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Hausdorff Measure and the Navier-Stokes Equations

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Abstract. Solutions to the Navier-Stokes equations are continuous except for a closed set whose Hausdorff dimension does not exceed two.

1. Informal Statement of Results

Let $v:R^3 \to R^3$ be a divergence free, square integrable vector field on 3-space. We will show that there exists a function $u:R^3 \times R^+ \to R^3$ ($R^+ = \{t:t>0\}$ is time) which is a weak solution to the Navier-Stokes equations of incompressible fluid flow with viscosity $=1$ and initial conditions v, and which satisfies the following: There exists a set $S \subset R^3 \times R^+$ such that the two dimensional Hausdorff measure of S is finite, $(R^3 \times R^+) - S$ is an open set, and the restriction of u to $(R^3 \times R^+) - S$ is a continuous function.

The above will be derived as a consequence of a more general theorem in which u satisfies a weak form of the Navier-Stokes equations with an external force $f:R^3 \times R^+ \to R^3$ which is divergence free with the property $f(x,t) \cdot u(x,t) \leqq 0$.

2. Notation and Complete Statement of Results

Hausdorff measure is defined in [2, p. 171]. We set $R^+ = \{t \in R : t > 0\}$ and $B(a,r) = \{x \in R^3 : |x-a| \leqq r\}$ for all $a \in R^3$ and $r>0$. The norm $|\ |$ is always euclidean norm and $\|\ \|_p$ is the L^p norm. Open and closed intervals are denoted (a,b) and $[a,b]$, respectively. If $f:X \to R$ and $A \subset X$ then $\sup(f,A)$ is the supremum of f over A and $\operatorname{spt}(f)$ is the closure of $\{x : f(x) \neq 0\}$. If f and g are functions defined on a subset of $R^3 \times R$, h is a function on R^3, and k is a function on R, then we set

$$(f*g)(x,t) = \iint f(y,s) g(x-y,t-s)\,dy\,ds\,,$$
$$(f*h)(x,t) = \int f(y,t) h(x-y)\,dy\,,$$
$$(f*k)(x,t) = \int f(x,s) k(t-s)\,ds$$

whenever the integrals make sense. If $X = R^3$, $X = R$, or $X = R^3 \times R^+$, we let $C^\infty(X,R)$ be the set of infinitely differentiable functions $f:X \to R$. In addition, $C_0^\infty(X,R)$ is the set of all functions in $C^\infty(X,R)$ which are zero outside of some compact set. We also set $D^\infty(R^3\times R^+,R)=\{f\in C^\infty(R^3\times R^+,R):\operatorname{spt}(f)\subset R^3\times[a,b]$ for some $0<a<b<\infty\}$. If f is a distribution defined on an open subset of $R^3\times R$ then D_if, $D_{ij}f$, etc. are the distribution partial derivatives $(\partial/\partial x_i)f$, $(\partial^2/\partial x_i\partial x_j)f$ with respect to the variables x_1,x_2,x_3 of R^3. The partial derivative of f with respect to the R variable of $R^3\times R$ is denoted D_tf. We also set $Df=(D_1f,D_2f,D_3f)$, $\Delta f=D_{ii}f$ (repeated indices are summed), and $\operatorname{div}(f)=D_if_i$ in case the range of f is R^3. Similar definitions are made for distributions defined on R^3 and R.

An absolute constant is a positive constant that is independent of all the parameters in this paper. The letter C always denotes an absolute constant. The value of C changes from line to line (e.g. $2C\leqq C$). When an absolute constant is denoted by a letter other than C, its value remains fixed.

The statements below (Parts 1 and 2) are called *Hypothesis I*:

Part. 1. We have a Lebesgue measurable function $u:R^3\times R^+\to R^3$ (a time dependent velocity vector field), a Lebesgue measurable and locally integrable function $p:R^3\times R^+\to R$ (pressure), and a constant $0<L<\infty$ such that

$$\operatorname{div}(u)=0, \tag{2.1}$$

$$\int_{R^3}|u(x,t)|^2dx\leqq L \quad \text{for almost every} \quad t\in R^+, \tag{2.2}$$

the distribution Du is a square integrable function satisfying

$$\int_0^\infty\int_{R^3}|Du(x,t)|^2dx\,dt\leqq L, \tag{2.3}$$

and for almost every $t\in R^+$ we have

$$\int_{R^3}p(x,t)\Delta\phi(x)dx=-\int_{R^3}D_iu_j(x,t)D_ju_i(x,t)\phi(x)dx \tag{2.4}$$

if $\phi\in C^\infty(R^3,R)$, ϕ is bounded, $|x|\,|D\phi(x)|$ is bounded, and $\Delta\phi\in C_0^\infty(R^3,R)$.

Part 2. We assume that the conditions

$$\begin{aligned}&\phi\in D^\infty(R^3\times R^+,R);\quad \phi(x,t)\geqq 0 \quad \text{for all} \quad (x,t);\\ &\text{and} \quad \phi, D\phi, \Delta\phi+D_t\phi \quad \text{are bounded}\end{aligned} \tag{2.5}$$

imply that (2.6) holds. Note that (2.2), (2.3), Lemma 3.2, and Lemma 3.6 can be used to show that the integrals in (2.6) exist.

$$-2^{-1}\left(\int|u|^2(D_t\phi+\Delta\phi)\right)+\int|Du|^2\phi\leqq\int u_i(2^{-1}|u|^2+p)D_i\phi. \tag{2.6}$$

Hypothesis II is the following: In addition to Hypothesis I, we assume

$$-\int u_i(D_t\phi+\Delta\phi)=\int u_ju_iD_j\phi+\int pD_i\phi \tag{2.7}$$

for every $i\in\{1,2,3\}$ and $\phi\in C_0^\infty(R^3\times R^+,R)$.

Hypothesis I is a weak form of the classical Navier-Stokes equations

$$D_tu_i=-u_jD_ju_i-D_ip+\Delta u_i+f_i, \qquad \operatorname{div}(u)=0 \tag{2.8}$$

where the external force f satisfies $\operatorname{div}(f)=0$ and $f(x,t)\cdot u(x,t)\leqq 0$. Hypothesis II is a weak form of (2.8) with $f=0$. We will prove

2.1. Theorem. *If Hypothesis I holds then there exist a function $u': R^3 \times R^+ \to R^3$ and a set $S \subset R^3 \times R^+$ satisfying the following: The functions u and u' are equal almost everywhere, the two dimensional Hausdorff measure of S is finite, $S \cap \{(x,t): t \geqq \varepsilon\}$ is compact for every $\varepsilon > 0$, and $|u'|$ is bounded on every compact set $K \subset R^3 \times R^+$ which satisfies $K \cap S = \emptyset$.*

The proof of this theorem includes a priori estimates on the size of $|u'|$. It is possible to show that the Hausdorff dimension of S is at most 7/4. We also have

2.2. Theorem. *If Hypothesis II holds and S is as in Theorem 2.1 then there exists a function $u'': R^3 \times R^+ \to R^3$ such that u and u'' are equal almost everywhere, and u'' is continuous on $(R^3 \times R^+) - S$.*

2.3. Theorem. *If $v: R^3 \to R^3$ is a square integrable function satisfying* $\operatorname{div}(v) = 0$ *then there exists u satisfying Hypothesis II and*

$$-\int_{R^3} v_i(x)\phi(x,0)dx - \int_{R^3 \times R^+} u_i(D_t\phi + \Delta\phi)$$

$$= \int_{R^3 \times R^+} u_j u_i D_j \phi + \int_{R^3 \times R^+} pD_i\phi \tag{2.9}$$

if $\phi: R^3 \times R \to R$ is smooth with compact support and $i \in \{1,2,3\}$.

