\varepsilon, \lambda), \tag{3.12}$$

where $\Lambda = \mathbb{R} \times \bar{\mathbb{R}}$; $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ with the obvious topology; $e^{-\infty} = 0$ and $dP(\varepsilon, \lambda)$ is the common spectral measure. Let Σ^{ω} denotes the support of the spectral measure dP. One may say that Σ^{ω} is the joint spectrum of H and $\log \Delta$.

Remark. The reader certainly noticed that the set Σ^{ω} can be constructed for any invariant state ω of any C^*-dynamical system $(\mathfrak{A}, \alpha)$.

The main result of this section is contained in the following theorem.

Theorem 3.1. *Assume that ω is a passive state of a C^*-dynamical system $(\mathfrak{A}, \alpha)$. Then*

$$\Sigma^{\omega} \subset \Sigma^{\text{passive}}, \tag{3.13}$$

where Σ^{ω} is the set constructed above and

$$\Sigma^{\text{passive}} = \{(\varepsilon, \lambda) \in \Lambda : \varepsilon\lambda \leqq 0\}. \tag{3.14}$$

Moreover the finite part of Σ^{ω}, i.e. the set $\Sigma^{\omega}_{\text{finite}} = \{(\varepsilon, \lambda) \in \Sigma^{\omega} : \lambda \neq -\infty\}$ is invariant under the mapping $(\varepsilon, \lambda) \mapsto (-\varepsilon, -\lambda)$.

Proof. At first we note that $(\varepsilon, \lambda) \in \Sigma^{\text{passive}}$ iff $\varepsilon e^{-\varepsilon^2}(1 - e^{\lambda}) \geqq 0$. Therefore, to obtain (3.13) it is sufficient to show that

$$He^{-H^2}(I - \Delta) \geqq 0. \tag{3.15}$$

To this end we set QE (where Q is any element of $\mathscr{A}$) instead of Q in (3.8). Since $E\Omega = \Omega$ and E commutes with H, we get

$$(Q\Omega | He^{-H^2} Q\Omega) + (EQ^*\Omega | He^{-H^2} Q^*\Omega) \geqq 0. \tag{3.16}$$

We shall transform the second term in this expression. One can easily check that the function $\varepsilon e^{-\varepsilon^2}$ is the Fourier transform of some purely imaginary measure $\sigma : \varepsilon e^{-\varepsilon^2} = \int_{-\infty}^{+\infty} e^{it\varepsilon} d\sigma(t)$. Therefore introducing an operator $R \in \mathscr{A}$ such that

$$R^* = \int_{-\infty}^{+\infty} T_t Q^* T_t^* d\sigma(t)$$

we have

$$R = -\int_{-\infty}^{+\infty} T_t Q T_t^* d\sigma(t)$$

and

$$R^*\Omega = He^{-H^2}Q^*\Omega$$
$$R\Omega = -He^{-H^2}Q\Omega.$$

Now, using (3.10) we obtain:

$$(EQ^*\Omega|He^{-H^2}Q^*\Omega) = (EQ^*\Omega|R^*\Omega) = (\Delta^{1/2}R\Omega|\Delta^{1/2}Q\Omega)$$
$$= -(\Delta^{1/2}He^{-H^2}Q\Omega|\Delta^{1/2}Q\Omega)$$

and combining with (3.16) we get

$$(Q\Omega|He^{-H^2}Q\Omega) - (\Delta^{1/2}He^{-H^2}Q\Omega|\Delta^{1/2}Q\Omega) \geqq 0$$

for any $Q \in \mathscr{A}$. Now, by the continuity argument ($\mathscr{A}\Omega$ is a core for $\Delta^{1/2}$) we have

$$(\psi|He^{-H^2}\psi) - (\Delta^{1/2}He^{-H^2}\psi|\Delta^{1/2}\psi) \geqq 0$$

for any $\psi \in D(\Delta^{1/2})$. In particular for any $\psi \in D(\Delta)$

$$(\psi|He^{-H^2}(I-\Delta)\psi) \geqq 0$$

and (3.15) follows. The remaining part of the theorem follows from the fact that on the subspace $E\mathscr{H}$ (which corresponds to $\Sigma^{\omega}_{\text{finite}}$) we have

$$J_E \Delta_E J_E = \Delta_E^{-1} \quad J_E H J_E = -H. \tag{3.17}$$

The first relation is known in the Tomita-Takesaki theory, the second is implied by the following calculus:

$$e^{itH}Q\Omega = e^{itH}Qe^{-itH}\Omega = J_E \Delta_E^{1/2} e^{itH}Q^* e^{-itH}\Omega$$
$$= J_E e^{itH} \Delta_E^{1/2} Q^*\Omega = J_E e^{itH} J_E Q\Omega$$

for all $Q \in E\mathscr{A}E$ and by the antilinear nature of J_E. Q.E.D.

