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A Sketch of Lie Superalgebra Theory

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Abstract. This article deals with the structure and representations of Lie superalgebras ($\mathbb{Z}_2$-graded Lie algebras). The central result is a classification of simple Lie superalgebras over $\mathbb{R}$ and $\mathbb{C}$.

Introduction

"Graded Lie algebras have recently become a topic of interest in physics in the context of supergauge symmetries relating particles of different statistics". See the review [22] from which this quotation is taken and where there is a voluminous bibliography. (See also the review [25].)

In this paper an attempt is made to develop Lie superalgebra theory. Lie superalgebras are often called $\mathbb{Z}_2$-graded Lie algebras. We prefer the term "superalgebra" inspired by physicists. In fact, a Lie superalgebra is not a Lie algebra either graded or not.

We call *superalgebra* any $\mathbb{Z}_2$-graded algebra $A=A_{\bar{0}}\oplus A_{\bar{1}}$, i.e. if $a\in A_\alpha$, $b\in A_\beta$, $\alpha, \beta\in\mathbb{Z}_2=\{\bar{0},\bar{1}\}$, then $ab\in A_{\alpha+\beta}$. A superalgebra $G=G_{\bar{0}}\oplus G_{\bar{1}}$ with product [,], satisfying the following axioms

$$[a,b]=-(-1)^{\alpha\beta}[b,a]; \quad [a,[b,c]]=[[a,b],c]+(-1)^{\alpha\beta}[b,[a,c]],$$
$$a\in G_\alpha, b\in G_\beta,$$

is called *Lie superalgebra.*

Note that these axioms are satisfied by the Whitehead product in homotopy groups. Lie superalgebras arise also in various cohomology theories, e.g. in deformation theory.

In paper [4] Lie superalgebras are initially introduced as Lie algebras of some generalized groups now called formal Lie supergroups. At present, there is a satisfactory theory analogous to Lie theory connecting Lie superalgebras and Lie supergroups, i.e. groups with functions taking value in some Grassmann algebra, [5].

Let us now enumerate in short the main points of finite-dimensional Lie superalgebra theory. Let G be a Lie superalgebra of finite dimension. Then there exists in G the unique maximal solvable ideal R (solvable radical). The Lie superalgebra G/R is semisimple (i.e. G/R does not contain any solvable ideals). Therefore finite-dimensional Lie superalgebras theory is in a sense reduced to the theories of semisimple Lie superalgebras and solvable Lie superalgebras.

Note that the analogue of the Levi theorem stating that G is semidirect product of R and G/R does not hold for Lie superalgebras.

The crucial point of solvable Lie algebra theory is the Lie theorem asserting that any irreducible finite-dimensional representation over $\mathbb{C}$ of a solvable Lie algebra is one-dimensional. That is not true for Lie superalgebras. In this paper there is given a classification of finite-dimensional irreducible representations of solvable Lie superalgebras (Theorem 7). In particular we obtain a necessary and sufficient condition for every representation to be 1-dimensional.

Further, it is known that a semisimple Lie algebra is a direct sum of simple Lie algebras. This is far from being so in the Lie superalgebra case. However, there is a construction which enables us to describe semisimple Lie superalgebras via simple ones (Theorem 6). The construction is analogous to the one introduced in the paper [21].

Thus we are driven to the fundamental problem of classification of finite-dimensional simple Lie superalgebras. The general purpose of this paper is to answer that question in case of an algebraically closed field of zero characteristic. The main difficulty lies in the possible degeneracy of the Killing form, which does not take place in the simple Lie algebras case. That is why the usual Killing-Cartan technique is not applicable. The classification is divided into two principal parts.

First, there is given the classification of Lie superalgebras of classical type. A Lie superalgebra is of *classical type* if it is simple and if the representation of the Lie algebra $G_{\bar{0}}$ in $G_{\bar{1}}$ is completely reducible.

This part is in turn divided into two parts, the Killing form being nondegenerate or zero respectively. Vanishing of the Killing form is used to obtain strong limitations on the indices of representations of $G_{\bar{0}}$ in $G_{\bar{1}}$.

The obtained classification of Lie superalgebras which are not Lie algebras is as follows (Theorem 2):

a) 4 series $\boldsymbol{A}(m, n)$, $\boldsymbol{B}(m, n)$, $\boldsymbol{C}(n)$, $\boldsymbol{D}(m, n)$ that are in many ways like the $\boldsymbol{A}-\boldsymbol{D}$ series of Lie algebras,

b) 2 exceptional Lie superalgebras, one being 40-dimensional $\boldsymbol{F}(4)$, the other being 31-dimensional $\boldsymbol{G}(3)$,

c) a family of 17-dimensional superalgebras $\boldsymbol{D}(2, 1; \alpha)$ that are deformations of $\boldsymbol{D}(2, 1)$,

d) 2 "strange" series $\boldsymbol{P}(n)$ and $\boldsymbol{Q}(n)$.

In the second part there is given a classification of simple Lie superalgebras of nonclassical type. For this purpose there is made a filtration $G=L_{-1}\supset L_0\supset L_1\supset\ldots$, L_0 being maximal subalgebra that contains G_0, $L_i=\{a\in L_{i-1}|[a, L]\subset L_{i-1}\}$ for $i>0$. Then we give a classification of $\mathbb{Z}$-graded Lie superalgebras with the same properties that the associated graded Lie superalgebra $\operatorname{Gr} G$ has notoriously (Theorem 4). In the proof are used methods developed by the author in his paper [11] for the classification of infinite-dimensional Lie algebras. Finally there remains only to restore filtered Lie superalgebra G via $\mathbb{Z}$-graded Lie superalgebra $\operatorname{Gr} G$.

The finished classification of simple Lie superalgebras is as follows (Theorem 5).

a) Lie superalgebras of classical type (enumerated above).

b) Lie superalgebras of Cartan type $\boldsymbol{W}(n)$, $\boldsymbol{S}(n)$, $\boldsymbol{H}(n)$, $\tilde{\boldsymbol{S}}(n)$, where the first three series are analogous to the corresponding series of simple infinite-dimensional Lie algebras of Cartan type, and $\tilde{\boldsymbol{S}}(n)$ is deformation of $\boldsymbol{S}(n)$.

Finite-dimensional irreducible representations of simple Lie algebras are described by the highest weight theorem. The simple Lie superalgebra case is the same (Theorem 8). Note that the complete reducibility of finite-dimensional representations of simple Lie superalgebras generally does not take place.

In this paper there are also enumerated all finite-dimensional simple real Lie superalgebras (Theorem 9).

Finally, there is made an attempt to extend Cartan's results on classification of complete infinite-dimensional primitive Lie algebras to Lie superalgebras. In this line we obtain only partial results (Theorem 10).

From this viewpoint one can see also the cause of finite-dimensionality of Lie superalgebras of Cartan type. The latter, being Lie superalgebras of vector fields in commuting and anticommuting variables, are finite-dimensional when there are no commuting variables and so there is no analogue to contact Lie algebra in the odd case.

The main results, i.e. Theorems 1, 2, 4–7 are formulated in [16].

All spaces and algebras discussed are over a field K which is usually considered to be algebraically closed and of zero characteristic. Denote $\langle M \rangle$ the linear span of the set M over K, $\oplus$ the sign of direct sum and $\otimes$ the sign of tensor product over K.

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I am obliged to D. A. Leites who translated the text.

Remarks. 1) This article is written for physicists. Consequently some proofs are sketched or omitted. The complete proofs are contained in my forthcoming paper "Lie superalgebras" submitted to the Russian journal Uspechi mat. nauk (see [31]).

2) The partial results of classification of simple $\mathbb{Z}_2$-graded Lie algebras were independently obtained by Kaplansky, Freund, Djoković, Pais, Rittenberg, Nahm, Scheunert [26–30].

Chapter I. Classification of Simple Lie Superalgebras

§ 1. Superalgebras and Lie Superalgebras

1. Superalgebras. Suppose $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a superalgebra. Call subalgebra or ideal of a superalgebra A a $\mathbb{Z}_2$-graded subalgebra or ideal. Call homomorphism the one that preserves $\mathbb{Z}_2$-grading. Direct and semidirect sum of superalgebras is defined as usual. The definition of tensor product is different. Suppose $A = A_0 \oplus A_1$;

$B=B_0\oplus B_1$ are superalgebras. Call tensor product superalgebra $A\otimes B$, the space $A\otimes B$, space $A\otimes B$ being tensor product of spaces A and B with the induced grading and multiplication defined as follows

$$(a_1\otimes b_1)(a_2\otimes b_2)=(-1)^{\deg b_1\cdot\deg a_2}a_1a_2\otimes b_1b_2,\ a_i\in A,\ b_i\in B\,.$$

For a superalgebra A it is natural to define the commutator (bracket) by the equation

$$[a,b]=ab-(-1)^{\deg a\cdot\deg b}ba\,. \tag{1.1}$$

A superalgebra is called commutative if $[a,b]=0$. Usually, permutability is understood in the sence of (1.1). Associativity is understood as in the algebra case.

Example 1. Assuming that M is abelian group, $V=\bigoplus_{i\in M}V_i$ is M-graded vector space. Then the associative algebra End V is supplied with the induced M-grading $\operatorname{End}V=\bigoplus_{i\in M}\operatorname{End}_i V$, $\operatorname{End}_i V=\{a\in\operatorname{End}V|aV_s\subset V_{s+i}\}$.

In particular, we obtain for an $M=\mathbb{Z}_2$ superalgebra $\operatorname{End}V=\operatorname{End}_{\bar 0}V\oplus\operatorname{End}_{\bar 1}V$.

Example 2. Denote $\Lambda(n)$ the Grassmann algebra of n variables $\xi_1,\ldots,\xi_n$. $\Lambda(n)$ is $\mathbb{Z}_2$-graded, if $\deg\xi_i=\bar 1$, $i=1,\ldots,n$. Call the obtained superalgebra a Grassmann superalgebra. $\Lambda(n)$ is a commutative associative superalgebra. Evidently, $\Lambda(n)\otimes\Lambda(m)=\Lambda(m+n)$.

2. Lie Superalgebras. Suppose that $G=G_{\bar 0}\oplus G_{\bar 1}$ is a Lie superalgebra. Then $G_{\bar 0}$ is a usual Lie algebra, left multiplication defines $G_{\bar 0}$-module $G_{\bar 1}$ and multiplication in $G_{\bar 1}$ defines homomorphism φ of $G_{\bar 0}$-modules $\varphi:S^2G_{\bar 1}\to G_{\bar 0}$. Therefore, a Lie superalgebra could be defined by three objects, namely, a Lie algebra $G_{\bar 0}$, a $G_{\bar 0}$-module $G_{\bar 1}$ and a homomorphism of $G_{\bar 0}$-modules $\varphi:S^2G_{\bar 1}\to G_{\bar 0}$ satisfying the unique condition

$$\varphi(a,b)c+\varphi(b,c)a+\varphi(c,a)b=0,\ a,b,c\in G_{\bar 1} \tag{1.2}$$

Example 1. Suppose A is associative superalgebra. Then bracket (1.1) defines the Lie superalgebra structure on A. Denote this Lie superalgebra A_L.

Example 2. Let G be Lie superalgebra, and let $\Lambda(n)$ be a Grassmann superalgebra. Then $G\otimes\Lambda(n)$ is Lie superalgebra.

The definition of solvable and nilpotent Lie superalgebras is the same as for Lie algebras. A Lie algebra is called simple (semisimple) if it does not contain nontrivial (resp. solvable) ideals.

3. Universal Enveloping Superalgebra. In [22] it is verified that the Poincaré-Birkhoff-Witt theorem holds.

Theorem (PBW). *Suppose $G=G_{\bar 0}\oplus G_{\bar 1}$ is Lie superalgebra, $a_1,\ldots,a_m$ being basis of $G_{\bar 0}$, $b_1,\ldots,b_n$ that of $G_{\bar 1}$. Then elements of the form*

$$a_1^{k_1}\ldots a_m^{k_m}b_{i_1}\ldots b_{i_s},\quad \text{for}\quad k_i>0,\ 1\leqq i_1\leqq\ldots\leqq i_s\leqq n$$

form a basis of the universal enveloping Lie superalgebra $U(G)$.

Define the *diagonal morphism* $\Delta: U(G)\to U(G)\otimes U(G)$ by the formula

$$\Delta(a)=a\otimes 1+(-1)^{\deg a}1\otimes a,\ a\in G\,.$$

4. Derivations and Automorphisms of Superalgebras. Call *derivation of degree s* of a superalgebra A, $s\in\mathbb{Z}_2$, an endomorphism $D\in\operatorname{End}_s A$ with the property

$$D(ab)=D(a)b+(-1)^{s\cdot\deg a}aD(b)\,.$$

Denote $\operatorname{der}_s A\subset\operatorname{End}_s A$ the space of all derivations of degree s, assuming that $\operatorname{der}A=\operatorname{der}_{\bar 0}A\oplus\operatorname{der}_{\bar 1}A$. The subspace $\operatorname{der}A\subset\operatorname{End}A$ is closed under the bracket (1.1), i.e. der A is Lie subalgebra of $(\operatorname{End}A)_L$. We call it the *Lie superalgebra of derivations of A.*

Example 1. Suppose that G is Lie superalgebra. Then the Jacobi identity implies that $\operatorname{ad}a: b\to[a,b]$ is derivation of G. Derivations of this kind are called *inner*; they form an ideal inner G in $\operatorname{der}G$, because

$$[D,\operatorname{ad}a]=\operatorname{ad}D(a),\quad\text{where}\quad D\in\operatorname{der}G\,.$$

Example 2. It is easily verified that for any $P_1,\dots,P_n\in\Lambda(n)$ there is unique derivation $D\in\operatorname{der}\Lambda(n)$ satisfying $D(\xi_i)=P_i$. Denote derivations $\partial/\partial\xi_i$ via the formula $\partial\xi_j/\partial\xi_i=\delta_{ij}$. The derivation wanted could be written in the form $D=\sum P_i\,\partial/\partial\xi_i$.