Here (2.9) states that v is the initial condition for the solution u.

This type of partial regularity is similar to that obtained by Almgren for solutions to generalized variational problems [1]. The study of the relationship between Hausdorff measure and the geometry of turbulence was started by Mandelbrot [6].

The next three sections contain the proof of Theorems 2.1 and 2.2. The proof of Theorem 2.3 is outlined in Section 6.

3. Preliminary Estimates

Throughout this section we assume that Part I of Hypothesis I holds.

3.1. Lemma. *If $f: R^3 \to R$, $f \in L^2$, and $Df \in L^2$, then*

(1) $\int |f|^6 \leqq C(\int |Df|^2)^3$

(2) $\int |f|^3 \leqq C\varepsilon^{-3}(\int |f|^2)^3 + C\varepsilon(\int |Df|^2)$ *whenever* $0 < \varepsilon < \infty$.

Proof. Part (1) is the first inequality in line 9, p. 127 of [9]. We use Hölder's inequality, part (1), and Young's inequality

$$ab \leqq (1/4)(\delta^{-1}a)^4 + (3/4)(\delta b)^{4/3} \quad \text{for} \quad a, b \geqq 0 \quad \text{and} \quad \delta = \varepsilon^{3/4}$$

to estimate

$$\begin{aligned}
\int |f|^3 &= \int |f|^{3/2}|f|^{3/2} \\
&\leqq \left(\int (|f|^{3/2})^{4/3}\right)^{3/4}\left(\int (|f|^{3/2})^4\right)^{1/4} \\
&= \left(\int |f|^2\right)^{3/4}\left(\int |f|^6\right)^{1/4} \\
&\leqq C\left(\int |f|^2\right)^{3/4}\left(\int |Df|^2\right)^{3/4} \\
&\leqq C\varepsilon^{-3}\left(\int |f|^2\right)^3 + C\varepsilon\left(\int |Df|^2\right).
\end{aligned}$$

3.2. Lemma. *If* $0<T<\infty$ *then* $\int_0^T \int_{R^3} |u(x,t)|^3 dx dt \leqq CL^{3/2} T^{1/4}$.

Proof. Using Lemma 3.1 with $\varepsilon = L^{1/2} T^{1/4}$, (2.2), and (2.3), we obtain

$$\begin{aligned}
&\int_0^T \int_{R^3} |u(x,t)|^3 dx dt \\
&\leqq C\varepsilon^{-3}\left(\int_0^T \left(\int_{R^3} |u(x,t)|^2 dx\right)^3 dt\right) + C\varepsilon\left(\int_0^T \int_{R^3} |Du(x,t)|^2 dx dt\right) \\
&\leqq CL^{3/2} T^{1/4}.
\end{aligned}$$

3.3. Definition. We fix $f_0 \in C_0^\infty(R^3, R)$ and $g_0 \in C_0^\infty(R, R)$ such that $\operatorname{spt}(g_0) \subset [-1, 1]$, $f_0 \geqq 0$, $g_0 \geqq 0$, $f_0(x) = f_0(-x)$, $g_0(t) = g_0(-t)$, and $\int f_0 = \int g_0 = 1$. For $n = 1, 2, 3, \ldots$ we set $f_n(x) = n^3 f_0(nx)$ and $g_n(t) = n g_0(nt)$. We let A consist of all $t \in R^+$ such that the function $p_t(x) = p(x,t)$ is locally integrable, (2.2) and (2.4) hold, the function $d_t(x) = Du(x,t)$ is square integrable, the divergence of the function $u_t(x) = u(x,t)$ is zero, and $\lim_n (|u|^2 * g_n)(x,t) = |u(x,t)|^2$ for almost every $x \in R^3$. Part 1 of Hypothesis I, Fubini's theorem, and [10, Theorem 1.25, p. 13] imply that A is almost all of R^+.

3.4. Lemma. *If* $t \in A$, $\alpha \in C_0^\infty(R^3, R)$, $\beta = 1 - \alpha$, $\beta(x) = 0$ *for all* x *in a neighborhood of* 0, $\alpha'(x) = -(4\pi|x|)^{-1}\alpha(x)$, *and* $\beta'(x) = -(4\pi|x|)^{-1}\beta(x)$, *then*

$$p(x,t) = -(D_i u_j D_j u_i * \alpha')(x,t) - (u_j u_i * D_{ij}\beta')(x,t)$$

holds for almost every $x \in R^3$.

Proof. Define $k: R^3 - \{0\} \to R$ by $k(x) = -(4\pi|x|)^{-1}$. Recalling Definition 3.3, we have $\Delta(k * f_n) = f_n$ (see [9, p. 126]). Hence 3.3 and Part 1 of Hypothesis I yield the following for all $x \in R^3$:

$$\begin{aligned}
(p * f_n)(x,t) &= (p * \Delta(k * f_n))(x,t) \\
&= ((-D_i u_j D_j u_i) * (k * f_n))(x,t) \\
&= ((-D_i u_j D_j u_i) * (\alpha' * f_n))(x,t) + ((-D_i u_j D_j u_i) * (\beta' * f_n))(x,t) \\
&= ((-D_i u_j D_j u_i) * (\alpha' * f_n))(x,t) + ((-u_j u_i) * (D_{ij}\beta' * f_n))(x,t).
\end{aligned}$$

Since $\alpha' \in L^1$, $\alpha' * f_n$ converges to α' in the L^1 norm (see [10, Theorem 1.18, p. 10]). Hence the assumption $t \in A$ and [10, Theorem 1.3, p. 3] imply

$$\lim_n \int_{R^3} |((-D_i u_j D_j u_i) * ((\alpha' * f_n) - \alpha'))(x,t)| dx = 0.$$

Hence [3, (11.26)] implies that there exists a subsequence n_k of the positive integers such that

$$\lim_k ((-D_i u_j D_j u_i) * (\alpha' * f_{n_k}))(x,t) = ((-D_i u_j D_j u_i) * \alpha')(x,t)$$

holds for almost every $x \in R^3$. Since $t \in A$ and $D_{ij}\beta' * f_n$ converges uniformly to $D_{ij}\beta'$, we have

$$\lim_k ((-u_j u_i) * (D_{ij}\beta' * f_{n_k}))(x,t) = ((-u_j u_i) * D_{ij}\beta')(x,t)$$

for all $x \in R^3$. Finally, [10, Theorem 1.25, p. 13] yields

$$\lim_k (p*f_{n_k})(x,t) = p(x,t) \quad \text{for almost every} \quad x \in R^3.$$

These statements yield the conclusion of the lemma.