4. The Main Proofs

This section is mainly devoted to the proofs of Theorems 1.3 and 1.4. We start with the following simple operation: for any Σ' and Σ'' contained in $\Lambda = \mathbb{R} \times \bar{\mathbb{R}}$ we set

$$\Sigma' + \Sigma'' = \{(\varepsilon' + \varepsilon'', \lambda' + \lambda''): (\varepsilon', \lambda') \in \Sigma', (\varepsilon'', \lambda'') \in \Sigma''\},$$

where $\lambda' + \lambda''$ denotes the usual sum if both λ' and λ'' are finite and $-\infty$ otherwise. A set $\tilde{\Sigma}$ is called additive iff $\tilde{\Sigma} + \tilde{\Sigma} \subset \tilde{\Sigma}$. It is called semi-additive iff $\tilde{\Sigma}_{\text{finite}} + \tilde{\Sigma} \subset \tilde{\Sigma}$. Here $\tilde{\Sigma}_{\text{finite}}$ denotes the finite part of $\tilde{\Sigma}$:

$$\tilde{\Sigma}_{\text{finite}} = \tilde{\Sigma} \cap \mathbb{R}^2.$$

Lemma 4.1. *Let ω be an invariant state of a C^*-dynamical system $(\mathfrak{A}, \alpha)$. Assume that there exists a semi-additive set $\tilde{\Sigma}$ such that*

$$\Sigma^{\omega} \subset \tilde{\Sigma} \subset \Sigma^{\text{passive}}. \tag{4.1}$$

Then either ω is a KMS-*state with some non-negative inverse temperature $\beta \geqq 0$ or ω is a ground state.*

Proof. We shall consider two cases

1. $\Sigma^\omega = \Sigma^\omega_{\text{finite}}$

Let $p, q \in \Sigma^\omega$. Then $-p, -q \in \Sigma^\omega$ (cf. Theorem 3.1) and $np + mq \in \tilde{\Sigma}$ for any integer n and m, because $\tilde{\Sigma}$ is semi-additive. In virtue of (3.14) p and q belong to the same straight line passing through (0,0) (otherwise $\{np + mq : n, m \in \mathbb{Z}\}$ would form a lattice in $\mathbb{R}^2$ which is never contained in Σ^{passive}). Since this fact holds for every pair of points of Σ^ω, Σ^ω itself is contained in a straight line passing through (0,0). If this line is vertical (this is a very degenerate case) then $H = 0$ and ω is a ground state (we remind that ω is called a ground state if $H \geqq 0$). If the line is not vertical then there exists $\beta \geqq 0$ such that $\lambda = -\beta\varepsilon$ for all $(\varepsilon, \lambda) \in \Sigma^\omega$. Then [cf. (3.11) and (3.12)] $\Delta = e^{-\beta H}$ and ω is a KMS-state with the inverse temperature β.

2. $\Sigma^\omega \neq \Sigma^\omega_{\text{finite}}$

Then there exists $\varepsilon_0 \in \mathbb{R}$ such that $(\varepsilon_0, -\infty) \in \Sigma^\omega$. Assume that $(\varepsilon, \lambda) \in \Sigma^\omega$ for some $\varepsilon < 0$. Then [cf. (4.1) and (3.14)] $\lambda > 0$ and $(\varepsilon, \lambda) \in \tilde{\Sigma}_{\text{finite}}$. Since $\tilde{\Sigma}$ is semi-additive, $n(\varepsilon, \lambda) + (\varepsilon_0, -\infty) = (n\varepsilon + \varepsilon_0, -\infty) \in \tilde{\Sigma}$ for any natural n and for n sufficiently large (such that $n\varepsilon + \varepsilon_0 < 0$) we get contradiction with (3.14). Therefore $\varepsilon \geqq 0$ for any $(\varepsilon, \lambda) \in \Sigma^\omega$. It means that $H \geqq 0$ and ω is a ground state. Q.E.D.

The proof of Theorem 1.4 is now very simple. It is based on the following obvious formula:

$$\Sigma^{\overset{N}{\otimes}\omega} = \underbrace{\Sigma^\omega + \Sigma^\omega + \ldots + \Sigma^\omega}_{N\text{-copies}}.$$

If ω is completely passive then $\overset{N}{\bigotimes}\omega$ are passive and according to Theorem 3.1: $\Sigma^{\overset{N}{\otimes}\omega} \subset \Sigma^{\text{passive}}$ for all integer N. Let

$$\tilde{\Sigma} = \bigcup_{N=1}^{\infty} \Sigma^{\overset{N}{\otimes}\omega}.$$

It is clear that this set is additive and $\Sigma^\omega \subset \tilde{\Sigma} \subset \Sigma^{\text{passive}}$. Now, the statement of Theorem 1.4 follows directly from Lemma 4.1.