Note that if D is an even derivation of superalgebra $A=A_{\bar 0}\oplus A_{\bar 1}$ then $\exp tD$, $t\in K$, is 1-parameter group of automorphisms. In particular, if A is Lie superalgebra then $\exp(\operatorname{ad}a)$ is an automorphism of A for $a\in A_{\bar 0}$. The group generated by all such automorphisms is called the *group of inner automorphisms*.

5. Superalgebra $l(V)$ and Supertrace. Let $V=V_{\bar 0}\oplus V_{\bar 1}$ be $\mathbb{Z}_2$-graded space. The algebra End V has a $\mathbb{Z}_2$-grading (see Section 1) and becomes therefore an associative superalgebra. Denote $l(V)$ or elsewise $l(m,n)$ the Lie superalgebra $(\operatorname{End}V)_L$ (cf. Section 2), where $m=\dim V_{\bar 0}$, $n=\dim V_{\bar 1}$. The role played by $l(V)$ in Lie superalgebras theory is the same as that of GL in Lie algebras theory. If the same decomposition of $V=V_0+V_1$ is considered as $\mathbb{Z}$-grading then there is corresponding Z-grading of $l(V)$ agreeing with $\mathbb{Z}_2$-grading: $l(V)=G_{-1}\oplus G_0\oplus G_1$.

Assume that $e_1,\dots,e_m$, $e_{m+1},\dots,e_n$ is basis of V that is the union of bases of $V_{\bar 0}$ and $V_{\bar 1}$. Such a basis is called homogeneous. In this basis the matrix of an operator $a\in l(V)$ is written in the form (1); (2) and (3) being for even and odd elements respectively;

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(1);\quad \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}(2),\quad \begin{pmatrix}0 & \beta\\ \gamma & 0\end{pmatrix}(3);\quad \begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}(4);\quad \begin{pmatrix}0 & 0\\ \gamma & 0\end{pmatrix}(5)$$

(4) and (5) for G_1 and G_{-1} in $\mathbb{Z}$-grading of $l(V)$ respectively. One can see that G_0-modules G_1 and G_{-1} are contragredient, G_0-module G_1 being isomorphic to $\mathrm{gl}(m)\otimes\mathrm{gl}(n)$.

Call *supertrace* the function str on $l(V)$,

$$\operatorname{str}a=\operatorname{tr}\alpha-\operatorname{tr}\delta\,.$$

Note that the supertrace of the matrix of an operator a does not depend on the choice of a basis. Hence, we can speak about supertrace of an operator.

Let $G = G_{\bar 0} \oplus G_{\bar 1}$ be a $\mathbb{Z}_2$-graded vector space, f be a bilinear form on G. Call f *agreed* or *consistent* if $f(a, b) = 0$ for any $a \in G_{\bar 0}$, $b \in G_{\bar 1}$, – and *supersymmetric* if $f(a, b) = (-1)^{(\deg a)(\deg b)} f(b, a)$. If G is a Lie superalgebra then f is called *invariant* if $f[a, b], c) = f(a, [b, c])$.

Proposition 1.1. a) *Bilinear form* $(a, b) = \operatorname{str}(ab)$ *on* $l(V)$ *is consistent, supersymmetric and invariant.*

b) $\operatorname{str}([a, b]) = 0$ *for all* $a, b \in l(V)$.

Proof. Consistency of the form follows from the inclusion $ab \in l_{\bar 1}(V)$, for $a \in l_{\bar 0}(V)$, $b \in l_{\bar 1}(V)$. Supersymmetry for $a, b \in l(V)$ only the verification for $a, b \in l_{\bar 1}(V)$, and that is done by simple computation. b) is the other expression for supersymmetry. Invariance holds due to the Jacobi identity.

6. *Linear Representations of Lie Superalgebras.* Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a $\mathbb{Z}_2$-graded vector space. Call *linear representation of Lie superalgebra* $G = G_{\bar 0} \oplus G_{\bar 1}$ in V a homomorphism $\varrho: G \to l(V)$.

We shall often say more shortly that V is a G-module and write $g(v)$ instead of $\varrho(g)(v)$ for $g \in G$, $v \in V$. Note that by definition $G_i(V_j) \subset V_{i+j}$, $i, j \in \mathbb{Z}_2$, and $[h_1, h_2]v = h_1(h_2 v) - (-1)^{\alpha_1 \alpha_2} h_2(h_1 v)$. Note that $\operatorname{ad}: G \to l(G)$, $(\operatorname{ad} g)(h) = [g, h]$ is a linear representation of G, called adjoint representation.

We call *submodule* of a G-module V a $\mathbb{Z}_2$-graded submodule. A G-module V is called irreducible if it does not contain nontrivial submodules. Call $\Phi: V \to V'$ a *homomorphism* of G-modules if for some bijection $\varphi: \mathbb{Z}_2 \to \mathbb{Z}_2$ $\Phi(V_i) \subset V_{\varphi(i)}$.

The Schur Lemma. *Let* $V = V_{\bar 0} \oplus V_{\bar 1}$, $\mathfrak{M}$ *be a irreducible set of operators of* $l(V)$, $C(\mathfrak{M}) = \{a \in l(V) \mid [a, m] = 0, m \in \mathfrak{M}\}$.

Then either

1) $C(\mathfrak{M}) = \langle 1 \rangle$ *or*

2) $\dim V_{\bar 0} = \dim V_{\bar 1}$, $C(\mathfrak{M}) = \langle \bar 1, A \rangle$ *where* A *is nonsingular operator in* V *permuting* $V_{\bar 0}$ *and* $V_{\bar 1}$.

Example. Consider Lie superalgebra $N = N_{\bar 0} \oplus N_{\bar 1}$ with $N_{\bar 0} = \langle e \rangle$, $N_{\bar 1} = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$, $[a_i, b_j] = \delta_{ij} e$ and all other brackets vanishing.

Let us introduce a family of representations ϱ_α, $\alpha \in K^*$ of a Lie superalgebra N in the space $\Lambda(n)$, setting

$$\varrho_\alpha(a_i)u = \partial u / \partial \xi_i , \quad \varrho_\alpha(b_i)u = \alpha \xi_i u , \quad \varrho_\alpha(e)u = \alpha u .$$

The dimension of this irreducible representation is 2^n. Now, let us consider the Lie superalgebra $N' = N \oplus \langle c \rangle$, $[N, c] = 0$, $[c, c] = e$ and the superalgebra $\Lambda'(n) = \Lambda(n) \otimes K[\varepsilon]$, $\deg \varepsilon = \bar 1$, $\varepsilon^2 = \alpha/2$, $\alpha \in K^*$.

Denote a family of representations ϱ'_α, $\alpha \in K^*$, of the Lie superalgebra N' in the space $\Lambda'(n)$, setting

$$\varrho'_\alpha(n)(u \otimes v) = \varrho_\alpha(n)(u \otimes v) , \quad \varrho'_\alpha(c)(u \otimes v) = \alpha(1 \otimes \varepsilon)(u \otimes v)$$
$$u \in \Lambda(n) , \quad v \in K[\varepsilon] , \quad n \in N .$$

The dimension of this irreducible representation is 2^{n+1}.

It is natural to call N and N' Lie *superalgebras of Geisenberg*. Note that cases 1) and 2) of Schur's lemma hold respectively for ϱ_α and ϱ'_α.

This example shows that in the Lie superalgebra case there are irreducible representations of solvable Lie superalgebras in dimensions more than 1. The Engel theorem still holds though, and the proof is the same as for Lie algebras [10].

The Engel Theorem. *Let $G \subset l(V)$ be a subalgebra, all elements G being nilpotent. Then there is $v \in V$, $v \neq 0$, annuled by each element of G.*

§2. $\mathbb{Z}$-Grading and Filtration

1. $\mathbb{Z}$-Grading. Call Lie superalgebra G *$\mathbb{Z}$-graded* if it is decomposed into a direct sum of finite-dimensional $\mathbb{Z}_2$-graded subspaces $G = \bigoplus_{i \in \mathbb{Z}} G_i$, $[G_i, G_j] \subset G_{i+j}$. $\mathbb{Z}$-grading is called *consistent* or *agreed* (with $\mathbb{Z}_2$-grading), if $G_{\bar{0}} = \oplus G_{2i}$, $G_{\bar{1}} = \oplus G_{2i+1}$.

If follows from the definition that if G is a $\mathbb{Z}$-graded Lie superalgebra then G_0 is a subalgebra and $[G_0, G_i] \subset G_i$. So, the restriction of the adjoint representation to G_0 induces a linear representation of the Lie superalgebra G_0 in each G_i.

A $\mathbb{Z}$-graded Lie superalgebra $G = \oplus G_i$ is called *irreducible* if the representation of G_0 in G_{-1} is irreducible. A $\mathbb{Z}$-graded Lie superalgebra $G = \oplus G_i$ is called *transitive* if

$$\text{for} \quad a \in G_i, i \geqq 0, [a, G_{-1}] = 0 \quad \text{implies} \quad a = 0, \tag{T1}$$

and *bitransitive* if both (T1) and

$$\text{for} \quad a \in G_i, i \leqq 0, [a, G_1] = 0 \quad \text{implies} \quad a = 0. \tag{T2}$$

2. Conditions for Simplicity. In this section we give some conditions for simplicity of Lie superalgebras. Their proof is evident and usual.

Proposition 2.1. *Necessary conditions for simplicity of the Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ are as follows*

1) *$G_{\bar{0}}$ acts faithfully in $G_{\bar{1}}$,*
2) $[G_{\bar{1}}, G_{\bar{1}}] = G_{\bar{0}}$.

If the following additional condition is satisfied:

3) *the representation of G in G is irreducible,*

then G is simple.

Proposition 2.2. *Necessary conditions for simplicity of $\mathbb{Z}$-graded Lie superalgebra $G = \bigoplus_{i \geqq -1} G_i$ are as follows*

1) *G is transitive,*
2) *G is irreducible,*
3) $[G_{-1}, G_1] = G_0$.

If the following additional conditions are satisfied

4) $\{a \in G_1 \mid [G_0, a] = 0\} = 0$,
5) $G_i = G_1^i, i > 0$,

then Lie superalgebra G is simple.

3. Filtration. The sequence of subspace $L = L_{-1} \supset L_0 \supset L_1 \supset \ldots$ is called a *filtration* if $[L_i, L_j] \subset L_{i+j}$ and $\cap L_i = 0$ $(i, j \in \mathbb{Z})$.

A filtered Lie superalgebra is called *transitive* if for any $a \in L_i \setminus L_{i+1}$, $i \geqq 0$, there is an element $b \in L$ such that $[a, b] \notin L_i$. This condition could be written also in the form

$$L_i = \{a \in L_{i-1} | [a, L] \subset L_{i-1}\}, \quad i > 0. \tag{F1}$$

Let L be a Lie superalgebra, L_0 be a subalgebra of L, that does not contain nontrivial ideals of L. Then (F1) defines a transitive filtration in L. In fact the first property of filtration is easily proved inductively via Jacoby identity.

$\cap L_i$ is ideal in L and therefore $\cap L_i$ being zero proves the second condition.

This filtration is called the *transitive filtration* of (L, L_0).

To a filtered Lie superalgebra $L = L_{-1} \supset L_0 \supset L_1 \supset \ldots$ there corresponds by the usual way an associated $\mathbb{Z}$-graded Lie superalgebra $\operatorname{Gr} L = \bigoplus_{i \geqq -1} \operatorname{Gr}_i L$, $\operatorname{Gr}_i L = L_i / L_{i+1}$. L_i are $\mathbb{Z}_2$-graded spaces and so is $\operatorname{Gr} L$, $\mathbb{Z}$-grading of $\operatorname{Gr} L$ usually is not agreed. The $\mathbb{Z}$-graded Lie superalgebra $G = \bigoplus_{i \geqq -1} G_i$ has a canonical filtration $L_i = \bigoplus_{s \geqq i} G_s$.

A filtered Lie superalgebra L is transitive iff $\operatorname{Gr} L$ is transitive. If $\operatorname{Gr} L$ is simple then so is L.

4. *On Relations between* L *and* $\operatorname{Gr} L$. Assume that in filtered Lie superalgebra $L = L_{-1} \supset L_0 \supset L_1 \supset \ldots$ there are given subspaces G_s such that $L_s = G_s \oplus L_{s+1}$ and $[G_i, G_j] \subset G_{i+j}$. In such a case we say that the grading of L is *agreed* with filtration or *consistent*. If L is of finite dimension and the grading is consistent, than $L \simeq \operatorname{Gr} L$.

Proposition 2.3. *Suppose* $L = L_{-1} \supset L_0 \supset L_1 \supset \ldots$ *are transitive filtered finite-dimensional Lie superalgebras and* $\operatorname{Gr}_0 L$ *acts irreducibly in* $\operatorname{Gr}_{-1} L$, $\operatorname{Gr}_0 L$ *being a Lie algebra with nontrivial centre. Then there is a grading of L agreed with filtration, and henceforth* $L \simeq \operatorname{Gr} L$.

5. Suppose $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is Lie superalgebra, L_0 being some own maximal subalgebra that contains $L_{\bar{0}}$. Suppose L_0 does not contain nontrivial ideals of L,

$$L_i = \{a \in L_{i-1} | [a, L] \subset L_{i-1}\}, \quad i > 0,$$

is transitive filtration of (L, L_0). Let $\operatorname{Gr} L = \bigoplus_{i \geqq -1} \operatorname{Gr}_i L$ be associated $\mathbb{Z}$-graded Lie superalgebra.