3.5. Lemma. *If* $t \in A$ *and* $0 < r < \infty$ *then*

$$\Big(\int_{R^3} |p(x,t)|^2 dx\Big)^{1/2}$$
$$\leqq C r^{1/2}\Big(\int_{R^3} |Du(x,t)|^2 dx\Big) + C r^{-3/2}\Big(\int_{R^3} |u(x,t)|^2 dx\Big).$$

Proof. Given r, we fix α and β as in Lemma 3.4 such that $\beta(x)=0$ for $x \in B(0,r)$, $\operatorname{spt}(\alpha) \subset B(0,2r)$, $0 \leqq \alpha(x) \leqq 1$, $|D\alpha(x)| \leqq C r^{-1}$, and $|D_{ij}\alpha(x)| \leqq C r^{-2}$. Then we have $\|\alpha'\|_2 \leqq C r^{1/2}$ and $\|D_{ij}\beta'\|_2 \leqq C r^{-3/2}$. Hence Lemma 3.4 and [10, Theorem 1.3, p. 3] yield the conclusion.

3.6. Lemma. *If* $0 < T < \infty$ *then* $\int_0^T \int_{R^3} |u(x,t)|\,|p(x,t)| dx dt \leqq C L^{3/2} T^{1/4}$.

Proof. Using the Schwarz inequality, Lemma 3.5 with $r = T^{1/2}$, (2.2), and (2.3), we estimate

$$\int_0^T \int_{R^3} |u(x,t)|\,|p(x,t)| dx dt$$
$$\leqq \int_0^T \Big(\int_{R^3} |u(x,t)|^2 dx\Big)^{1/2} \Big(\int_{R^3} |p(x,t)|^2 dx\Big)^{1/2} dt$$
$$\leqq C L^{1/2}\Big(\int_0^T T^{1/4}\Big(\int_{R^3} |Du(x,t)|^2 dx\Big) + T^{-3/4}\Big(\int_{R^3} |u(x,t)|^2 dx\Big) dt\Big)$$
$$\leqq C L^{3/2} T^{1/4}.$$

3.7. Lemma. *If Hypothesis I holds,* $s \in A$, $B = R^3 \times [0,s]$, *and* ϕ *satisfies* (2.5) *then*

$$2^{-1}\int_{R^3} |u(x,s)|^2 \phi(x,s) dx - 2^{-1}\int_B |u|^2 (D_t\phi + \Delta\phi) + \int_B |Du|^2 \phi$$
$$\leqq \int_B u_i (2^{-1}|u|^2 + p) D_i \phi.$$

Proof. Let $h_n : R^3 \times R \to R$ be the function that satisfies $D_t h_n(x,t) = -g_n(t-s)$ (see Definition 3.3) and $h_n(x,t) = 1$ for $t < s - n^{-1}$. Then $h_n(x,t) = 0$ for $t > s + n^{-1}$. We obtain the conclusion by substituting ϕh_n for ϕ in (2.6), taking the limit inferior over n, using Fatou's lemma, and observing that

$$\lim_n \int_{R^+} |u(x,t)|^2 \phi(x,t) g_n(t-s) dt = |u(x,s)|^2 \phi(x,s)$$

holds for almost every $x \in R^3$ [this is a consequence of $s \in A$ and the relation $g_n(t) = g_n(-t)$].

3.8. Lemma. *If $f:R^3\to R$, $f\in L^2$, and $Df\in L^2$ then for every $a\in R^3$ and $0<r<\infty$ we have*

$$\left(\int_{B(a,r)}|f|^4\right)^{1/2}\leqq Cr^{1/2}\left(\int_{B(a,2r)}|Df|^2\right)+Cr^{-3/2}\left(\int_{B(a,2r)}|f|^2\right).$$

Proof. Let $g\in C_0^\infty(R^3,R)$ satisfy $\operatorname{spt}(g)\in B(a,2r)$, $g(x)=1$ *if* $x\in B(a,r)$, $0\leqq g(x)\leqq 1$, and $|Dg(x)|\leqq Cr^{-1}$. We apply the Schwarz inequality, Lemma 3.1 (1), Young's inequality [4, p. 11], and the estimate $|D(fg)|\leqq|Df|\,|g|+|f|\,|Dg|$ to write

$$\begin{aligned}\int_{B(a,r)}|f|^4&\leqq\int|fg|^4\\&=\int|fg|^3|fg|\\&\leqq\left(\int|fg|^6\right)^{1/2}\left(\int|fg|^2\right)^{1/2}\\&\leqq C\left(\int|D(fg)|^2\right)^{3/2}\left(\int|fg|^2\right)^{1/2}\\&\leqq Cr\left(\int|D(fg)|^2\right)^2+Cr^{-3}\left(\int|fg|^2\right)^2\\&\leqq Cr\left(\int_{B(a,2r)}|Df|^2\right)^2+Cr^{-3}\left(\int_{B(a,2r)}|f|^2\right)^2.\end{aligned}$$

4. The Basic Estimate

In this section we assume that Hypothesis I (Parts 1 and 2) holds. The section is devoted to proving the following:

4.1. Theorem. *There exist absolute constants ε and K satisfying the following: If $a\in R^3$, $b\in A$ (see Definition 3.3), $\gamma>0$, $b-\gamma^2>0$, and*

$$\int_{b-\gamma^2}^{b}\int_{R^3}|u(x,t)|(2^{-1}|u(x,t)|^2+|p(x,t)|)(|x-a|+\gamma)^{-4}dxdt\leqq\varepsilon\gamma^{-2}\tag{4.1}$$

then

$$\int_{B(a,\tau\gamma)}|u(x,b)|^2dx\leqq K\tau^3\gamma\quad\textit{for}\quad 0<\tau\leqq 1/2.\tag{4.2}$$

We fix $a\in R^3$, $b\in A$, and $\gamma>0$ with $b-\gamma^2>0$. For integers k we set

$$E_k=\{(x,t):|x-a|<\gamma 2^{-k},\,b-\min(\gamma^2 2^{-2k},\gamma^2)<t<b\}.\tag{4.3}$$

4.2. Lemma. *There exist $\phi_n\in D^\infty(R^3\times R^+,R)$ for $n=1,2,3,\ldots$ such that $\phi_n(x,t)\geqq 0$, the functions ϕ_n, $D\phi_n$, and $\Delta\phi_n+D_t\phi_n$ are bounded,*

$$\phi_n(x,t)=0\quad\textit{if}\quad t\leqq b-\gamma^2,\tag{4.4}$$

$$\sup(\phi_n+\gamma 2^{-n}|D\phi_n|,E_n)\leqq C\gamma^{-3}2^{3n},\tag{4.5}$$

$$\sup(\phi_n+\gamma 2^{-k}|D\phi_n|,E_k-E_{k+1})\leqq C\gamma^{-3}2^{3k}\quad\textit{if}\quad 0\leqq k<n,\tag{4.6}$$

$$\sup(\phi_n+\gamma 2^{-k}|D\phi_n|,E_k-E_{k+1})\leqq C\gamma^{-3}2^{4k}\quad\textit{if}\quad k<0,\tag{4.7}$$

$$\sup(|D_t\phi_n+\Delta\phi_n|,E_0)\leqq C\gamma^{-5},\tag{4.8}$$

$$\sup(|D_t\phi_n+\Delta\phi_n|,E_k-E_{k+1})\leqq C\gamma^{-5}2^{4k}\quad\textit{if}\quad k<0,\tag{4.9}$$

and

$$(|x-a|+\gamma 2^{-n})^{-4} \leqq C\gamma^{-1}2^n\phi_n(x,t) \quad \textit{if} \quad b-\gamma^2 2^{-2n} \leqq t \leqq b. \tag{4.10}$$