According to the same lemma, in order to demonstrate Theorem 1.3 it is sufficient to prove the following

Proposition 4.2. *Let* $(\mathfrak{A}, \alpha)$ *be a* C^**-dynamical system and* G *be a locally compact amenable group of automorphisms of* $\mathfrak{A}$ *commuting with* α. *Assume that* ω *is a* G*-weakly clustering and passive state of* $(\mathfrak{A}, \alpha)$. *Then* Σ^ω *is semi-additive.*

Proof. It is known that G-weakly clustering states are G-invariant. Therefore there exists a strongly continuous unitary representation of G acting on $\mathscr{H}$ which implements the action of G on $\mathfrak{A}$:

$$\pi(\varkappa_x(A)) = U_x \pi(A) U_x^* \quad x \in G, A \in \mathfrak{A} \tag{4.2}$$

and leaves Ω invariant: $U_x\Omega=\Omega$. Clearly, U_x commutes with H and Δ and $U_x\mathscr{A}U_x^*=\mathscr{A}$.

Let $\eta_x(f(x))$ denotes the mean value of a continuous function f on G. It follows immediately from Definition 1.2 that

$$\eta_x((\psi|U_x\phi))=(\psi|\Omega)(\Omega|\phi) \quad \psi,\phi\in\mathscr{H}. \tag{4.3}$$

We shall use the method of Arveson [5]. Let $Q\in\mathscr{A}$ and $\mathcal{O}$ be a region in $\Lambda=\mathbb{R}\times\bar{\mathbb{R}}$. We say that the spectrum of Q is contained in $\mathcal{O}$ (and write $\operatorname{Sp}Q\subset\mathcal{O}$) iff the following three conditions are satisfied:

a) Q vanishes on $(E\mathscr{H})^\perp$; $QE=Q$.

b) $$\int_{\mathbb{R}^2} T_t\Delta_E^{i\tau}EQ\Delta_E^{-i\tau}ET_t^*\varphi(t,\tau)dtd\tau=0$$

for any function $\varphi\in\mathscr{S}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} e^{it\varepsilon}e^{i\tau\lambda}\varphi(t,\tau)dtd\tau=0$$

for all $(\varepsilon,\lambda)\in\mathcal{O}_{\text{finite}}$.

c) $$\int_{\mathbb{R}} T_t(I-E)QT_t^*\psi(t)dt=0$$

for any function $\psi\in\mathscr{S}(\mathbb{R}^1)$ such that

$$\int_{\mathbb{R}} e^{it\varepsilon}\psi(t)dt=0$$

for all ε such that $(\varepsilon,-\infty)\in\mathcal{O}$.

The following simple facts are very useful:

A) If $p\in\Sigma^\omega$ then for any neighbourhood $\mathcal{O}$ of p (in Λ) there exists $Q\in\mathscr{A}$ such that $\operatorname{Sp}Q\subset\mathcal{O}$ and $Q\Omega\neq0$.

B) If $Q,Q'\in\mathscr{A}$ and $\operatorname{Sp}Q\subset\mathcal{O}$, $\operatorname{Sp}Q'\subset\mathcal{O}'$ then $\operatorname{Sp}QQ'\subset\mathcal{O}+\mathcal{O}'$.

C) If $Q\in\mathscr{A}$, $\operatorname{Sp}Q\subset\mathcal{O}$ and $Q\Omega\neq0$, then the intersection of $\mathcal{O}$ with the support of $dP(\varepsilon,\lambda)$ is not empty.

The proofs of these facts, although very simple are rather annoying and will be omitted. We refer to [5, 7] where they are considered in a slightly simpler setting.

Now, let $p'\in\Sigma^\omega$ and $p''\in\Sigma^\omega_{\text{finite}}$ and $p=p'+p''$. Then, for any neighbourhood $\mathcal{O}$ of p one can find neighbourhoods $\mathcal{O}'$ of p' and $\mathcal{O}''$ of p'' such that $\mathcal{O}'+\mathcal{O}''\subset\mathcal{O}$. We may assume that $\mathcal{O}''$ does not contain any point at infinity. According to (A) we can find Q' and $Q''\in\mathscr{A}$ such that $\operatorname{Sp}Q'\subset\mathcal{O}'$ and $\operatorname{Sp}Q''\subset\mathcal{O}''$ and

$$Q'\Omega\neq0\neq Q''\Omega. \tag{4.4}$$

Since $\mathcal{O}''$ contains no points of the form $(\varepsilon,-\infty)$, then [cf. c)] $(I-E)Q''=0$ and Q''^* vanishes on $(E\mathscr{H})^\perp$.