Proposition 2.4. *The described $\mathbb{Z}$-graded Lie superalgebra* $\operatorname{Gr} L$ *satisfies the following conditions:*

a) $\operatorname{Gr} L$ *is transitive,*

b) *Z-grading of* $\operatorname{Gr} L$ *is agreed with* $\mathbb{Z}_2$*-grading,*

c) $\operatorname{Gr} L$ *is irreducible,*

d) *If the representation of* $L_{\bar{0}}$ *in* $L_{\bar{1}}$ — *is not irreducible then* $\operatorname{Gr}_1 L \neq 0$.

Proof. a) follows from the transitivity of the filtered Lie superalgebra L. The inclusion $\operatorname{Gr}_{-1} L \subset (\operatorname{Gr} L)_{\bar{1}}$ – follows from the inclusion $L_{\bar{0}} \subset L_0$. The transitivity of $\operatorname{Gr} L$ and a induction implies b). To prove c) suppose the opposite. Then there is a $\mathbb{Z}$-graded subspace $\tilde{L} \subset L$ containing L_0 that is neither L nor L_0 and $[L_0, \tilde{L}] \subset \tilde{L}$. We have $\tilde{L} = L_0 \oplus V$, $V \subset L_{\bar{1}}$. $L_0 \supset L_{\bar{0}}$, so $[V, V] \subset L_0$. $[\tilde{L}, \tilde{L}] = [L_0 \oplus V, L_0 \oplus V] =$

$[L_0, L_0]+[L_0, V]+[V, V]\subset \tilde{L}$ in contradiction to maximality of L_0. d) If $\mathrm{Gr}_1 L=0$, then $\mathrm{Gr}_0 L=L_{\bar{0}}$, on account of c), $L_{\bar{1}}$ – is irreducible representation of $L_{\bar{0}}$.

§3. The Description of Classical Type Lie Superalgebras

We say that a finite-dimensional Lie superalgebra $G=G_{\bar{0}}\oplus G_{\bar{1}}$ is of *classical type* if it is simple and the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ is completely reducible.

1. Lie Superalgebras $A(m, n)$. The basic property of supertrace, i.e. $\mathrm{str}([a, b])=0$, implies that $\mathrm{sl}(m, n)=\{a\in l(m, n) \mid \mathrm{str}(a)=0\}$ is the 1-codimensional ideal in $l(m, n)$. $\mathbb{Z}_2$- and $\mathbb{Z}$-gradings of $l(m, n)$ induce the same gradings on $\mathrm{sl}(m, n)$. The Lie superalgebra $\mathrm{sl}(n, n)$ contains a 1-dimensional ideal that consists of scalar matrices $\lambda 1_{2n}$. The Lie superalgebra $\mathrm{sl}(1, 1)$ is 3-dimensional nilpotent Lie algebra. Put

$$A(m, n)=\mathrm{sl}(m+1, n+1) \quad \text{for} \quad m \neq n, \quad m, n, \geqq 0\,,$$

$$A(m, m)=\mathrm{sl}(m+1, m+1)/\lambda 1_{2m+2}\,, \quad \mathrm{m}>0\,.$$

$\mathbb{Z}$-grading of $\mathrm{sl}(m+1, n+1)$ induces a $\mathbb{Z}$-grading of $A(m, n)$, $A(m, n)_i$ being zero for $|i|>1$.

2. Lie Superalgebras $B(m, n)$, $D(m, n)$, $C(n)$. Let $i=\sqrt{-1}$, T denote the sign of transposition, B be the matrix of order $m+2n$;

$$B=\left(\begin{array}{c|cc} i1_m & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & 1_n \\ & -1_n & 0 \end{array}\right)$$

Denote in $l(m, 2n)$ the subalgebra $\mathrm{osp}(m, n)$ putting $\mathrm{osp}(m, n)_\alpha=\{a\in l(m, n) \mid aB+i^\alpha Ba^{\mathrm{T}}=0, \alpha\in\mathbb{Z}_2\}$. Then $\mathrm{osp}(m, n)_{\bar{0}}$ consists of matrices of the form

$$\left(\begin{array}{c|cc} a & \multicolumn{2}{c}{0} \\ \hline & b & c \\ 0 & d & -b^{\mathrm{T}} \end{array}\right),$$

a being skew symmetric, c and d symmetric and b arbitrary, $\mathrm{osp}(m, n)_{\bar{1}}$ – consists of matrices of the form

$$\left(\begin{array}{c|cc} 0 & x & y \\ \hline -y^{\mathrm{T}} & \multicolumn{2}{c}{0} \\ x^{\mathrm{T}} & & \end{array}\right), \ x \text{ and } y \text{ being arbitrary.}$$

$\mathrm{osp}(m, n)$ is called the *orthogonal-symplectic* superalgebra. Put

$$\boldsymbol{B}(m, n)=\mathrm{osp}(2m+1, 2n)\,, \quad m\geqq 0\,, \quad n>0\,,$$

$$\boldsymbol{D}(m, n)=\mathrm{osp}(2m, 2n)\,, \quad m\geqq 2\,, \quad n>0\,,$$

$$\boldsymbol{C}(n+1)=\mathrm{osp}(2, 2n)\,, \quad n>0\,.$$

Another realisation of $\mathrm{osp}(m, n)$ follows. Let $V_{\bar{0}}$- be an m-dimensional space with symmetric form $(\,,)_0$, $V_{\bar{1}}$ be an n-dimensional space with skew symmetric form $(,)_1$, both form being bilinear and nondegenerate (that implies $n=2k$). Then put

$$\operatorname{osp}(m,k)_{\bar{0}}=\Lambda^2 V_{\bar{0}}\oplus S^2 V_{\bar{1}}\,, \quad \operatorname{osp}(m,k)_{\bar{1}}=V_{\bar{0}}\otimes V_{\bar{1}}$$

with multiplication

$$[a\wedge b, c]=(a,c)_0 b-(b,c)_0 a\,, \quad a\wedge b\in\Lambda^2 V_{\bar{0}}\,, \quad c\in V_{\bar{0}}$$
$$[a\circ b, c]=(a,c)_1 b+(b,c)_1 a\,, \quad a\circ b\in S^2 V_{\bar{1}}\,, \quad c\in V_{\bar{1}}\,.$$

These brackets define brackets in $\Lambda^2 V_{\bar{0}}$ and $S^2 V_{\bar{1}}$ in the usual way:

$$[ab, cd]=[ab, c]d+c[ab, d]\,.$$

Finally for $a\otimes c, b\otimes d\in V_{\bar{0}}\otimes V_{\bar{1}}$, put

$$[a\otimes c, b\otimes d]=(a,b)_0 c\otimes d+(c,d)_1 a\wedge b\,.$$

Such a realisation admits an interesting agreed $\mathbb{Z}$-grading, all $\operatorname{osp}(m,k)_i$ vanishing for $|i|>2$. Let $V_{\bar{1}}$ be the sum of isotropic spaces $V_{\bar{1}}=V'\oplus V''$, $\dim V'=\dim V''=k$. Then the following decomposition is $\mathbb{Z}$-grading:

$$\operatorname{osp}(m,k)=S^2(V')\oplus(V_{\bar{0}}\otimes V')\oplus(V'\otimes V''\oplus \Lambda^2 V_{\bar{0}})\oplus(V_{\bar{0}}\otimes V'')\oplus S^2 V''\,.$$

It is evident that $G_0\simeq \mathrm{gl}_k\oplus \mathrm{so}_m$, the representations of G_0 in G_i and in G_{-i} are contragredient. The G_0-module G_1 is isomorphic to $\mathrm{gl}_k\otimes \mathrm{so}_m$, the G_0-module G_2 to $S^2\mathrm{gl}_k$.

3. Lie Superalgebras $P(n)$. Let $G_0=\mathrm{sl}(n+1)$, G_1 (resp. G_{-1}) be the space of all symmetric (resp. skewsymmetric) matrices of order $n+1$, $n\geqq 2$. The $\mathbb{Z}$-graded Lie superalgebra structure on $\boldsymbol{P}(n)=G_{-1}\oplus G_0\oplus G_1$ is introduced by formulae

$$[c_1, c_2]=c_1c_2-c_2c_1\,, \quad [c,a]=ca+ac^{\mathrm{T}}$$
$$[c,b]=-c^{\mathrm{T}}b-bc\,, \quad [a,b]=ab \quad (\text{note, that } \operatorname{tr}ab=0)\,,$$
$$[a,a_1]=[b,b_1]=0\,, \quad c,c_1,c_2\in G_0\,, \quad a,a_1\in G_1\,, \quad b,b_1\in G_{-1}\,.$$

$\mathbb{Z}$-grading is induced by $\mathbb{Z}$-grading, i.e. $\boldsymbol{P}(n)_{\bar{0}}=G_0$, $\boldsymbol{P}(n)_{\bar{1}}=G_{-1}\oplus G_1$. By simple computation one verifies that $\boldsymbol{P}(n)$ is a Lie superalgebra.

4. Lie Superalgebras $Q(n)$. Let $G_{\bar{0}}$ and $G_{\bar{1}}$ be duplicates of $\mathrm{sl}(n+1)$, $n\geqq 2$. The Lie superalgebra structure on $\boldsymbol{Q}(n)$ is introduced by formulae

$$[a_1, a_2]=a_1a_2-a_2a_1\,, \quad [a,b]=ab-ba\,,$$
$$[b_1, b_2]=b_1b_2+b_2b_1-2/(n+1)\cdot\operatorname{tr}(b_1b_2)1_{n+1}\,,$$
$$a,a_1,a_2\in G_{\bar{0}}\,, \quad b,b_1,b_2\in G_{\bar{1}}\,.$$

5. Lie Superalgebras $F(4)$, $G(3)$, $D(2,1;\alpha)$

Proposition 3.1. a) *There is unique* 40-*dimensional classical type Lie superalgebra* $\boldsymbol{F}(4)$, $\boldsymbol{F}(4)_{\bar{0}}$ *being the Lie algebra of type* $\boldsymbol{B}_3\oplus\boldsymbol{A}_1$ *and* $\boldsymbol{F}(4)_{\bar{1}}$ *as* $\boldsymbol{F}(4)_{\bar{0}}$-*module is isomorphic to* $\mathrm{spin}_7\oplus\mathrm{sl}_2$.

b) *There is a unique* 31-*dimensional classical type Lie superalgebra* $\boldsymbol{G}(3)$, $\boldsymbol{G}(3)_{\bar{0}}$ – *being the Lie algebra of type* $\boldsymbol{G}_2 \oplus \boldsymbol{A}_1$ *while* $\boldsymbol{G}(3)_{\bar{1}}$ *as a* $\boldsymbol{G}(3)_{\bar{0}}$-*module is isomorphic to* $\boldsymbol{G}_2 \otimes \mathrm{sl}_2$.[1]

c) *There is a* 1-*parameter family of 17-dimensional Lie superalgebras* $\boldsymbol{D}(2,1;\alpha)$, $\alpha \in K - \{0, -1\}$ *that consists of simple Lie superalgebras.* $\boldsymbol{D}(2,1;\alpha)_{\bar{0}}$ – *being Lie algebra of type* $\boldsymbol{A}_1 \oplus \boldsymbol{A}_1 \oplus \boldsymbol{A}_1$, $\boldsymbol{D}(2,1;\alpha)_{\bar{1}}$ *as* $\boldsymbol{D}(2,1;\alpha)_{\bar{0}}$-*module is isomorphic to* $sl_2 \otimes sl_2 \otimes sl_2$.

The proof could be obtained by simple computation of epimorphism of $G_{\bar{0}}$-modules $S^2 G_{\bar{1}} \to G_{\bar{0}}$ satisfying (1.2). On the other hand it follows from contragredient Lie algebras theory, see §5.

6. **Proposition 3.2.** a) *All Lie superalgebras* $\boldsymbol{A}(m,n)$, $\boldsymbol{B}(m,n)$, $\boldsymbol{C}(n)$, $\boldsymbol{D}(m,n)$, $\boldsymbol{D}(2,1;\alpha)$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$, $\boldsymbol{P}(n)$, $\boldsymbol{Q}(n)$ *are classical type Lie superalgebras.*

b) The $G_{\bar{0}}$-*module* $G_{\bar{1}}$ *is isomorphic in cases* $\boldsymbol{B}(m,n)$, $\boldsymbol{D}(m,n)$, $\boldsymbol{D}(2,1;\alpha)$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$ *and* $\boldsymbol{Q}(n)$ *to the following one:*

Table 1

G	$G_{\bar{0}}:G_{\bar{1}}$	G	$G_{\bar{0}}:G_{\bar{1}}$
$\boldsymbol{B}(m,n)$	$\mathrm{so}_{2n+1} \otimes \mathrm{sp}_{2n}$	$\boldsymbol{F}(4)$	$\mathrm{spin}_7 \otimes \mathrm{sl}_2$
$\boldsymbol{D}(m,n)$	$\mathrm{so}_{2n} \otimes \mathrm{sp}_{2n}$	$\boldsymbol{G}(3)$	$G_2 \otimes \mathrm{sl}_2$
$\boldsymbol{D}(2,1;\alpha)$	$\mathrm{sl}_2 \otimes \mathrm{sl}_2 \otimes \mathrm{sl}_2$	$\boldsymbol{Q}(n)$	$\mathrm{ad}\,\mathrm{sl}_{n+1}$

c) *Lie superalgebras* $\boldsymbol{A}(m,n)$, $\boldsymbol{C}(n)$, $\boldsymbol{P}(n)$ *admit the unique agreed* $\mathbb{Z}$-*grading of the form* $G_{-1} \oplus G_0 \oplus G_1$. *The* G_0-*modules* $G_{\pm 1}$ *are irreducible and in cases* $A(m,n)$ *and* $\boldsymbol{C}(n)$ *contragredient. They are enumerated below*

Table 2

G	$G_0:G_{-1}$	$G_0:G_1$
$A(m,n), m \neq n$	$\mathrm{gl}_{m+1} \otimes \mathrm{sl}_{n+1}$	$(G_{-1})^*$
$A(n,n)$	$\mathrm{sl}_{n+1} \otimes \mathrm{sl}_{n+1}$	$(G_{-1})^*$
$\boldsymbol{C}(n)$	csp_{2n-2}	$(G_{-1})^*$
$\boldsymbol{P}(n)$	$\Lambda^2\, \mathrm{sl}_{n+1}^*$	$S^2\, \mathrm{sl}_{n+1}$

d) *If* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *is a simple Lie superalgebra, the representation of* $G_{\bar{0}}$ *in* $G_{\bar{1}}$ *being the same as one of a), then G is isomorphic to one of these superalgebras.*

7. *The Two Cases in the Classification of Classical Type Lie Superalgebras.* Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a classical type Lie superalgebra. Then $G_{\bar{0}} = G'_{\bar{0}} \oplus C$, $G'_{\bar{0}}$ being semisimple Lie algebra, C being the centre of $G_{\bar{0}}'$.