Proof. We fix n. Let $h_n: R^3 \to R^+$ be defined by $h(x)=\gamma 2^{-n}(|x|+\gamma 2^{-n})^{-4}$. We define $F: R^3 \times \{t: t<0\} \to R^+$ by

$$F(x,t)=(2\sqrt{\pi})^{-3}(-t)^{-3/2}\exp(|x|^2/(4t)).$$

The function F is the fundamental solution to the heat equation with time reversed. We define $\psi_n: R^3 \times (-\infty, b+\gamma^2 2^{-2n}) \to R^+$ by

$$\psi_n(x,t)=(F*h_n)(x-a, t-(b+\gamma^2 2^{-2n})).$$

We have $D_t\psi_n + \Delta\psi_n = 0$, and the properties (4.5), (4.6), (4.7), (4.10) are satisfied if ϕ_n is replaced by ψ_n. Let $g: R \to [0,1]$ be a C^∞ function such that $g(t)=0$ if $t \leqq b-\gamma^2$, $g(t)=1$ if $b-\gamma^2/2 \leqq t \leqq b$, $g(t)=0$ if $b+\gamma^2 2^{-2n-1} \leqq t$, and $|D_t g(t)| \leqq C\gamma^{-2}$ if $b-\gamma^2 \leqq t \leqq b-\gamma^2/2$. Define $\phi_n \in D^\infty(R^3 \times R^+, R)$ by $\phi_n(x,t)=g(t)\psi_n(x,t)$ if $t<b+\gamma^2 2^{-2n}$, and $\phi_n(x,t)=0$ if $t>b+\gamma^2 2^{-2n-1}$. Then (4.4) is clear, and (4.5), (4.6), (4.7), (4.10) follow from the corresponding estimates on ψ_n. We have $(D_t\phi_n + \Delta\phi_n)(x,t) = D_t g(t)\psi_n(x,t)$ if $t \leqq b$. In particular, we have $(D_t\phi_n + \Delta\phi_n)(x,t)=0$ if $b-\gamma^2/2 \leqq t \leqq b$. Now (4.8) and (4.9) follow from the estimates on ψ_n.

4.3. Definition. For $k=1,2,3,\ldots$ we fix C^∞ functions r_k on a neighborhood of $R^3 \times [b-\gamma^2, b]$ such that $r_k(x,t) \in [0,1]$, $r_k(x,t)=1$ if $(x,t) \in R^3 \times (b-\gamma^2, b)$ and $(x,t) \notin E_{k+1}$, $r_k(x,t)=0$ if $(x,t) \in E_{k+2}$, and $\sup(|Dr_k|, E_{k+1}-E_{k+2}) \leqq C\gamma^{-1}2^k$. For $n=1,2,3,\ldots$ and $\delta>0$ the inequalities (4.11), (4.12), (4.13) will be known as Property $P(n,\delta)$:

$$\int_{b-\gamma^2}^{b} \int_{R^3} |u(x,t)|^2(|x-a|+\gamma)^{-4}\,dx\,dt \leqq \delta\gamma^{-1} \tag{4.11}$$

$$\int_{b-\gamma^2}^{b} \left| \int_{R^3} u_i(x,t)(2^{-1}|u(x,t)|^2 + p(x,t))D_i\phi_n(x,t)\,dx \right| dt \leqq \delta\gamma^{-2} \tag{4.12}$$

$$\int_{b-\gamma^2}^{b} \left| \int_{R^3} u_i(x,t)(2^{-1}|u(x,t)|^2 + p(x,t))D_i(r_n\phi_q)(x,t)\,dx \right| dt$$
$$\leqq \delta\gamma^{-2} \quad \text{if} \quad q>n. \tag{4.13}$$

4.4. Lemma. *There exists an absolute constant M with the following property: If $P(n,\delta)$ holds (see Definition 4.3) then $P(n+1, \delta+M\delta^{3/2}2^{-n})$ holds.*

Proof. Suppose that $P(n,\delta)$ holds for some n and δ. Let $s \in A$ (see Definition 3.3) with $b-\gamma^2 2^{-2n} \leqq s \leqq b$ and set $B_s = R^3 \times [0,s]$. Using Lemma 3.7, Lemma 4.2, and $P(n,\delta)$ [Parts (4.12) and (4.11)] we obtain

$$\begin{aligned}
&2^{-1}\int_{R^3} |u(x,s)|^2\phi_n(x,s)\,dx + \int_{B_s} |Du|^2\phi_n \\
&\leqq 2^{-1}\int_{B_s} |u|^2(D_t\phi_n + \Delta\phi_n) + \int_{B_s} u_i(2^{-1}|u|^2+p)D_i\phi_n \\
&\leqq C\gamma^{-1}\int_{b-\gamma^2}^{s}\int_{R^3} |u(x,t)|^2(|x-a|+\gamma)^{-4}\,dx\,dt + \delta\gamma^{-2} \\
&\leqq C\delta\gamma^{-2}.
\end{aligned} \tag{4.14}$$

Now (4.10) and (4.14) yield

$$\int_{R^3} |u(x,s)|^2(|x-a|+\gamma 2^{-n})^{-4}dx$$

$$\leqq C\gamma^{-1}2^n \int_{R^3} |u(x,s)|^2\phi_n(x,s)dx$$

$$\leqq C\delta\gamma^{-3}2^n \quad \text{if} \quad s\in A \quad \text{and} \quad b-\gamma^2 2^{-2n} \leqq s \leqq b. \tag{4.15}$$

Using (4.3), (4.10), and (4.14) (with $s=b$) we obtain

$$\int_{E_n} |Du|^2 \leqq C\gamma^3 2^{-3n} \int_{B_b} |Du|^2\phi_n \leqq C\delta\gamma 2^{-3n}. \tag{4.16}$$

For $q=n+1, n+2, n+3,\ldots$ we define $h_q: R^3\times(b-\gamma^2,b)\to R$ by (see Definition 4.3)

$$h_{n+1}=(1-r_n)\phi_{n+1}, h_q=(r_{n+1}-r_n)\phi_q \quad \text{if} \quad q>n+1. \tag{4.17}$$

From Definition 4.3 and Lemma 4.2 we obtain

$$\begin{gathered} h_q(x,t)=0 \quad \text{if} \\ (x,t)\notin E_{n+1}, \|h_q\|_\infty \leqq C\gamma^{-3}2^{3n}, \|Dh_q\|_\infty \leqq C\gamma^{-4}2^{4n}. \end{gathered} \tag{4.18}$$

Let $s\in A$ such that $b-\gamma^2 2^{-2n-2}<s<b$. Using (4.18), the Schwarz inequality, (4.15), Lemma 3.8, and (4.15) again, we obtain

$$\left|\int_{R^3} u_i(x,s)(2^{-1}|u(x,s)|^2)D_ih_q(x,s)dx\right|$$

$$\leqq C\gamma^{-4}2^{4n}\left(\int_{B(a,\gamma 2^{-n-1})} |u(x,s)|^2dx\right)^{1/2}\left(\int_{B(a,\gamma 2^{-n-1})} |u(x,s)|^4dx\right)^{1/2}$$

$$\leqq C\delta^{1/2}\gamma^{-7/2}2^{5n/2}\left(\int_{B(a,\gamma 2^{-n-1})} |u(x,s)|^4dx\right)^{1/2}$$

$$\leqq C\delta^{1/2}\gamma^{-3}2^{2n}\left(\int_{B(a,\gamma 2^{-n})} |Du(x,s)|^2dx\right)$$

$$+C\delta^{1/2}\gamma^{-5}2^{4n}\left(\int_{B(a,\gamma 2^{-n})} |u(x,s)|^2dx\right)$$

$$\leqq C\delta^{1/2}\gamma^{-3}2^{2n}\left(\int_{B(a,\gamma 2^{-n})} |Du(x,s)|^2dx\right)+C\delta^{3/2}\gamma^{-4}2^n. \tag{4.19}$$