Since U_x commutes with H and Δ, we have $\operatorname{Sp}U_xQ''U_x^*\subset\mathcal{O}''$. Therefore [cf. (B)]

$$\operatorname{Sp}(Q'U_xQ''U_x^*)\subset\mathcal{O}'+\mathcal{O}''\subset\mathcal{O}$$

and if for some $x\in G$

$$Q'U_xQ''U_x^*\Omega\neq0 \tag{4.5}$$

then [cf. (C)] the intersection $\mathcal{O}\cap\Sigma^\omega$ is not empty. To prove (4.5) assume that for all $x\in G: Q'U_xQ''U_x^*\Omega=0$. Let us notice that the operator $Q'U_xQ''U_x^*$ belonging to $\mathscr{A}$ vanishes on $(E\mathscr{H})^\perp$. For such operators Ω is a separating vector and we have $Q'U_xQ''U_x^*=0$. Therefore $Q'U_xQ''=0$ and

$$(Q'\Omega|Q'U_xQ''Q''^*\Omega)=0$$

for all $x\in G$. Calculating the mean value on G and using (4.3) we obtain

$$\|Q'\Omega\|^2\|Q''^*\Omega\|^2=0$$

According to (4.4) $Q'\Omega\neq 0$. Therefore $Q''\Omega=0$ and, since Q''^* vanishes on $(E\mathscr{H})^\perp$, $Q''^*=0$. Then $Q''=0$, $Q''\Omega=0$ and we obtain the contradiction with (4.4). Therefore (4.5) holds at least for one $x\in G$. This way we proved that $\Sigma^\omega\cap\mathcal{O}$ is not empty for any neighbourhood $\mathcal{O}$ of $p'+p''$. Therefore (Σ^ω is closed) $p'+p''\in\Sigma^\omega$ and Σ^ω is semiadditive. Q.E.D.

We would like to end our considerations with the following interesting result:

Theorem 4.3. *Let ω be a non-central passive state of a C^*-dynamical system $(\mathfrak{A},\alpha)$. We assume that ω is weakly clustering with respect to the time translation group. Then either*

1) *ω is a ground state and the* GNS *representation associated with it is irreducible, or*

2) *ω is a* KMS-*state with the inverse temperature $\beta>0$ and the von Neumann algebra generated by the* GNS *representation associated with ω is a factor of type* III_1.

Proof. We use the notation introduced in the previous sections. Let $\mathscr{T}$ be the von Neumann algebra generated by $\{T_t\}_{t\in\mathbb{R}}$. Since ω is weakly clustering, Ω is the only vector invariant under $T_t(t\in\mathbb{R})$ and therefore $|\Omega)(\Omega|\in\mathscr{T}$. Remembering that Ω is cyclic for $\mathscr{A}$ we see that the von Neumann algebra generated by $\mathscr{T}$ and $\mathscr{A}$ coincides with $B(\mathscr{H})$:

$$\mathscr{A}\vee\mathscr{T}=B(\mathscr{H}). \tag{4.6}$$

We have to consider two cases:

1. ω is a ground state. Then $H\geqq 0$ and using the Kadison result [9] we get $\mathscr{T}\subset\mathscr{A}$. Now (4.6) shows that $\mathscr{A}=B(\mathscr{H})$ i.e., the representation π is irreducible.

2. ω is a KMS-state with the inverse temperature $\beta\geqq 0$. Since ω is not central, β must be strictly positive. Then the modular automorphism group σ_t^ω is related to the time translations in a very simple way; $\sigma_t^\omega=\alpha_{-\beta t}$. Moreover, using (3.17) we see that $J\mathscr{T}J=\mathscr{T}$. Therefore, in virtue of (4.6)

$$\mathscr{A}\cap\mathscr{T}'=J\mathscr{A}'J\cap J\mathscr{T}'J=J(\mathscr{A}'\cap\mathscr{T}')J=\{\lambda I\}.$$

It shows that

$$\{A\in\mathscr{A}:\sigma_t^\omega(A)=A \text{ for all } t\in\mathbb{R}\}=\{\lambda I\}$$

and this property is characteristic for the type III_1 factors ([7]). Q.E.D.

Remark. It is known, that if the fix-point algebra is a factor, then the invariant $\Gamma(\mathscr{A})$ coincides with the spectrum of $\log\Delta$. It implies that in the second case the spectrum of H coincides with $\mathbb{R}$.

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Communicated by H. Araki

Received September 30, 1977