I. Case. The representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ is irreducible. Then $G_{\bar{0}}$ is semisimple. In fact, it being not so implies the existence of a central element $z \in G_{\bar{0}}$ such that $[z,g] = g, g \in G_{\bar{1}}$. $[G_{\bar{1}}, G_{\bar{1}}] = G_{\bar{0}}$ hence $[z,g] = 2g$, $g \in G_{\bar{0}}$. That is contradiction.

---

[1] Here by spin_7 is denoted the spin representation of $\boldsymbol{B}_3$, sl_n, sp_n, so_n stands for standard representations of these Lie algebras, *csp* is *sp* plus the 1-dimensional centre, ad sl_n stands for the adjoint representation of sl_n and asteric denotes the dual module

II. Case. The representation $G_{\bar{0}}$ in $G_{\bar{1}}$ is reducible. Let us consider some own maximal subalgebra $L_0 \subset G$, such that $G_{\bar{0}} \subset L_0$. Let us consider the transitive filtration corresponding to L_0. The following proposition is easily deduced from Proposition 2.4.

Proposition 3.3. *Suppose* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *is classical type Lie superalgebra,* $G_{\bar{1}}$ *being the space of reducible representation of* $G_{\bar{0}}$. *Then there is filtration of* G, $G = L_{-1} \supset L_0 \supset L_1$, *such that* $\operatorname{Gr} L = \operatorname{Gr}_{-1} L \oplus \operatorname{Gr}_0 L \oplus \operatorname{Gr}_1 L$ *is simple* $\mathbb{Z}$*-graded Lie superalgebra,* $\operatorname{Gr}_{\pm 1} L$ *being spaces of faithful and irreducible representations of* $\operatorname{Gr}_0 L$, $(\operatorname{Gr} L)_{\bar{0}} = \operatorname{Gr}_0 L \cong G_{\bar{0}}$ *and the representations of* $G_{\bar{0}}$ *in* $G_{\bar{1}}$ *is equivalent to the one of* $\operatorname{Gr}_0 L$ *in* $\operatorname{Gr}_{-1} L \oplus \operatorname{Gr}_1 L$.

§4. Classification of Classical Type Lie Superalgebras

1. Definition of the Killing Form and Its Properties. Call the *Killing form* of a Lie superalgebra G the bilinear form

$$(a, b) = \operatorname{str}((\operatorname{ad} a)(\operatorname{ad} b)),$$

str being supertrace on $l(G)$. Properties of supertrace (cf. Prop. 1.1) imply the same properties for the Killing form:

Proposition 4.1. *The Killing form on Lie superalgebra* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *– is consistent, supersymmetric and invariant, i.e.*

$$(a, b) = 0 \quad \text{for} \quad a \in G_{\bar{0}}, b \in G_{\bar{1}}$$

$$(a, b) = (-1)^{\deg a \cdot \deg b}(b, a)$$

$$([a, b], c) = (a, [b, c]).$$

The following proposition is proved as the analogous one in the Lie algebra case (cf. [10]).

Proposition 4.2. *Lie superalgebra with nondegenerate Killing form is decomposed into the direct sum of simple Lie algebras each having nondegenerate Killing form.*

Recall, that which each representation ϱ of a Lie algebra $G_{\bar{0}}$ is connected the invariant bilinear symmetric form on $G_{\bar{0}}$:

$$(a, b)_V = \operatorname{tr}(\varrho(a)\varrho(b)).$$

In particulear, ϱ being ad, we obtain the Killing form $(\,,\,)_0$. If G_0 is semisimple then $(\,,\,)_V$ is always nondegenerate. If G is simple then $(\,,\,)_V = l_V(\,,\,)_0$, l_V being a positive rational number. l_V is called the index of the representation of G in V. The index of a direct sum of representations is evidently equal to the sum of indices.

Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra. There are two bilinear forms on G:

$$(a, b)_0 = \operatorname{tr}(\operatorname{ad}_{G_{\bar{0}}} a \cdot \operatorname{ad}_{G_{\bar{0}}} b), \quad (a, b)_1 = \operatorname{tr}(\operatorname{ad}_{G_{\bar{1}}} a \cdot \operatorname{ad}_{G_{\bar{1}}} b). \tag{4.1}$$

From the definition of the Killing form

$$(a, b) = (a, b)_0 - (a, b)_1, \quad a, b \in G_{\bar{0}}. \tag{4.2}$$

If $G_{\bar{0}}$ is a direct sum of Lie algebras $G'_{\bar{0}}$ and $G''_{\bar{0}}$, $G'_{\bar{0}}$ being simple, then

$$(a, b) = (1 - l)(a, b)_0 \tag{4.3}$$

for $a, b \in G_{\bar{0}}$, l being the index of the representation of $G'_{\bar{0}}$ in $G_{\bar{1}}$. This follows from (4.1), (4.2).

Proposition 4.3. *A simple Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with a nondegenerate Killing form is of classical type.*

Proof. The unipotent radical N of a subalgebra G is contained (cf. [9]) in the kernel of the form $(a, b)_V$. Thus, for $a \in N$, $b \in G$, formula (4.2) implies $(a, b) = (a, b)_0 - (a, b)_1 = 0$. Hence, a is contained in the kernel of the Killing form of G. This fact in turn implies that $N = 0$. Thus G is of classical type.

With the use of the Killing form the Jacobi identity could be written in a very suitable form. Assume that $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is a Lie superalgebra with a nondegenerate Killing form. Pick some basis u_i in $G_{\bar{0}}$ and its dual v_i with respect to the restriction to $G_{\bar{0}}$ of the Killing form. Let $a, b, c \in G$. Then

$$[a, b] = \sum_i \alpha_i v_i, \qquad \alpha_i = ([a, b], u_i) = -(a, [u_i, b]).$$

Hence $[a, b] = -\sum (a, [u_i, b]) v_i$. The Jacobi identity implies:

$$\sum_i (a, [u_i, b])[v_i, c] + (b, [u_i, c])[v_i, a] + (c, [u_i, a])[v_i, b] = 0. \qquad (*)$$

2. The following lemma is quite usefull in the classification of Lie superalgebras of classical type.

Lemma 4.4. *Assume that ϱ is faithfull irreducible finite dimensional representation of semisimple Lie superalgebra G in V. Let Δ be the root system of G, L be the weight system of the representation ϱ, Λ being the highest weight. Then*
a) *if $2\Lambda \in \Delta$, then the G-module V is isomorphic to* sp_n,
b) *if for any $\mu \in L$ one has $\Lambda - \mu \in \Delta$, then the G-module V is isomorphic either to* sl_n *or* sp_n,
c) *if for any $\mu \in L$, $\mu \neq -\Lambda$, $\Lambda - \mu \in \Delta$ then the G-module V is isomorphic to one of* sl_n, sp_n, so_n, spin_7 *or* $\boldsymbol{G}_2$ *(7-dim. module of* $\boldsymbol{G}_2$*).*

3. **Proposition 4.5.** *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a simple Lie superalgebra with nondegenerate Killing form, and let the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ be irreducible. Then G is isomorphic to one of $\boldsymbol{B}(m, n)$, $\boldsymbol{D}(m, n)$ for $m - n \neq 1$, $\boldsymbol{F}(4)$ or $\boldsymbol{G}(3)$.*

As was shown in Section 3.7 the Lie algebra $G_{\bar{0}}$ is semisimple. Let H be a Cartan subalgebra of $G_{\bar{0}}$, and let Δ be the root system. Assume that L is the weight system of the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$, $G_1 = \bigoplus_\lambda V_\lambda$ being the weight decomposition. Evidently

$$(v_\lambda, v_\mu) = 0, \qquad \lambda \neq -\mu \qquad (4.4)$$

if $\lambda \in L$ then

$$-\lambda \in L \quad \text{and} \quad (v_\lambda, V_{-\lambda}), \qquad v_\lambda \in V_\lambda \qquad (4.5)$$

Let $G_{\bar{0}} = \bigoplus_{s=1}^{N} G_0^{(s)}$ be the decomposition of $G_{\bar{0}}$ into a direct sum of simple components. The summands are orthogonal with repsect to (,) and $(,)_0$. Denote by $(,)_0^{(s)}$ the restriction of $(,)_0$ to $G_0^{(s)}$. Denote l_s the index of the representation of $G_0^{(s)}$ in $G_{\bar{1}}$. Assume that $h_1,\dots,h_r$ is a basis in the Cartan subalgebra H, and the union of bases of Cartan subalgebras $H \cap G_0^{(s)}$. Let $\hat{h}_1,\dots,\hat{h}_r$ be the dual basis with respect to $(\,,)$ and $\bar{h}_1,\dots,\bar{h}_r$ be the dual basis with respect to $(\,,)_0$. Section 4.1 implies that

$$\bar{h}_i=(1-l_s)\hat{h}_i\,, \qquad h_i \in G_{\bar{0}}^s\ . \tag{4.6}$$

We prove the following lemma first.

Lemma 4.6. a) *If* $\lambda \in \Delta$, $2\lambda \notin \Delta$, *then*

$$(\lambda, \lambda)= \sum_{s=1}^{N} (\lambda, \lambda)_0^{(s)}(1-l_s)^{-1}=0\,.$$

b) *If* $\lambda, \mu \in L$, $\lambda \pm \mu \in \Delta$, *then*

$$(\lambda, \mu)= \sum (\lambda, \mu)_0^{(s)}(1-l_s)^{-1}=0\,.$$

Proof. Consider the following basis of G: $\{u_i\}=\{e_\alpha, \alpha \in \Delta \setminus 0, h_i, i=1,\dots,r\}$, $\{v_i\}=\{e_{-\alpha}, \alpha \in \Delta \setminus 0, \hat{h}_i, i=1,\dots,r\}$ being its dual with respect to $(\,,)$.

Prove a). Assume $\lambda \in L$, $a=c=v_\lambda$, $b=v_{-\lambda}$, $(v_\lambda, v_{-\lambda})=1$. (Such v_λ does exist due to (4.5).) Let us write

the identity $(*)$ for a, b, c in chosen bases $\{u_i\}$ and $\{v_i\}$.

With respect to (4.4) we obtain $\sum \lambda(h_i)\lambda(\hat{h}_i)=0$.

The formulae $(\lambda, \mu)= \sum \lambda(h_i)\mu(\hat{h}_i)$ and $(\lambda, \mu)_0=\sum \lambda(h_i)\mu(\bar{h}_i)$ together with (4.6) imply a).

b) As in a) put $a=v_\lambda$, $b=v_{-\lambda}$, $c=v_\mu$.

Proof of the Proposition 4.5. If the Lie algebra $G_{\bar{0}}$ is simple then $(\lambda, \lambda)_0 \neq 0$, so after Lemma 4.6a) $\lambda \in L$ implies $2\lambda \in \Delta$. Lemma 4.4a) implies then that the $G_{\bar{0}}$-module $G_{\bar{1}}$ is isomorphic to sp_n. It follows from Proposition 3.2d) that G is isomorphic to $\boldsymbol{B}(0, n)$. Let now $G_{\bar{0}}$ be semisimple but not simple. Decompose $G_{\bar{0}}$ in the direct sum of G_0' and G_0''; G_0' (resp. G_0'') being simple components of G with positive (resp. negative) numbers $1-l_i$. Let $\Lambda=\Lambda'+\Lambda''$ be the highest weight of a representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ (the upper index denoting on the restriction of the weight to the corresponding direct summand). Consider the weight $\mu=\mu'+\Lambda''$ where $\mu' \neq \pm \Lambda'$. Note that

$$\Lambda+\mu \notin \Delta \tag{4.7}$$

Furthermore, $(\Lambda, \mu)=(\Lambda', \mu')+(\Lambda'', \Lambda'')=(\Lambda', \mu')+(\Lambda, \Lambda)-(\Lambda', \Lambda')$. As $2\Lambda \notin \Delta$ then via Lemma 4.6

$$(\Lambda, \mu)=(\Lambda', \mu')-(\Lambda', \Lambda')= \sum_s ((\Lambda, \mu)_0^{(s)}-(\Lambda, \Lambda)_0^{(s)})(1-l_s)^{-1}\,. \tag{4.8}$$

The sum in (4.8) is taken with respect to simple components of G_0'. As Λ is the highest weight, $(\Lambda, \Lambda)_0^{(s)} \geqq (\mu, \mu)_0^{(s)}$ for every s. Therefore, after using the Cauchy-Bounjakovsky inequality, all summands of (4.8) are nonpositive.