Now we integrate (4.19) over s (recall Definition 3.3) and apply (4.16) and (4.3) to obtain

$$\int_{b-\gamma^2 2^{-2n-2}}^{b} \left|\int_{R^3} u_i(x,s)(2^{-1}|u(x,s)|^2)D_ih_q(x,s)dx\right|ds \tag{4.20}$$

$$\leqq C\delta^{3/2}\gamma^{-2}2^{-n}.$$

We choose α and β as in Lemma 3.4 such that $0 \leqq \alpha(x) \leqq 1$, $\alpha(x)=0$ for $|x| \geqq \gamma 2^{-n-1}$, $\beta(x)=0$ for $|x| \leqq \gamma 2^{-n-2}$, $|D\beta(x)| \leqq C\gamma^{-1}2^n$, $|D_{ij}\beta(x)| \leqq C\gamma^{-2}2^{2n}$, and $|D_{ijk}\beta(x)| \leqq C\gamma^{-3}2^{3n}$. Then we have (see Lemma 3.4)

$$\|\alpha'\|_2 \leqq C\gamma^{1/2}2^{-n/2}, |D_{ijk}\beta'(x)| \leqq C(|x|+\gamma 2^{-n})^{-4} \quad \text{if} \quad x \in R^3 . \tag{4.21}$$

Let $s \in A$ such that $b-\gamma^2 2^{-2n-2} < s < b$. We set $g_s(x)=(D_i u_j D_j u_i)(x,s)$ for $|x-a| < \gamma 2^{-n}$, and $g_s(x)=0$ for $|x-a| \geqq \gamma 2^{-n}$. Then the property $\operatorname{spt}(\alpha') \subset B(0, \gamma 2^{-n-1})$, [10, Theorem 1.3, p. 3], and (4.21) yield

$$\begin{aligned}
&\Big(\int_{B(a,\gamma 2^{-n-1})} |((D_i u_j D_j u_i) * \alpha')(x,s)|^2 dx\Big)^{1/2} \\
&= \Big(\int_{B(a,\gamma 2^{-n-1})} |(g_s * \alpha')(x)|^2 dx\Big)^{1/2} \\
&\leqq \|g_s * \alpha'\|_2 \leqq \|g_s\|_1 \|\alpha'\|_2 \leqq C\Big(\int_{B(a,\gamma 2^{-n})} |Du(x,s)|^2 dx\Big)(\gamma^{1/2}2^{-n/2}) .
\end{aligned} \tag{4.22}$$

Using (4.18), the Schwarz inequality, (4.15), and (4.22) we obtain

$$\begin{aligned}
&\Big|\int_{R^3} u_k(x,s)((D_i u_j D_j u_i) * \alpha')(x,s) D_k h_q(x,s) dx\Big| \\
&\leqq C\gamma^{-4}2^{4n}\Big(\int_{B(a,\gamma 2^{-n-1})} |u(x,s)|\,|((D_i u_j D_j u_i) * \alpha')(x,s)| dx\Big) \\
&\leqq C\delta^{1/2}\gamma^{-3}2^{2n} \int_{B(a,\gamma 2^{-n})} |Du(x,s)|^2 dx .
\end{aligned} \tag{4.23}$$

If $|x-a| \leqq \gamma 2^{-n-1}$ then (4.21) and (4.15) yield

$$\begin{aligned}
&|((u_j u_i) * D_{ijk}\beta')(x,s)| \leqq C \int_{R^3} |u(y,s)|^2(|x-y|+\gamma 2^{-n})^{-4} dy \\
&\leqq C \int_{R^3} |u(y,s)|^2(|y-a|+\gamma 2^{-n})^{-4} dy \leqq C\delta\gamma^{-3}2^n .
\end{aligned} \tag{4.24}$$

Hence Definition 3.3, integration by parts, (4.24), (4.18), the Schwarz inequality, and (4.15) yield

$$\begin{aligned}
&\Big|\int_{R^3} u_k(x,s)((u_j u_i) * D_{ij}\beta')(x,s) D_k h_q(x,s) dx\Big| \\
&= \Big|\int_{R^3} u_k(x,s)((u_j u_i) * D_{ijk}\beta')(x,s) h_q(x,s) dx\Big| \\
&\leqq C(\delta\gamma^{-3}2^n)(\gamma^{-3}2^{3n}) \int_{B(a,\gamma 2^{-n-1})} |u(x,s)| dx \\
&\leqq C\delta\gamma^{-6}2^{4n}\Big(\int_{B(a,\gamma 2^{-n-1})} |u(x,s)|^2 dx\Big)^{1/2}(\text{measure}(B(a,\gamma 2^{-n-1})))^{1/2} \\
&\leqq C\delta^{3/2}\gamma^{-4}2^n .
\end{aligned} \tag{4.25}$$

Now (4.23), (4.25), and Lemma 3.4 yield

$$\begin{aligned}
&\Big|\int_{R^3} u_k(x,s) p(x,s) D_k h_q(x,s) dx\Big| \\
&\leqq C\delta^{1/2}\gamma^{-3}2^{2n}\Big(\int_{B(a,\gamma 2^{-n})} |Du(x,s)|^2 dx\Big) + C\delta^{3/2}\gamma^{-4}2^n .
\end{aligned} \tag{4.26}$$

Integration of (4.26) with respect to s, Definition 3.3, (4.16), (4.3), and (4.20) yield

$$\int_{b-\gamma^2 2^{-2n-2}}^{b} \left| \int_{R^3} u_i(x,s)(2^{-1}|u(x,s)|^2 + p(x,s))D_i h_q(x,s)dx \right| ds$$

$$\leqq M\delta^{3/2}\gamma^{-2}2^{-n}, \tag{4.27}$$

where M is an absolute constant.

Setting $q=n+1$ in (4.13) and (4.27), and using (4.17), (4.18), and (4.3), we obtain

$$\int_{b-\gamma^2}^{b} \left| \int_{R^3} u_i(x,s)(2^{-1}|u(x,s)|^2 + p(x,s))D_i \phi_{n+1}(x,s)dx \right| ds$$

$$\leqq (\delta + M\delta^{3/2}2^{-n})\gamma^{-2}. \tag{4.28}$$

Using (4.13) and (4.27) with $q>n+1$, and using (4.17), (4.18), and (4.3) once again, we obtain

$$\int_{b-\gamma^2}^{b} \left| \int_{R^3} u_i(x,s)(2^{-1}|u(x,s)|^2 + p(x,s))D_i(r_{n+1}\phi_q)(x,s)dx \right| ds$$

$$\leqq (\delta + M\delta^{3/2}2^{-n})\gamma^{-2} \quad \text{if} \quad q>n+1. \tag{4.29}$$

Now (4.28), (4.29), and property $P(n,\delta)$ imply that $P(n+1, \delta + M\delta^{3/2}2^{-n})$ holds. Lemma 4.4 has been proved.

Now we can prove Theorem 4.1. Choose an absolute constant δ_0 such that $M(2\delta_0)^{3/2} \leqq \delta_0$ (see Lemma 4.4). We have

$$(\delta_0(2-2^{-n+1})) + M(\delta_0(2-2^{-n+1}))^{3/2}2^{-n}$$
$$< 2\delta_0 - 2^{-n+1}\delta_0 + M(2\delta_0)^{3/2}2^{-n} \leqq \delta_0(2-2^{-(n+1)+1}).$$

Hence Lemma 4.4 and the definition of $P(n,\delta)$ yield that $P(n, \delta_0(2-2^{-n+1}))$ implies $P(n+1, \delta_0(2-2^{-(n+1)+1}))$. Hence induction yields that $P(1,\delta_0)$ implies $P(n, \delta_0(2-2^{-n+1}))$ for all n. Now the definition of $P(n,\delta)$ yields

$$P(1,\delta_0) \quad \text{implies} \quad P(n, 2\delta_0) \quad \text{for all} \quad n. \tag{4.30}$$

There is an absolute constant η satisfying

$$\eta \int_{b-\gamma^2}^{b} \int_{R^3} (|x-a|+\gamma)^{-4} dx dt \leqq \gamma.$$

Young's inequality (see [4, p. 11]) yields

$$|u|^2 \leqq (2/3)((\delta_0\eta\gamma^{-2})^{-1/3}|u|^2)^{3/2} + (1/3)((\delta_0\eta\gamma^{-2})^{1/3})^3.$$

Hence we have

$$\int_{b-\gamma^2}^{b} \int_{R^3} |u(x,t)|^2(|x-a|+\gamma)^{-4} dx dt$$

$$\leqq C\gamma \left(\int_{b-\gamma^2}^{b} \int_{R^3} |u(x,t)|^3(|x-a|+\gamma)^{-4} dx dt \right) + (1/3)\delta_0\gamma^{-1}. \tag{4.31}$$

Now the estimates $|D_i\phi_1(x,t)| \leqq C(|x-a|+\gamma)^{-4}$ and $|D_i(r_1\phi_q)(x,t)| \leqq C(|x-a| + \gamma)^{-4}$ for $b-\gamma^2 \leqq t \leqq b$ and $q>1$ (see Lemma 4.2 and Definition 4.3), and (4.31) yield the existence of an absolute constant ε such that (4.1) implies $P(1,\delta_0)$. Hence (4.30) yields

$$\text{Inequality (4.1) implies } P(n, 2\delta_0) \text{ for all } n. \tag{4.32}$$

The assumption $b \in A$, (4.32), and the argument that yielded (4.15) can be used to show that (4.1) implies

$$\int_{R^3} |u(x,b)|^2(|x-a|+\gamma 2^{-n})^{-4}dx \leqq C(2\delta_0)\gamma^{-3}2^n .$$

Hence we have that (4.1) implies

$$\int_{B(a,\gamma 2^{-n})} |u(x,b)|^2 dx \leqq C\delta_0 \gamma 2^{-3n} . \tag{4.33}$$

For $0<\tau \leqq 1/2$ we choose n such that $2^{-n} \geqq \tau > 2^{-n-1}$. Then (4.33) yields

$$\int_{B(a,\tau\gamma)} |u(x,b)|^2 dx \leqq \int_{B(a,\gamma 2^{-n})} |u(x,b)|^2 dx$$
$$\leqq C\delta_0\gamma 2^{-3n} \leqq C\delta_0 \gamma (2\tau)^3 = K\tau^3\gamma$$

where K is an absolute constant. Theorem 4.1 has been proved.

5. The Connection with Hausdorff Measure

Throughout this section we assume that Hypothesis I holds.

5.1. Definition. We define $V: R^3 \times R^+ \to R$ by $V=|u|(2^{-1}|u|^2+|p|)$. For every integer n we define $Q_n: R^3 \times R \to R$ by $Q_n(x,t)=(|x|+2^{-n})^{-4}$ if $-2^{-2n} \leqq t \leqq 2^{-2n}$, and $Q_n(x,t)=0$ otherwise. For $t \geqq 2^{-2n}$ we set

$$V_n(x,t)=\int_0^\infty \int_{R^3} V(y,s)Q_n(x-y,t-s)dyds .$$

We define $B(n,p_1,p_2,p_3,p_4)$ to be the set of all $(x,t) \in R^3 \times R$ satisfying $p_i 2^{-n} \leqq x_i \leqq (p_i+1)2^{-n}$ for $i \in \{1,2,3\}$, and $p_4 2^{-2n} \leqq t \leqq (p_4+1)2^{-2n}$. We set $B(n)=\{B(n,p_1,p_2,p_3,p_4): p_i$ is an integer for all i, and $p_4 \geqq 1\}$.

From Lemma 3.2 and Lemma 3.6 we obtain

$$\int_0^T \int_{R^3} V(x,t)dxdt \leqq CL^{3/2}T^{1/4} \quad \text{if} \quad 0<T<\infty . \tag{5.1}$$

If $2^{-2n} \leqq a < b$, [10, Theorem 1.3, p. 3] yields

$$\int_a^b \int_{R^3} V_n(x,t)dxdt \leqq C2^{-n} \int_{a-2^{-2n}}^{b+2^{-2n}} \int_{R^3} V(x,t)dxdt . \tag{5.2}$$

5.2. Lemma. *There exists an absolute constant θ such that the conditions $B \in B(n)$ and $\int_B V_n \leqq \theta 2^{-3n}$ imply that $|u|$ is essentially bounded on a neighborhood of B.*

Proof. Let $B=B(n,p_1,p_2,p_3,p_4)$ and $\gamma=2^{-n-2}$. We set

$$U=\{(x,t)\in R^3 \times R: (p_i-1)2^{-n} < x_i < (p_i+2)2^{-n} \text{ for } i \in \{1,2,3\} ,$$
$$\text{and } p_4 2^{-2n}-2^{-2n-4} < t < (p_4+1)2^{-2n}+2^{-2n-4}\} ,$$
$$D=\{(x,t)\in R^3 \times R: p_i 2^{-n} \leqq x_i \leqq (p_i+1)2^{-n} \text{ for } i \in \{1,2,3\} ,$$
$$\text{and } p_4 2^{-2n}+2^{-2n-2} \leqq t \leqq (p_4+1)2^{-2n}-2^{-2n-2}\} .$$

Now let $(a,b)\in U$. For every $(y,s)\in D$ we have (see Definition 5.1)

$$\int_{b-\gamma^2}^{b}\int_{R^3} V(x,t)(|x-a|+\gamma)^{-4}dxdt \leqq CV_n(y,s)\,.$$

Averaging over D and using the fact $D\subset B$, we obtain

$$\int_{b-\gamma^2}^{b}\int_{R^3} V(x,t)(|x-a|+\gamma)^{-4}dxdt$$
$$\leqq C(\text{measure}(D))^{-1}\left(\int_D V_n\right) \leqq C\gamma^{-5}\int_B V_n\,.$$

Since $2^{-3n}=2^6\gamma^3$, there exists an absolute constant θ such that the property $\int_B V_n \leqq \theta 2^{-3n}$ implies that (4.1) holds for $(a,b)\in U$. Then we can use Theorem 4.1, Definition 3.3, and [9, Corollary 1, p. 5] to conclude that $|u(a,b)|^2 \leqq K(4\pi/3)^{-1}\gamma^{-2}$ holds for almost every $(a,b)\in U$.