Hence,

$$(\Lambda, \mu) \neq 0\,. \tag{4.9}$$

According to the Lemma 4.6.b) it follows from (4.7) and (4.9) that $\Lambda-\mu\in\Delta'$, so if $\mu'\neq\pm\Lambda'$ then $\Lambda'-\mu'\in\Delta'$. The same is true for G_0''. With respect to the Lemma 4.4 this means that the linear representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ could be equivalent only to a tensor product of two standard representations of Lie algebras each being sp_n, $n\geqq 2$ or sl_n, $n\geqq 3$, or so_n, $n\geqq 3$ or spin_7 or $\boldsymbol{G}_2$. As the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ admits an invariant nondegenerate supersymmetric bilinear form, than one multiple of tensor product admits an invariant symmetric form while the other an invariant skew form. Thus, only the three possibilities remain: 1) $\mathrm{sp}_n\otimes\mathrm{so}_m$, 2) $\mathrm{sp}_n\otimes\mathrm{spin}_7$, 3) $\mathrm{sp}_n\otimes\boldsymbol{G}_2$. In the case 1) G is isomorphic to $\boldsymbol{B}(m-1)/2, n/2)$, m being odd and $m>1$ or $\boldsymbol{D}(m/2, n/2)$, m being even, $m>2$, cf. Proposition 3.2d). In cases 2 and 3 the identity $(\Lambda,\Lambda)=0$ implies $n=2$. Hence, by Proposition 3.2d) we find that G is isomorphic to $\boldsymbol{F}(4)$ and $\boldsymbol{G}(3)$ respectively.

3. The proof of the following proposition is the same as that of Proposition 4.5.

Proposition 4.7. *Let* $G=G_{-1}\oplus G_0\oplus G_1$ *be a simple Lie superalgebra with consistent* $\mathbb{Z}$*-grading, and let the representations of* G_0 *in* G_{-1} *and* G_1 *be faithful and irreducible and the Killing form be nondegenerate. Then the Lie superalgebra* G *is isomorphic either to the Lie superalgebra* $\boldsymbol{A}(m,n)$, $m\neq n$ *or to* $\boldsymbol{C}(n)$.

The following theorem is a corollary of Propositions 3.3, 4.5 and 4.7.

Theorem 1. *A simple finite-dimensional Lie superalgebra* $G=G_{\bar{0}}\oplus G_{\bar{1}}$ *with nondegenerate Killing form is isomorphic to one of the following:* $\boldsymbol{A}(m,n)$, $m\neq n$; $\boldsymbol{B}(m,n)$, $\boldsymbol{C}(n)$, $\boldsymbol{D}(m,n)$, $m-n\neq 1$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$.

4. The classification of Classical Type Lie Superalgebras is as Follows.

Theorem 2. *A classical type Lie superalgebra with* $G_{\bar{1}}\neq 0$ *is isomorphic to one of the following:* $\boldsymbol{A}(m,n)$, $\boldsymbol{B}(m,n)$, $\boldsymbol{C}(n)$, $\boldsymbol{D}(m,n)$, $\boldsymbol{D}(2,1;\alpha)$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$, $\boldsymbol{P}(n)$, $\boldsymbol{Q}(n)$.

The Theorem 2 is proved because the Proposition 4.8. holds and the Killing form is either nondegenerate or zero on a simple Lie superalgebra.

Proposition 4.8. *A classical type Lie superalgebra with zero Killing form is isomorphic to one of the following Lie superalgebras:* $\boldsymbol{A}(n,n)$, $\boldsymbol{D}(n+1,n)$, $\boldsymbol{D}(2,1;\alpha)$, $\boldsymbol{P}(n)$, $\boldsymbol{Q}(n)$.

Proof. From (4.3) it follows that the index of the representation of $G_{\bar{0}}^{(i)}$ on $G_{\bar{1}}$ is equal to 1 because $(a,b)=0$. In particular, the index of $G_0^{(i)}$ in every irreducible component of $G_{\bar{1}}$ is no more then 1. All irreducible representations of simple Lie algebras having index $\leqq 1$ are listed in [1]. The proof is based on this list and results of Section 3.7.

§5. Contragredient Lie Superalgebras

1. Lie Superalgebras $G(A,\tau)$. Let $A=(a_{ij})$ be a matrix of order r with elements from K, and let τ be a subset of the set $I=\{1,\dots,r\}$. Denote by $\tilde{G}(A,\tau)$ the Lie superalgebra with generators e_i, f_i, h_i, $i\in I$ and the following defining relations:

$$[e_i,f_j]=\delta_{ij}h_i\,,\qquad [h_i,h_j]=0$$

$$[h_i,e_j]=a_{ij}e_j\,,\qquad [h_i,f_j]=-a_{ij}f_j$$

$$\deg h_i=\bar{0}\,;\qquad \deg e_i=\deg f_i=\bar{0},\quad i\notin\tau\,;\qquad \deg e_i=\deg f_i=\bar{1}\,,\quad i\in\tau\,.$$

By the methods of [11] it is possible to demonstrate that setting $\deg e_i = -\deg f_i = 1$, $\deg h_i = 0$, $i \in I$, we obtain a $\mathbb{Z}$-grading of Lie superalgebra $\tilde{G}(A, \tau) = \bigoplus_{i \in \mathbb{Z}} \tilde{G}_i$, $\{e_i\}$, $\{f_i\}$, and $\{h_i\}$ being bases of the space $\tilde{G}_1$, $\tilde{G}_{-1}$ and $\tilde{G}_0$ respectively. Let J be the unique maximal $\mathbb{Z}$-graded ideal in $\tilde{G}(A, \tau)$ satisfying $J \cap (G_{-1} \oplus G_0 \oplus G_1) = 0$. The $\mathbb{Z}$-graded Lie superalgebra $G(A, \tau) = \tilde{G}(A, \tau)/J = \oplus G_i$ is called a *contragredient Lie superalgebra*, the matrix A is called the *Cartan matrix* of $G(A, \tau)$ and r is called the *rank* of $G(A, \tau)$.

If $\tau = \emptyset$ we obtain contragredient Lie algebras; their theory is developed in [11].

Some propositions concerning contragredient Lie algebras from [8] and [11] are valid in the superalgebra case (with the same proof, too).

We quote only what is needed in the following.

Proposition 5.1. *The centre C of the Lie superalgebra $G(A, \tau)$ consists of elements $\sum a_i h_i$, where $\sum_i a_{ij} a_i = 0$.*

Proposition 5.2. *Let $G(A, \tau)$ be a finite-dimensional contragredient Lie superalgebra with the centre C. The Lie superalgebra $G(A, \tau)/C$ is simple iff the Cartan matrix satisfies the following conditions:*

$$\begin{array}{l} \text{for any } i, j \in I \text{ there is a sequence } i_1, \ldots, i_r \in I, \\ \text{such that } a_{ii_1} a_{i_1 i_2} \ldots a_{i_r j} \neq 0. \end{array} \tag{m}$$

With the concept of contragredient Lie superalgebras the proof of existence of exceptional Lie superalgebras becomes easy.

We will demonstrate the example of $D(2, 1; \alpha)$. Let us take the matrix

$$D_\alpha = \begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \alpha \neq 0, -1,$$

and put $\tau = \{1\}$. Let us find a basis of the contragredient Lie superalgebra $G(D_\alpha, \tau) = \oplus G_i$. Note that if $g \in G_i$, $i \geqq 1$, then $g \neq 0$ iff $[g, G_{-1}] \neq 0$. Therefore we obviously have $G_2 = \langle [e_1, e_2], [e_1, e_3] \rangle$, $G_3 = \langle [[e_1, e_2], e_3] \rangle$, $G_4 = \langle [[[e_1, e_2], e_3], e_1] \rangle$, $G_i = 0$ for $i > 4$.

The case $i < 0$ is treated in the same way. From the above it is clear that the representation of $G(D_\alpha, \tau)_{\bar{0}}$ in $G(D_\alpha, \tau)_{\bar{1}}$ is $sl_2 \otimes sl_2 \otimes sl_2$.

Note that $D(2, 1; -1) = D(2, 1)$.

For the $F(4)$ and $G(3)$ cases it is necessary to pick the matrices

$$F_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

and $\tau = \{1\}$.

Theorem 3. *Let* $G(A,\tau)$ *be a finite-dimensional contragredient Lie superalgebra with a Cartan matrix* A *satisfying* (*m*) *and with a centre* C. *Then* $G(A,\tau)/C$ *is one of classical type Lie superalgebras* $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$, $G(3)$.

2. Properties of the Root Decomposition of a Classical Type Lie Superalgebras. We define a *Cartan superalgebra* of a finite dimensional Lie superalgebra $G=G_{\bar{0}}\oplus G_{\bar{1}}$ to be a Cartan subalgebra of $G_{\bar{0}}$. It is evident that any inner automorphism of $G_{\bar{0}}$ continues to an automorphism of G. It is known that Cartan subalgebras are conjugate in G. Hence, Cartan subalgebras of Lie superalgebra G are conjugate too.

Let G be a classical type Lie superalgebra, H being its Cartan subalgebra. Then we have a root decomposition $G=\bigoplus_{\alpha\in H^*} G_\alpha$ where $G_\alpha=\{a\in G\,|\,[h,a]=\alpha(h)a,\, h\in H\}$. The set $\Delta=\{\alpha\in H^*\,|\,G_\alpha\neq 0\}$ is called the root system. It is evident that $\Delta=\Delta_0\cup\Delta_1$ where Δ_0 is the root system of $G_{\bar{0}}$ and Δ_1 is the weight system of the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$. Call Δ_0 and Δ_1 the even root system and the odd root system respectively. The root system $\Pi=\{\alpha_1,\ldots,\alpha_r\}$ is called a simple root system if there are vectors $e_i\in G_{\alpha_i}$, $f_i\in G_{\alpha_i}$ such that $[e_i,f_j]=\delta_{ij}h_i\in H$, vectors e_i, f_i being generators of G.

The consideration of classical type Lie superalgebras together with the usual reasonings from Lie algebra theory give directly the following information about the root decompositions.

Proposition 5.3. *Assume that* G *is a classical type Lie superalgebra, and that* $G=\oplus G_\alpha$ *is its root decomposition with respect to the Cartan subalgebra* H. *Then*

a) $G_0=H$ *except in the* $Q(n)$ *cases.*

b) $\dim G_\alpha=1$ *when* $\alpha\neq 0$ *except for the* $A(1,1)$, $P(2)$, $P(3)$ *and* $Q(n)$ *cases.*

c) *There is unique (up to scalar multiple) nondegenerate invariant symmetric (in "super" sense) bilinear form* $(\,,)$ *on* G *except in the* $P(n)$ *and* $Q(n)$ *cases.*

d) *If* G *is not one of* $A(1,1)$, $P(n)$ *or* $Q(n)$ *then*

1) $[G_\alpha,G_\beta]\neq 0$ *iff* $\alpha,\beta,\alpha+\beta\in\Delta$,

2) $(G_\alpha,G_\beta)=0$ *for* $\alpha\neq-\beta$,

3) $[e_\alpha,e_{-\alpha}]=(e_\alpha,e_{-\alpha})h_\alpha$ *where* h_α *is nonzero vector defined by* $(h_\alpha,h)=\alpha(h)$, $h\in H$,

4) $(\,,)$ *defines a nondegenerate pairing of* G_α *and* $G_{-\alpha}$,

5) Δ_0 *and* Δ_1 *are invariant under the action of the Weil group* W *of* $G_{\bar{0}}$,

6) *if* $\alpha\in\Delta$ *(resp.* Δ_0,Δ_1*) then* $-\alpha\in\Delta$ *(resp.* Δ_0,Δ_1*)*

7) $k\alpha\in\Delta$ *when* $\alpha\neq 0$, $k\neq 0$, ± 1 *iff* $\alpha\in\Delta_1$, $(\alpha,\alpha)\neq 0$. *In this case* $k=\pm 2$.

3. Dynkin Diagrams of Finite Dimensional Contragredient Lie Superalgebras. Suppose G is one of Lie superalgebras $\mathrm{sl}(m+1,n+1)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$ or $G(3)$. Suppose that H is a Cartan subalgebra, and that Π is one of the simple root systems. Let e_i and f_i be vectors defining Π. Then the vectors $[e_i,f_i]=h_i$ are a basis of H. We defined $\mathbb{Z}$-grading on G by putting $\deg e_i=-\deg f_i=1$, $\deg h_i=0$. It follows from the simplicity of G (module its centre C) that G is a contragredient Lie superalgebra. Its Cartan matrix is $A=(\alpha_j(h_i))$,

$\tau=\{i\in I,\ \alpha_i\in\Delta_1\}$. It follows from Theorem 3 that these examples exhaust all simple (mod. C) finite dimensional contragredient Lie superalgebras.

It is not difficult to enumerate all such pairs (A, τ) up to equivalence. Here we list only pairs of rank 1 and 2 with indecomposable Cartan matrix, corresponding pairs (A, τ), and Dynkin diagrams (Tables 3 and 4) and the "simplest" Dynkin diagrams for arbitrary G (Table 5).

Nodes ○, ⊗ and ● are called white, grey and black respectively. The contragredient Lie algebras of rank r are denoted by diagram consisting of r modes, the i-th node white, if $i\notin\tau$ and grey or black if $i\in\tau$ and $a_{ii}=0$ or 2 respectively. Given two distinct nodes i-th and j-th do not join them if $a_{ij}=a_{ji}=0$, otherwise do so as in Table 4.

Proposition 5.4. *Each of Lie superalgebras* $\boldsymbol{A}(m, n)$, $\boldsymbol{B}(m, n)$, $\boldsymbol{C}(n)$, $\boldsymbol{D}(m, n)$, $\boldsymbol{D}(2, 1; \alpha)$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$ *could be represented in the form* $G(A, \tau)/C$, ($C\neq 0$ *only in the* $\boldsymbol{A}(n, n)$ *case*) *where* τ *consists only of one element.*

The corresponding diagrams are enumerated below with the coefficients of the decomposition of the highest root with respect to simple roots s being the number of exceptional nonwhite node, r being the total number of nodes.