5.3. *Definition.* The 2 dimensional Hausdorff measure of a set $S\subset R^3\times R$ is denoted by $\mathscr{H}^2(S)$. For the definition of Hausdorff measure, see [2, p. 171] (where $X=R^3\times R$ and the metric on X is the usual metric on R^4).

5.4. Lemma. *For each integer k there exists a compact set S_k contained in $R^3\times[2^{-k},2^{-k+1}]$ such that*

$$\mathscr{H}^2(S_k)\leqq C\int_{2^{-k-1}}^{2^{-k+2}}\int_{R^3} V(x,t)dxdt \tag{5.3}$$

and for every $(x,t)\in(R^3\times[2^{-k},2^{-k+1}])-S_k$ there exists a neighborhood U of (x,t) such that $|u|$ is essentially bounded on U.

Proof. Let k be given. For each integer n satisfying $n\geqq k+1$ and $n\geqq 0$ we set (see Lemma 5.2) $D(n)=\left\{B\in B(n): B\subset R^3\times[2^{-k},2^{-k+1}] \text{ and } \int_B V_n>\theta 2^{-3n}\right\}$. We then set

$$S_k=\cap\{(\cup\{B:B\in D(n)\}): n\geqq k+1 \text{ and } n\geqq 0\}\,.$$

For each n, (5.2) yields

$$\sum_{B\in D(n)}\int_B V_n \leqq C2^{-n}\int_{2^{-k-1}}^{2^{-k+2}}\int_{R^3} V(x,t)dxdt\,.$$

Hence the number of elements in $D(n)$ is at most

$$C2^{2n}\int_{2^{-k-1}}^{2^{-k+2}}\int_{R^3} V(x,t)dxdt\,.$$

Hence (5.1) implies that S_k is compact, and we also have (using $n\geqq 0$)

$$\sum_{B\in D(n)}(\text{diameter}(B))^2 \leqq \sum_{B\in D(n)} C2^{-2n}$$
$$\leqq C\int_{2^{-k-1}}^{2^{-k+2}}\int_{R^3} V(x,t)dxdt\,.$$

Since the diameter of the sets in $D(n)$ can be made arbitrarily small by taking n sufficiently large, and S_k is contained in $\cup\{B: B\in D(n)\}$ for sufficiently large n, [2, p. 171] yields (5.3).

Now take $(x,t)\in(R^3\times[2^{-k},2^{-k+1}])-S_k$. There exists $n\geqq\max(k+1,0)$ such that $(x,t)\notin B$ for every $B\in D(n)$. However, there exists $B\in B(n)$ such that $B\subset R^3\times[2^{-k},2^{-k+1}]$ and $(x,t)\in B$. Hence Lemma 5.2 implies that $|u|$ is essentially bounded on a neighborhood of B, and hence on a neighborhood of (x,t).

Now we can prove Theorem 2.1. For any integer n, (5.2) and (5.1) yield

$$\int_{2^{-2n}}^{2^{-2n+2}}\int_{R^3} V_n(x,t)\,dx\,dt\leqq C2^{-n}\int_0^{2^{-2n+3}}\int_{R^3} V(x,t)\,dx\,dt$$
$$\leqq WL^{3/2}2^{-3n/2} \tag{5.4}$$

where W is an absolute constant. Let m be the integer that satisfies $WL^{3/2}\leqq\theta 2^{-3m/2}<2^{3/2}WL^{3/2}$ (see Lemma 5.2). If n, p_1, p_2, p_3 are integers such that $n\leqq m$ then, setting $B_i=B(n,p_1,p_2,p_3,i)$ for $i\in\{1,2,3\}$, we obtain that (5.4) yields

$$\int_{B_i} V_n\leqq WL^{3/2}2^{-3n/2}\leqq\theta 2^{-3m/2}2^{-3n/2}\leqq\theta 2^{-3n}\quad\text{for } i=1,2,3\ .$$

Hence Lemma 5.2 yields that $|u|$ is essentially bounded on B_1, B_2, and B_3. By varying n and p_j, $j=1,2,3$, we obtain that $|u|$ is locally essentially bounded on the set $\{(x,t): x\in R^3 \text{ and } t\geqq 2^{-2m}\}$. Actually, the proof of Lemma 5.2 shows that $|u|$ is essentially bounded on that set. We define $S=\cup\{S_k: k\geqq 2m+1\}$. The above and Lemma 5.4 yield that $|u|$ is locally essentially bounded outside of S. Finally, the countable subadditivity of $\mathscr{H}^2$, (5.3), (5.1), and the definition of m yield

$$\mathscr{H}^2(S)\leqq\sum_{k\geqq 2m+1}\mathscr{H}^2(S_k)\leqq 3C\int_0^{2^{-2m+1}}\int_{R^3} V(x,t)\,dx\,dt\leqq CL^2\ .$$

Theorem 2.1 has been proved.

We can prove Theorem 2.2 as follows: First, use Hypothesis II to imitate the proof of [7, Lemma 1.1] and derive identity (1.8) of [7] for almost every x, t_1, and t_2. Then use Theorem 2.1 to adapt the proof in the last paragraph of [7, Section 2] to our case.

6. Outline of Proof of Theorem 2.3

Let v be given as in Theorem 2.3. From [5] we obtain that there exist $0<L<\infty$ and $(u,n)\in C^\infty(R^3\times R^+,R^3)$ for $n=1,2,3,\ldots$ such that (see Definition 3.3)

$$\operatorname{div}(u,n)=0\ , \tag{6.1}$$

$$\int_{R^3}|(u,n)(x,t)|^2dx\leqq L\quad\text{for all}\quad t\in R^+\ , \tag{6.2}$$

$$\int_0^\infty\int_{R^3}|D(u,n)(x,t)|^2dx\,dt\leqq L\ , \tag{6.3}$$

$$-\int_{R^3} v_i(x)\phi_i(x,0)\,dx-\int_{R^3\times R^+}(u,n)_i(D_t\phi_i+\Delta\phi_i)$$
$$=\int_{R^3\times R^+}((u,n)_j*f_n)(u,n)_iD_j\phi_i \tag{6.4}$$

whenever $\phi\in C_0^\infty(R^3\times R,R^3)$ satisfies $\operatorname{div}(\phi)=0$. We also obtain from [5] that there exists an increasing sequence n_1, n_2, n_3,... of positive integers and a Lebesgue measurable function $u:R^3 \times R^+ \to R^3$ such that (2.1), (2.2), and (2.3) are satisfied, and we have

$$\lim_k \int_{R^3} |u(x,t)-(u,n_k)(x,t)|^2 dx = 0 \tag{6.5}$$

for almost every $t \in R^3$, and

$$D(u,n_k) \text{ converges weakly in } L^2 \text{ to } Du\,. \tag{6.6}$$

If $0<T<\infty$ then the Lebesgue dominated convergence theorem, (2.2), (6.2), and (6.5) yield

$$\lim_k \int_0^T \left(\int_{R^3} |u(x,t)-(u,n_k)(x,t)|^2 dx\right)^3 dt = 0\,. \tag{6.7}$$