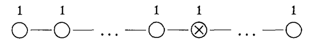
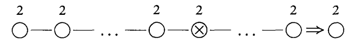
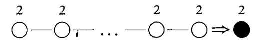
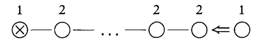
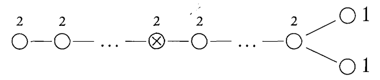
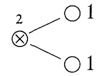
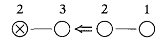
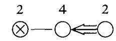
Table 3

$G(A, \tau)$	A	τ	diagram	dim.
$\boldsymbol{A}_1$	(2)	$\emptyset$	○	3
sl(1, 1)	(0)	{1}	⊗	3
$\boldsymbol{B}(0, 1)$	(2)	{1}	●	5

Table 4

$G(A, \tau)$	A	τ	diagram	dim.
$\boldsymbol{A}_2$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\emptyset$	○—○	8
$\boldsymbol{B}_2$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	$\emptyset$	○⇒○	10
$\boldsymbol{G}_2$	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	$\emptyset$	○⇛○	14
$\boldsymbol{A}(1, 0)$	$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$	{2}	○—⊗	8
$\boldsymbol{B}(1, 1)$	$\begin{pmatrix} 2 & -2 \\ -1 & 0 \end{pmatrix}$	{2}	○⇐⊗	12
$\boldsymbol{B}(0, 2)$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	{2}	○⇒●	14
$\boldsymbol{A}(1, 0)$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	{1, 2}	⊗—⊗	8
$\boldsymbol{B}(1, 1)$	$\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$	{1, 2}	⊗⇒●	12

Table 5

G	diagram	s	r
$A(m, n)$		$m+1$	$m+n+1$
$B(m, n), m>0$		n	$m+n$
$B(0, n)$		n	n
$C(n), n>2$		1	n
$D(m, n)$		n	$m+n$
$D(2, 1; \alpha)$		1	3
$F(4)$		1	4
$G(3)$		1	3

§6. Cartan Type Lie Superalgebras

1. Definition of $W(n)$. Let $\Lambda(n)$ be the Grassmann superalgebra, $\xi_1,\ldots,\xi_n$ being its generators. Denote by $W(n)$ the Lie superalgebra $\operatorname{der}\Lambda(n)$. Recall that any $D\in W(n)$ could be written in the form $D=\sum P_i\partial/\partial\xi_i$, $P_i\in\Lambda(n)$, derivations $\partial/\partial\xi_i$ being defined by $\partial\xi_j/\partial\xi_i=\delta_{ij}$.

Put $\deg\xi_i=1$, $i=1,\ldots,n$. Then we obtain an agreed $\mathbb{Z}$-grading of the superalgebra $\Lambda(n)$. This $\mathbb{Z}$-grading induces an agreed $\mathbb{Z}$-grading in the Lie superalgebra $W(n)=\bigoplus_{k\geqq-1} W(n)_k$, where

$$W(n)_k=\{\sum P_i\partial/\partial\xi_i|\deg P_i=k+1, i=1,\ldots,n\}^{k\geqq-1}\,.$$

The $\mathbb{Z}$-graded Lie superalgebra $W(n)$ is naturally filtered.

$W(n)$ has the following universal property.

Proposition 6.1. *Let* $L=L_{-1}\supset L_0\supset L_1\supset\ldots$ *be a filtered transitive Lie superalgebra,* $\dim L/L_0=n$, $L_0\supset L_{\bar{0}}$. *Then there is a monomorphism* $\alpha: L\to W(n)$ *that preserves filtration. If* β *is some other monomorphism that preserves filtration, then there exists unique automorphism* Φ *of* $W(n)$ *induced from* $\Lambda(n)$ *such that* $\alpha=\Phi\circ\beta$.

The proof follows almost literally from the one in [20], the corresponding definitions being changed for those of Section 1.3.

Proposition 6.2. *Let* $L=L_{-1}\supset L_0\supset L_1\supset\ldots$ *be a subalgebra of the Lie superalgebra* $W(n)$ *with induced filtration, and let* $\dim L_{-1}/L_0=n$. *Then any automorphism of* L *preserving filtration is induced by an automorphism of* $\Lambda(n)$.

This is a corollary of Proposition 6.1.

2. Superalgebras $\Delta(n)$ *and* $\Theta(n)$. Denote by $\Delta(n)$ an associative superalgebra over $\Lambda(n)$ with even generators $\delta\xi_1,\ldots,\delta\xi_n$; $\xi_1,\ldots,\xi_n$ being generators of $\Lambda(n)$. Let $\Delta(n)$

be a commutative superalgebra, i.e. $\delta\xi_i \circ \delta\xi_j = \delta\xi_j \circ \delta\xi_i$ and $\xi_i \cdot \delta\xi_j = \delta\xi_j \cdot \xi_i$. Define the differential δ on $\Delta(n)$ of degree $\bar{1}: \delta(\xi_i) = \delta\xi_i$, $\delta(\delta\xi_i) = 0$.

Proposition 6.3. *Differential δ has the following properties*

a) $\delta(\alpha \circ \beta) = (\delta\alpha) \circ \beta + (-1)^{\deg\alpha} \alpha \circ (\delta\beta)$, $\alpha, \beta \in \Delta(n)$,

b) $\delta(f) = \sum \frac{\partial f}{\partial \xi_i} \delta\xi_i$, $f \in \Lambda(n)$,

c) $\delta^2 = 0$.

d) *Any derivation D of $\Lambda(n)$ is uniquely extendable to a derivation $\tilde{D}$ of $\Delta(n)$ commuting with δ.*

e) *Any automorphism of $\Lambda(n)$ is uniquely extendable to an automorphism of $\Delta(n)$ commuting with δ.*

The following analogue of the Poincaré lemma holds.

Proposition 6.4. *If a differential form $\alpha \in \Delta(n)$ is closed, i.e. $\delta\alpha = 0$, then α is exact, i.e. $\alpha = \delta\beta$ for some $\beta \in \Delta(n)$.*

Denote by $\Theta(n)$ the associative superalgebra over $\Lambda(n)$ with generators $\theta\xi_1, \ldots, \theta\xi_n$ and defining relations

$$\theta\xi_i \wedge \theta\xi_j = -\theta\xi_j \wedge \theta\xi_i\,, \quad \theta\xi_i \cdot \xi_j = -\xi_j \theta\xi_i\,, \quad \deg\theta\xi_i = \bar{1}\,, \quad i, j = 1, \ldots, n\,.$$

In fact, $\Theta(n)$ is commutative superalgebra.

Define a differential θ of degree 0 on $\Theta(n)$ via

$$\theta(\xi_i) = \theta\xi_i\,, \quad \theta(\theta\xi_i) = 0\,, \qquad i = 1, \ldots, n\,.$$

Proposition 6.5. *The differential θ has the following properties:*

a) $\theta(\omega_1 \wedge \omega_2) = \theta(\omega_1) \wedge \omega_2 + \omega_1 \wedge \theta(\omega_2)$,

b) $\theta(f) = \sum \theta\xi_i \cdot \frac{\partial f}{\partial \xi_i}$, $f \in \Lambda(n)$,

c) *Any derivation D of $\Lambda(n)$ is uniquely extendable to a derivation $\tilde{D}$ of $\Theta(n)$ such that $\tilde{D}(\theta(f)) = \theta(Df)$, $f \in \Lambda(n)$. If $\theta^2 D(\xi_i) = 0$, $i = 1, \ldots, n$, then $\tilde{D}\theta = \theta\tilde{D}$,*

d) *Any automorphism Φ of $\Lambda(n)$ is uniquely extendable to an automorphism $\tilde{\Phi}$ of $\Theta(n)$, $\tilde{\Phi}(\theta(f)) = \theta(\Phi(f))$.*

Note that $\theta^2 \neq 0$. For example, $\theta^2(\xi_1\xi_2) = 2\theta\xi_1 \wedge \theta\xi_2$. It is not also true for every $D \in W(n)$ that $[\tilde{D}, \theta] = 0$. Nevertheless, c) gives us something of an action of $W(n)$ on $\Theta(n)$.

3. Forms of Volume, Hamiltonian Forms and Lie Superalgebras $S(n)$, $\tilde{S}(n)$, $H(n)$ and $\tilde{H}(n)$. We call a form ω of $\Theta(n)$ of the kind $\omega = f\theta\xi_1 \wedge \ldots \wedge \theta\xi_n$, $f \in \Lambda(n)_{\bar{0}}$, $f(0) \neq 0$, *form of volume.*

To ω there corresponds a superalgebra $S(\omega) \subset W(n)$, $S(\omega) = \{D \in W(n) | D\omega = 0\}$. From all these we pick out two: for every n $S(n) = S(\theta\xi_1 \wedge \ldots \wedge \theta\xi_n)$ and for $n = 2k$ $\tilde{S}(n) = S((1 + \xi_1 \ldots \xi_n)\theta\xi_1 \wedge \ldots \wedge \theta\xi_n)$. $D = \sum P_i \frac{\partial}{\partial \xi_i} \in S(\omega)$ iff $\sum \partial f P_i / \partial \xi_i = 0$. Hence, $S(\omega)$ is spanned by

$$f^{-1}\left(\frac{\partial a}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial a}{\partial \xi_j}\frac{\partial}{\partial \xi_i}\right), \qquad a \in \Lambda(n)\,.$$

$\boldsymbol{S}(\omega)$ has the filtration induced from $\boldsymbol{W}(n)$ and $\boldsymbol{S}(n)$ the induced $\mathbb{Z}$-grading, too.

Call *Hamiltonian form* a closed differential 2-form ω of $\Delta(n)$, $\omega=\sum\omega_{ij}\delta\xi_i\circ\delta\xi_j$, $\omega_{ij}\in\Lambda(n)_{\bar{0}}$, $\omega_{ij}=\omega_{ji}$, $\det(\omega_{ij}(0))\neq 0$, $\delta\omega=0$.

To a Hamiltonian form ω there corresponds the subalgebra $\tilde{\boldsymbol{H}}(\omega)=\{D\in\boldsymbol{W}(n)|$ $D\omega=0\}$. Denote $\boldsymbol{H}(\omega)=[\tilde{\boldsymbol{H}}(\omega),\tilde{\boldsymbol{H}}(\omega)]$, $\tilde{\boldsymbol{H}}(n)=\tilde{\boldsymbol{H}}(\sum(\delta\xi_i)^2)$, $\boldsymbol{H}(n)=[\tilde{\boldsymbol{H}}(n),\tilde{\boldsymbol{H}}(n)]$.

$$D=\sum P_i\frac{\partial}{\partial\xi_i}\in\tilde{\boldsymbol{H}}(\omega)\quad\text{iff}\quad\frac{\partial}{\partial\xi_j}\sum_t\omega_{it}P_t+\frac{\partial}{\partial\xi_i}\sum_t\omega_{jt}P_t=0\,.$$

Denote by $(\bar{\omega}_{ij})$ the matrix opposite to (ω_{ij}). $\tilde{\boldsymbol{H}}(\omega)$ consists of all elements of the form

$$D_f=\sum_{i,j}\left(\bar{\omega}_{ij}\frac{\partial f}{\partial\xi_j}\right)\frac{\partial}{\partial\xi_i},\qquad f\in\Lambda(n)\,,\qquad f(0)=0$$

and $[D_f,D_g]=D_{\{f,g\}}$, where $\{f,g\}=(-1)^{\deg f}\sum\limits_{i,j}\bar{\omega}_{ij}\dfrac{\partial f}{\partial\xi_i}\dfrac{\partial g}{\partial\xi_j}$.

In particular, $\tilde{\boldsymbol{H}}(n)$ consists of elements of the kind $D_f=\sum\dfrac{\partial f}{\partial\xi_i}\dfrac{\partial}{\partial\xi_i}$, $f\in\Lambda(n)$, $f(0)=0$ with Poisson bracket $\{f,g\}=(-1)^{\deg f}\sum\limits_i\dfrac{\partial f}{\partial\xi_i}\dfrac{\partial g}{\partial\xi_i}$.
$\boldsymbol{W}(n)$ induces filtration on $\tilde{\boldsymbol{H}}(\omega)$ and $\mathbb{Z}$-grading on $\boldsymbol{H}(n)$ and $\tilde{\boldsymbol{H}}(n)$.

4. Properties of Cartan Type Lie Superalgebras.

Proposition 6.6. a) *Assume that* $G=\bigoplus\limits_{i\geqq-1}G_i$ *is one of* $\boldsymbol{W}(n)$, $\boldsymbol{S}(n)$, $\boldsymbol{H}(n)$, $\tilde{\boldsymbol{H}}(n)$. *Then* G *is transitive,* $G_k=G_1^k$, $k\geqq 1$, *and the* G_0*-module* G_{-1} *is isomorphic to* gl_n, sl_n, so_n, so_n *respectively.*

b) *If* $G'=\bigoplus\limits_{i\geqq-1}G'_i$ *is a transitive* $\mathbb{Z}$*-graded Lie superalgebra and* G'_0*-module* G'_{-1} *is isomorphic to one of* gl_n, sl_n, *or* so_n *then there is an monomorphism of* G' *in* $\boldsymbol{W}(n)$, $\boldsymbol{S}(n)$, *or* $\tilde{\boldsymbol{H}}(n)$ *respectively,* $\mathbb{Z}$*-grading being preserved.*

c) *The Lie superalgebras* $\boldsymbol{W}(n)$, $\boldsymbol{S}(n)$, $\tilde{\boldsymbol{S}}(n)$, $\boldsymbol{H}(n)$ *are simple when* $n\geqq 2$, $n\geqq 3$, $n\geqq 4$, $n\geqq 4$ *respectively*

d) *Any automorphism of a Lie superalgebras* $\boldsymbol{W}(n)$, $n\geqq 3$, *or* $\boldsymbol{S}(\omega)$, $n\geqq 3$, *or* $\boldsymbol{H}(\omega)$, $n\geqq 5$ *is induced by an automorphism of* $\Lambda(n)$ *that multiplies the form by an element of* K.

Proposition 6.7. *Let* $L=L_{-1}\supset L_0\supset L_1\supset\ldots$ *be a filtered Lie superalgebra.*

a) *If* $\operatorname{Gr}L\simeq\boldsymbol{S}(n)$ *then* $L\simeq\boldsymbol{S}(n)$, *n being odd, and* $L\simeq\boldsymbol{S}(n)$ *or* $\tilde{\boldsymbol{S}}(n)$, *n being even.*

b) *If* $\operatorname{Gr}L\simeq\boldsymbol{H}(n)$ *or* $\operatorname{Gr}L\simeq\tilde{\boldsymbol{H}}(n)$ *then* $L\simeq\boldsymbol{H}(n)$ *or* $\tilde{\boldsymbol{H}}(n)$ *respectively.*

Proof. Due to Proposition 6.1 we assume that $L\subset\boldsymbol{W}(n)$. Due to the Levi and Malcev theorems the semisimple part of $L_{\bar{0}}$ (which is isomorphic to sl_n or so_n for a) and b) respectively) is included in $\boldsymbol{W}(n)_0$. One easily deduces that $L\simeq\operatorname{Gr}L$ in b) case and $L_0=\bigoplus\limits_{i\geqq 0}\boldsymbol{S}(n)_i$ in a) case, which in turn implies $L\simeq\boldsymbol{S}(n)$ or $\tilde{\boldsymbol{S}}(n)$.

Corollary. a) *Some automorphism of* $\Lambda(n)$ *reduces any form of the volume to the form* $(\alpha+\beta\xi_1\ldots\xi_n)\theta\xi_1\wedge\ldots\wedge\theta\xi_n$, *n being even if* $\beta\neq 0$.

b) *Any Hamiltonian form is reduced to the form* $\sum_{i=1}^{n}(\delta\xi_i)^2$ *by an appropriate automorphism of* $\Lambda(n)$.

5. Call $\boldsymbol{W}(n)$ for $n\geqq 3$, $\boldsymbol{S}(n)$ and $\tilde{\boldsymbol{S}}(n)$ for $n\geqq 4$ and $\boldsymbol{H}(n)$ for $n\geqq 5$ *Cartan type Lie superalgebras.* [There are the following identifications $\boldsymbol{W}(2)\simeq\boldsymbol{A}(1,0)\simeq\boldsymbol{C}(2)$; $\boldsymbol{S}(3)\simeq\boldsymbol{P}(2)$; $\boldsymbol{H}(4)\simeq\boldsymbol{A}(1,1)$].

§ 7. Completion of the Classification of Simple Lie Superalgebra

1. **Proposition 7.1.** *Let* $G=\oplus G_i$ *be a finite dimensional bitransitive Lie superalgebra with consistent* $\mathbb{Z}$*-grading, the following properties being fulfilled:*

a) G_0 *is semisimple Lie algebra,*

b) *representations of* G_0 *in* G_{-1} *and* G_1 *are irreducible,*

c) *representations of* G_0 *in* G_{-1} *and* G_1 *are not contragredient,*

d) $G_{-1}\oplus G_0\oplus G_1$ *generates G.*

Then G is isomorphic (as a $\mathbb{Z}$*-graded superalgebra) to one of the following Lie superalgebras:* $\boldsymbol{S}(n)$ *or* $\boldsymbol{H}(n)$ *for* $n>4$ *or* $\boldsymbol{P}(n)$.

In the proof, the technique of [11] is used.

2. Classification of $\mathbb{Z}$*-Graded Lie Superalgebras* $G=\bigoplus_{i\geqq -1}G_i$. We produce two constructions of transitive $\mathbb{Z}$-graded Lie superalgebras.

Any $\mathbb{Z}$-graded Lie superalgebra could be extended with the use of the even derivation z defined by $[z,x]=kx$, $x\in G_k$. Denote $G^z=\oplus G_i^z$, $G_i^z=G_i$, $i\neq 0$ and $G_0^z=G_0\oplus\langle z\rangle$, the extended Lie superalgebra. If the $\mathbb{Z}$-graded Lie superalgebra is transitive and the centre of G is trivial then, evidently, G is transitive.

The another construction is as follows. Assume that the Lie algebra H does not contain the centre. We put $H^\xi=G_{-1}\oplus G_0\oplus G_1$ where $G_{-1}=\xi H$, $G_0=H$, $G_1=\langle d/d\xi\rangle$, the commutators being defined as follows

$$[d/d\xi,\xi h]=h,\ [\xi h_1,h_2]=\xi[h_1,h_2],\quad [d/d\xi,h]=0.$$

It is evident that H^ξ is a transitive $\mathbb{Z}$-graded Lie superalgebra.

Proposition 7.1 yields the following theorem.

Theorem 4. *A transitive irreducible Lie superalgebra* $G=\bigoplus_{i\geqq -1}G_i$ *with consistent* $\mathbb{Z}$*-grading such that* $G_1\neq 0$ *is isomorphic as a* $\mathbb{Z}$*-graded superalgebra to one of the following* $\mathbb{Z}$*-graded Lie superalgebras*

I. $\boldsymbol{A}(m,n)$, $\boldsymbol{C}(n)$, $\boldsymbol{P}(n)$;

II. $\boldsymbol{W}(n)$, $\boldsymbol{S}(n)$, $\boldsymbol{H}(n)$, $\tilde{\boldsymbol{H}}(n)$;

III. H^ξ, *H being a simple Lie algebra;*

IV. G^z, *G being a Lie superalgebra of type I-III,* G_0 *having trivial centre.*

3. The Following Theorem is the Main Result of the Paper.

Theorem 5. *A simple finite-dimensional Lie superalgebra* $G=G_{\bar{0}}\oplus G_{\bar{1}}$, $G_{\bar{1}}\neq 0$, *over an algebraically closed field K of zero characteristic is isomorphic to one of* *the following:* $\boldsymbol{A}(m,n)$, $\boldsymbol{B}(m,n)$, $\boldsymbol{C}(n)$, $\boldsymbol{D}(m,n)$, $\boldsymbol{D}(2,1;\alpha)$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$, $\boldsymbol{P}(n)$, $\boldsymbol{Q}(n)$, $\boldsymbol{W}(n)$, $\boldsymbol{S}(n)$, $\tilde{\boldsymbol{S}}(n)$, $\boldsymbol{H}(n)$.

Proof. Assume that $L=L_{\bar{0}}\oplus L_{\bar{1}}$ is a simple finite-dimensional Lie superalgebra over K, and that the representation of $L_{\bar{0}}$ in $L_{\bar{1}}$ is irreducible. Then by the Theorem 2, L is one of the following: $\boldsymbol{B}(m,n)$, $\boldsymbol{D}(m,n)$, $\boldsymbol{D}(2,1;\alpha)$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$, $\boldsymbol{Q}(n)$.

If, on the contrary, the representation of $L_{\bar{0}}$ in $L_{\bar{1}}$ is reducible then by Proposition 2.4. there is a filtration $L=L_{-1}\supset L_0\supset\dots$ such that $\operatorname{Gr}L=\bigoplus_{i\geqq -1}\operatorname{Gr}_i L$ satisfies Theorem 4. Therefore $\operatorname{Gr}L$ could be isomorphic only to one of the Lie superalgebras of the mentioned type I – IV. Evidently if $\operatorname{Gr}L=H^{\xi}$, then L is not simple, so type III is impossible.

Proposition 2.3 yields $L\simeq\operatorname{Gr}L$ if the centre of $\operatorname{Gr}_0 L$ is not trivial. Hence type IV is impossible as all Lie superalgebras of this kind are not simple. If $\operatorname{Gr}L=\boldsymbol{W}(n)$ then by Proposition 2.3 $L\simeq\boldsymbol{W}(n)$.

If $\operatorname{Gr}L\simeq\boldsymbol{A}(m,n)$, $\boldsymbol{C}(n)$ or $\boldsymbol{P}(n)$ then the representation of $L_{\bar{0}}$ in $L_{\bar{1}}$ is the same as the one of $\operatorname{Gr}L$. Thus, Proposition 3.2d) yields $L\simeq\operatorname{Gr}L$, i.e. L is of the type I.

If $\operatorname{Gr}L\simeq\boldsymbol{S}(n)$, $\boldsymbol{H}(n)$ or $\tilde{\boldsymbol{H}}(n)$ then due to Proposition 6.7 either $L\simeq\operatorname{Gr}L$ or $L\simeq\tilde{\boldsymbol{S}}(n)$. Hence, the $\tilde{\boldsymbol{H}}(n)$ case is impossible for it is not simple and the only possibilities left are $\boldsymbol{S}(n)$, $\tilde{\boldsymbol{S}}(n)$, $\boldsymbol{H}(n)$. The theorem is proved.

Chapter II. Further Development of the Theory

§ 1. The Description of Semisimple Lie Superalgebras via Simple Ones

1. Definitions. Let $A=A_{\bar{0}}\oplus A_{\bar{1}}$ be a superalgebra, $\operatorname{der}A$ be the Lie superalgebra of derivations of A, and let $L\subset\operatorname{der}A$ be some subset. Lie superalgebra is called *L-simple* if A does not contain nontrivial ideals which are invariant under all derivations from L. If a superalgebra A is $\operatorname{der}A$-simple and $A^2\neq 0$ then A is called a *differentiable simple superalgebra.*

Denote operators l_s and r_s on A, $s\in A$ via formulae

$$l_s(a)=sa\,,\qquad r_s(a)=(-1)^{(\deg a)(\deg s)}as\,.$$

It is easily seen that if $D\in\operatorname{der}A$ then

$$[D,l_s]=l_{D(s)}\,,\qquad [D,r_s]=r_{D(s)}\,.$$

Denote by $T(A)$ the associative superalgebra spanned by all l_s and r_s, $s\in A$. $T(A)$ is a subalgebra of $L(A)$, $L(A)$ being the superalgebra of all endomorphisms of A.

2. Differentiably Simple Superalgebras. Arguments of paper [21] if repeated literally with the substitution of the corresponding definitions by the definitions of item 1 above give the following proposition.

Proposition 1.1. *Let G be finite-dimensional differentiably simple superalgebra. Then $G\simeq S\otimes\Lambda(n)$, where S is simple superalgebra and $\Lambda(n)$ is the Grassmann superalgebra.*

3. Description of Semisimple Lie Superalgebras. Recall, that a Lie superalgebra is called semisimple if $A^2 \neq 0$ and A does not contain nontrivial solvable ideals. In [21] from the description of differentiably simple algebras follows the description of semisimple Lie algebras over an arbitrary field. The same reasoning holds for Lie superalgebras.

Denote by $\operatorname{inder} S$ the set of inner derivations of a superalgebra S.

Theorem 6. *Let $S_1, \ldots, S_r$ be finite-dimensional simple Lie superalgebras, let $n_1, \ldots, n_r$ be whole positive numbers, and let*

$$S = \bigoplus_{i=1}^{r} S_i \otimes \Lambda(n_i).$$

Then

$$\begin{aligned} S = \operatorname{inder} S &= \bigoplus_{i=1}^{r} (\operatorname{inder} S) \otimes \Lambda(n_i) \leqq \operatorname{der} S \\ &= \bigoplus_{i=1}^{r} ((\operatorname{der} S) \otimes \Lambda(n_i) \oplus 1 \otimes \operatorname{der} \Lambda(n_i)). \end{aligned}$$

Let L be a subalgebra in $\operatorname{der} S$ containing S. Denote by L_i the projection of L on $1 \otimes \operatorname{der} \Lambda(n_i)$. Then

a) *L is a semisimple Lie superalgebra iff the superalgebra $\Lambda(n_i)$ is L-simple for each i.*

b) *All finite dimensional semisimple Lie superalgebras arise in the way described above.*

c) *The superalgebra* $\operatorname{der} L$ *coincides with the normalizer of* L in $\operatorname{der} S$ *provided L is semisimple.*

Example. Let $S_1, \ldots, S_r$ be simple Lie algebras, V a linear space with basis $e_1, \ldots, e_r$, $L \subset V^*$ a subspace, not belonging to any kernel of e_i, $i = 1, \ldots, r$. Define Lie superalgebra with agreed $\mathbb{Z}$-graduation by setting

$$G(S_1, \ldots, S_r; L) = G_{-1} \oplus G_0 \oplus G_1 = L \oplus \left(\bigoplus_{i=1}^{r} S_i \right) \oplus \left(\bigoplus_{i=1}^{r} e_i S_i \right),$$

where $[G_{-1}, G_{-1}] = [G_1, G_1] = [G_0, G_{-1}] = 0$, and the other brackets are: $[\sum s_i, \sum e_i s_i'] = \sum e_i [s_i, s_i']$, $[v^*, \sum e_i s_i] = \sum v^*(e_i) s_i$. Then $G(S_1, \ldots, S_r; L)$ can not be expanded into a direct sum and is a semisimple (but not simple) Lie superalgebra.

4. Description of Lie Superalgebras $G = G_{\bar{0}} \oplus G_{\bar{1}}$, for which the Representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ is Completely Reducible. Let a Lie superalgebra $\tilde{G} = G \oplus V$ be a semidirect sum of an ideal G and an odd commutative subalgebra $V \neq 0$, and let $[G_{\bar{1}}, V] = 0$; then $\tilde{G}$ will be called an elementary extension of G.

It follows from Theorem 5 and 6 the

Proposition 1.2. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra, with semisimple $G_{\bar{0}}$. Then G is an elementary extension of a direct sum of the following Lie superalgebras:*

simple Lie algebras or one of Lie superalgebras $\boldsymbol{A}(n, n)$, $\boldsymbol{B}(m, n)$, $\boldsymbol{D}(m, n)$, $\boldsymbol{D}(2, 1; \alpha)$, $\boldsymbol{F}(4)$, $\boldsymbol{G}(3)$, $\boldsymbol{P}(n)$, $\boldsymbol{Q}(n)$, $\mathrm{der}\,\boldsymbol{Q}(n)$ *or* $G(S_1,\ldots,S_r; L)$.