From Lemma 3.1, (6.2), (2.2), (6.3), and (2.3) we obtain

$$\begin{aligned}&\int_0^T \int_{R^3} |u(x,t)-(u,n_k)(x,t)|^3 dxdt \\ &\quad \leqq C\varepsilon^{-3} \int_0^T \left(\int_{R^3} |u(x,t)-(u,n_k)(x,t)|^2 dx\right)^3 dt + C\varepsilon L\end{aligned} \tag{6.8}$$

for every $0<\varepsilon<\infty$. *Combining* (6.7) *and* (6.8) (*with varying* ε) *we obtain*

$$\lim_k \int_0^T \int_{R^3} |u(x,t)-(u,n_k)(x,t)|^3 dxdt = 0\,. \tag{6.9}$$

Let α, α', β, β' be as in Lemma 3.4. Define $(p,n):R^3 \times R^+ \to R$ and $p:R^3 \times R^+ \to R$ by

$$\begin{aligned}(p,n)(x,t) &= -(D_i((u,n)_j * f_n)D_j(u,n)_i * \alpha')(x,t) \\ &\quad -(((u,n)_j * f_n)(u,n)_i * D_{ij}\beta')(x,t)\end{aligned} \tag{6.10}$$

$$p(x,t) = -(D_i u_j D_j u_i * \alpha')(x,t) - (u_j u_i * D_{ij}\beta')(x,t)$$

for almost all (x,t). The argument in Lemma 3.5, the Schwarz inequality, (2.2), (6.2), (2.3), (6.3), Young's inequality, and [10, Theorem 1.3, p. 3] yield

$$\begin{aligned}&\left(\int_{R^3} |(p,n_k)(x,t)-p(x,t)|^2 dx\right)^{1/2} \\ &\leqq Cr^{1/2} \int_{R^3} |(D(u,n_k)*f_{n_k})(x,t)|^2 + |D(u,n_k)(x,t)|^2 + |Du(x,t)|^2 dx \\ &\quad + Cr^{-3/2} \int_{R^3} |((u,n_k)*f_{n_k})(x,t)-u(x,t)|\,|(u,n_k)(x,t)| dx \\ &\quad + Cr^{-3/2} \int_{R^3} |u(x,t)|\,|(u,n_k)(x,t)-u(x,t)| dx \\ &\leqq Cr^{1/2} \int_{R^3} |D(u,n_k)(x,t)|^2 + |Du(x,t)|^2 dx \\ &\quad + Cr^{-3/2} L^{1/2} \left(\int_{R^3} |((u,n_k)*f_{n_k})(x,t)-u(x,t)|^2 dx\right)^{1/2} \\ &\quad + Cr^{-3/2} L^{1/2} \left(\int_{R^3} |(u,n_k)(x,t)-u(x,t)|^2 dx\right)^{1/2}\end{aligned} \tag{6.11}$$

for almost all $t\in R^+$. The Schwarz inequality, the argument in Lemma 3.5, (2.2), (2.3), (6.2), (6.3), (6.11), Young's inequality, and [10, Theorem 1.3, p. 3] yield

$$\begin{aligned}
&\int_0^T\int_{R^3} |(u,n_k)_i(x,t)(p,n_k)(x,t)-u_i(x,t)p(x,t)|dxdt\\
&\quad\leqq \int_0^T\int_{R^3} |(u,n_k)(x,t)-n(x,t)|\,|(p,n_k)(x,t)|dxdt\\
&\quad\quad+\int_0^T\int_{R^3} |u(x,t)|\,|(p,n_k)(x,t)-p(x,t)|dxdt\\
&\quad\leqq \int_0^T\left(\int_{R^3}|(u,n_k)(x,t)-u(x,t)|^2dx\right)^{1/2}\left(\int_{R^3}|(p,n_k)(x,t)|^2dx\right)^{1/2}dt\\
&\quad\quad+\int_0^T\left(\int_{R^3}|u(x,t)|^2dx\right)^{1/2}\left(\int_{R^3}|(p,n_k)(x,t)-p(x,t)|^2dx\right)^{1/2}dt\\
&\quad\leqq Cr^{1/2}\int_0^T\left(\int_{R^3}|(u,n_k)(x,t)-u(x,t)|^2dx\right)^{1/2}\left(\int_{R^3}|D(u,n_k)(x,t)|^2dx\right)dt\\
&\quad\quad+Cr^{-3/2}\int_0^T\left(\int_{R^3}|(u,n_k)(x,t)-u(x,t)|^2dx\right)^{1/2}\left(\int_{R^3}|(u,n_k)(x,t)|^2dx\right)dt\\
&\quad\quad+L^{1/2}\int_0^T\left(\int_{R^3}|(p,n_k)(x,t)-p(x,t)|^2dx\right)^{1/2}dt\\
&\quad\leqq Cr^{1/2}L^{3/2}+Cr^{-3/2}L\int_0^T\left(\int_{R^3}|(u,n_k)(x,t)-u(x,t)|^2dx\right)^{1/2}dt\\
&\quad\quad+Cr^{1/2}L^{3/2}+Cr^{-3/2}L\int_0^T\left(\int_{R^3}|((u,n_k)*f_{n_k})(x,t)-u(x,t)|^2dx\right)^{1/2}dt\\
&\quad\quad+Cr^{-3/2}L\int_0^T\left(\int_{R^3}|(u,n_k)(x,t)-u(x,t)|^2dx\right)^{1/2}dt
\end{aligned}\tag{6.12}$$

for $0<T<\infty$. Now we make r small and use (6.12), (2.2), (2.3), (6.2), (6.3), (6.5), the fact

$$\lim_k \int_{R^3} |((u,n_k)*f_{n_k})(x,t)-u(x,t)|^2dx=0$$

for almost every $t\in R^+$ [see (6.5)], and the Lebesgue dominated convergence theorem to conclude

$$\lim_k \int_0^T\int_{R^3} |(u,n_k)_i(x,t)(p,n_k)(x,t)-u_i(x,t)p(x,t)|dxdt=0 \tag{6.13}$$

Let ϕ satisfy (2.5). From (6.1), (6.2), (6.3), (6.4), (6.10), and the usual arguments we conclude

$$\begin{aligned}
&-2^{-1}\left(\int|(u,n)|^2(D_t\phi+\Delta\phi)\right)+\int|D(u,n)|^2\phi\\
&=2^{-1}\int((u,n)_i*f_n)|(u,n)|^2D_i\phi+\int(u,n)_i(p,n)D_i\phi\,.
\end{aligned}\tag{6.14}$$

Now (2.2), (6.2), and (6.5) yield

$$\lim_k \int |(u, n_k)|^2 (D_t\phi + \Delta\phi) = \int |u|^2 (D_t\phi + \Delta\phi).$$

Properties (2.3), (6.3), and (6.6) yield (recall $\phi \geqq 0$)

$$\liminf_k \int |D(u, n_k)|^2 \phi \geqq \int |Du|^2 \phi.$$

From (6.9) and (6.13) we obtain

$$\lim_k 2^{-1} \int ((u, n_k)_i * f_{n_k}) |(u, n_k)|^2 D_i\phi = \int u_i (2^{-1}|u|^2) D_i\phi,$$

$$\lim_k \int (u, n_k)_i (p, n_k) D_i\phi = \int u_i p D_i\phi.$$

Hence (6.14) yields (2.6). Properties (2.7) and (2.9) are a more immediate consequence of (6.1), (6.4), (6.10), and the usual estimates.

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