The description of Lie superalgebras with completely reducible $G_{\bar{0}}$ is analogous (though more complicated).

§2. Irreducible Finite Dimensional Representations of Lie Superalgebras

1. Induced Modules. Let G be a Lie superalgebra with universal enveloping $U(G)$. Let H be subalgebra of G, let V be an H-module. Note that V is also a $U(H)$-module. Consider the $\mathbb{Z}_2$-graded space $U(G)\otimes_{U(H)} V$, i.e. the factor space of $U(G)\otimes H$ by the $\mathbb{Z}_2$-graded subspace spanned by $gh\otimes v - g\otimes h(v)$, $g\in U(G)$, $h\in U(H)$. The space $U(G)\otimes_{U(H)} V$ is a G-module, $g(u\otimes v)=g(u)\otimes v$, $g\in G$, $u\in U(G)$, $v\in V$. Denote $\mathrm{Ind}_H^G V = U(G)\otimes_{U(H)} V$ and call it the G-module *induced from H-module V.*

The following simple properties of induced modules follow from the Poincaré-Birchoff-Witt theorem.

Proposition 2.1. a) *Let G be a Lie superalgebra with subalgebra H, let V be a simple G-module, W be an H-submodule of the H-module V. Then V is a quotient of* $\mathrm{Ind}_H^G V$.

b) *If $H_2 \subset H_1 \subset G$ are Lie superalgebras, and W is an H_2-module then* $\mathrm{Ind}_{H_1}^G \cdot (\mathrm{Ind}_{H_2}^{H_1} W) = \mathrm{Ind}_{H_2}^G W$.

c) *Let $H\subset G$ be Lie superalgebras, $G_0 \subset H$, $g_1,\ldots,g_t$ be odd elements of G such that their images under projection onto G/H is a basis and let W be an H-module. Then*

$$\mathrm{Ind}_H^G W = \bigoplus_{1\leqq i_1<\ldots<i_s\leqq t} g_{i_1}\ldots g_{i_s} W$$

is a direct sum of subspaces, therefore $\dim \mathrm{Ind}_H^G W = 2^t \dim W$.

Proposition 2.1c) combined with the Ado theorem for Lie algebras imply the following theorem.

Ado Theorem. *Any finite dimensional Lie superalgebra admits a faithful finite-dimensional representation.*

2. Representations of Solvable Lie Superalgebras. Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra. Call a linear form $l\in G^*$ *distinguished* if $l([G_{\bar{0}}, G_{\bar{0}}]) = l(G_{\bar{1}}) = 0$. Denote by $\mathscr{L}$ the space of distinguished linear forms, $\mathscr{L}_0$ the space of linear forms l satisfying $l([G, G]) = l(G_{\bar{1}}) = 0$ and by $\mathscr{L}_1$ the subgroup of the additive group G^* spanned by linear forms which give 1-dimensional quotients of the adjoint representation of G. It is evident that $\mathscr{L}\supset\mathscr{L}_0\supset\mathscr{L}_1$.

Let ϱ be a representation of the Lie superalgebra G in V and let $\mathscr{M}$ be a subgroup in $\mathscr{L}_0$. Define the representation $\tilde{\varrho}$ via $\tilde{\varrho}(g)v = \varrho(g)v + \lambda(g)v$. Representations ϱ and $\tilde{\varrho}$ are called $\mathscr{M}$-equivalent, if $\lambda\in\mathscr{M}$. Let $l\in\mathscr{L}$ a be distinguished linear form considered mod $\mathscr{L}_0$. Set $G_l = \{g\in G | l([g, g_1]) = 0$ for any $g_1 \in G\}$. Obviously, G is a subalgebra of G containing $G_{\bar{0}}$ and $l([G_l, G_l]) = 0$. $P\subset G$ is called a *subalgebra submitted to the linear form l* if $G_l \subset P$ and $l([P, P]) = 0$. These definitions are correct.

We pick out the important class of solvable Lie superalgebras, namely completely solvable Lie superalgebras with all irreducible quotients of the adjoint representation being 1-dimensional. By the Engel theorem a nilpotent Lie superalgebra is completely solvable and $\mathscr{L}_1 = 0$. Denote by $\{H, l\}$ the 1-dimensional

H-module defined by the linear form $l \in \mathscr{L}_0$ through the formula $h(v) = l(h)v$. Now we are able to describe finite-dimensional irreducible representations of solvable Lie superalgebras.

Theorem 7. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a solvable Lie superalgebra.*

a) *If V is an irreducible finite-dimensional G-module then all irreducible quotients of the $G_{\bar{0}}$-module V are 1-dimensional and the corresponding linear forms (extended by zero on $G_{\bar{1}}$) belong to the same $\bar{l}_V \in \mathscr{L}/\mathscr{L}_0$.*

b) *Let $\bar{l} \in \mathscr{L}/\mathscr{L}_0$, P be the maximal subalgebra submitted to $\bar{l}$, $\{P, \bar{l}\}$ be the 1-dimensional submodule defined by $l \in \bar{l}$. Then the G-module $V = \operatorname{Ind}_P^G\{P, \bar{l}\}$ is finite-dimensional and irreducible, and $\bar{l}_V = \bar{l}$. Such G-modules V_1 and V_2 are $\mathscr{L}_0$-equivalent iff $\bar{l}_1 = \bar{l}_2$.*

c) *Each finite-dimensional irreducible G-module V is isomorphic to one of the modules $\operatorname{Ind}_P^G\{P, l\}$ with $l \in \bar{l}_V$, P being a maximal subalgebra submitted to l.*

d) *If G is completely solvable then it is possible to replace $\mathscr{L}_0$ by $\mathscr{L}_1$ in the above propositions. In particular, when G is nilpotent we have the bijection $V \mapsto \bar{l}_V$ of the set of classes of isomorphic finite-dimensional irreducible G-modules and the set $\mathscr{L}$ of distinguished linear forms.*

The following two propositions are corollaries of Theorem 7.

Proposition 2.3. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be the space of irreducible finite-dimensional representations of the solvable Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$. Then either* $\dim V_{\bar{0}} = \dim V_{\bar{1}}$ *and* $\dim V = 2^s$, $0 < s \leqq \dim G_{\bar{1}}$, *or* $\dim V = 1$.

Proposition 2.4. *All irreducible representations of the solvable Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ are 1-dimensional iff $[G_{\bar{1}}, G_{\bar{1}}] \subset [G_{\bar{0}}, G_{\bar{0}}]$.*

Example 1. It follows from Theorem 7 that the families of representations ϱ_α and ϱ'_α of Geisenberg superalgebras of Section I.1.6. contain all their nontrivial finite-dimensional irreducible representations once each.

Example 2. $G = l(1, 1)$ is a completely solvable Lie superalgebra with basis

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The family of irreducible representations of dimension not equal to 1 are parametered by $\alpha = l(h)$ and $\beta = l(z) \neq 0$; $P = \langle z, h, e \rangle$:

$$z \mapsto \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad h \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha - 1 \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

When $\beta = 0$, we obtain the 1-dimensional representation $h \mapsto \alpha$; $z, e, f \mapsto 0$.

Remark. It is possible also to give a classification of infinite-dimensional representations of completely solvable Lie superalgebra, more exactly the primary ideals of the enveloping superalgebra. There is a $1-1$ correspondence between

$\mathscr{L}_0$-equivalent primary ideals and the set of $\mathscr{I}$-orbits in $\tilde{\mathscr{L}}/\mathscr{L}_0$, where $\mathscr{I}$ is the closure of $\operatorname{Aut} G_{\bar{0}}$ in the Zarisky topology of $GL(G_{\bar{0}})$, $\tilde{\mathscr{L}}=\{l\in G^* | l(G_1)=0\}$.

3. Representations of Simple Lie Superalgebras

Theorem 8. *Let G be one of the contragredient Lie superalgebras (enumerated in Table 5), and V let be a finite dimensional G-module. Then there exists (unique up to multiplication by a scalar) a vector $v_\Lambda \in V$, $\Lambda\in H^*$, such that $e_i(v_\Lambda)=0$, $h_i(v_\Lambda)=\Lambda(h_i)v_\Lambda$. Two G-modules V_1 and V_2 are isomorphic iff $\Lambda_1=\Lambda_2$. The set of numbers $a_i=\Lambda(h_i)\in K$, $i=1,\dots,r$, of the highest weight Λ of an irreducible finite-dimensional module are described by the following conditions (we consider $a_{s,s+1}=1$, if $a_{ss}=0$):*

1) *$a_i\in\mathbb{Z}_+$ if $i\neq s$;*
2) *$k\in\mathbb{Z}_+$, where k is done in Table 6.*

Table 6

G	k	b
$\boldsymbol{B}(0,n)$	$1/2a_n$	0
$\boldsymbol{B}(m,n), m>0$	$a_n-a_{n+1}-\dots-a_{m+n-1}-1/2a_{m+n}$	m
$\boldsymbol{D}(m,n)$	$a_n-a_{n+1}-\dots-1/2(a_{m+n-1}+a_{m+n})$	m
$\boldsymbol{D}(2,1;\alpha)$	$(1+\alpha)^{-1}(2a_1-a_2-\alpha a_3)$	2
$\boldsymbol{F}(4)$	$1/3(2a_1-3a_2-4a_3-2a_4)$	4
$\boldsymbol{G}(3)$	$1/2(a_1-2a_2-3a_3)$	3

3) *if $k<b$ (from the table) then there are additional conditions:*
$\boldsymbol{B}(m,n)$ $a_{n+k+1}=\dots=a_{m+n}=0$,
$\boldsymbol{D}(m,n)$ $a_{n+k+1}=\dots=a_{m+n}=0$ *if* $k\leqq m-2$; $a_{m+n-1}=a_{m+n}$ *if* $k=m-1$,
$\boldsymbol{D}(2,1;\alpha)$ $a_i=0$ *if* $k=0$; $(a_3+1)\alpha=\pm(a_2+1)$ *if* $k=1$,
$\boldsymbol{F}(4)$ $a_i=0$ *if* $k=0$; $k\neq 1$; $a_2=a_4=0$ *if* $k=2$; $a_2=2a_4+1$ *if* $k=3$,
$\boldsymbol{G}(3)$ $a_i=0$ *if* $k=0$; $k\neq 1$; $a_2=0$ *if* $k=2$.

Remark. The analogous theorem is valid for all simple Lie superalgebras (cf. [7]).

Now let G be a simple finite-dimensional contragredient Lie superalgebra, and (,) be some non-degenerate invariant bilinear form in G. Let ϱ be a difference between the halfsum of positive even roots and the halfsum of positive odd roots; then $\varrho(h_{\alpha_i})=(\alpha_i,\alpha_i)/2$. Define the Kasimir's operator (from the center of enveloping superalgebra):

$$\Gamma=\sum(-1)^{\deg u_i}u_iu^i \tag{2.1}$$

where $\{u_i\}$ and $\{u^i\}$ – dual bases in G.

Let V be a finite-dimensional irreducible G-module with the highest weight Λ. Then

$$\Gamma(u)=(\Lambda,\Lambda+2\varrho)u\,,\qquad u\in V\,. \tag{2.2}$$

The form of the supertrace is defined as usual: $(a,b)_V=\operatorname{str} ab$. Then

$$(a,b)_V=l_V(a,b)\,,\quad\text{where}\quad l_V\in K\,.$$

The two ways of computing $\operatorname{str}\Gamma$, using (2.1) and (2.2) give:

$$l_V(\dim G_{\bar{0}} - \dim G_{\bar{1}}) = (\dim V_{\bar{0}} - \dim V_{\bar{1}})(\Lambda, \Lambda + 2\varrho).$$

From this formula it follows that the form of the supertrace is nondegenerate iff (under the condition $\dim G_{\bar{0}} \neq \dim G_{\bar{1}}$)

$$\dim V_{\bar{0}} \neq \dim V_{\bar{1}} \quad \text{and} \quad (\Lambda, \Lambda + 2\varrho) \neq 0.$$

§ 3. Classification of Simple Finite Dimensional Real Lie Superalgebras

At first, we introduce some examples. Let $\mathbb{R} \subset \boldsymbol{C} \subset \mathbb{H}$ be the usual inclusions. Henceforth let $i = \sqrt{-1}$, let stroke be the usual conjugation in $\boldsymbol{C}$ or $\mathbb{H}$, and let $\top$ denote the matrix transposition.

a) The special linear Lie superalgebra $\operatorname{sl}(m, n; k)$, $k = \mathbb{R}, \boldsymbol{C}, \mathbb{H}$. Let $l(m, n; k)$ the space of all matrices of order $m+n$ over k,

$$l(m, n; k)_{\bar{0}} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right\}, \qquad l(m, n; k)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \right\},$$

α being an $m \times m$ matrix, etc. The brackets are defined as usual.

Define the special linear superalgebra $\operatorname{sl}(m, n; k)$ to be the real subalgebra of $l(m, n; k)$ defined as follows:

$$\operatorname{sl}(m, n; k) = \{a \in l(m, n; k) \mid \operatorname{str}(a) = 0\} \quad \text{for} \quad k = \mathbb{R} \quad \text{or} \quad \boldsymbol{C}$$

$$\operatorname{sl}(m, n; \mathbb{H}) = \{a \in l(m, n; \mathbb{H}) \mid \operatorname{Re} \operatorname{str}(a) = 0\}$$

where str is for the supertrace.

Let us introduce the following matrices of order $m+n$:

$$S_{p,q} = \left[\begin{array}{cc|cc} 1_p & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -1_{m-p} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1_q & 0 \\ & & 0 & -1_{n-q} \end{array}\right] \qquad T_p = \left[\begin{array}{cc|cc} i1_p & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & i1_{m-p} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1_r \\ & & -1_r & 0 \end{array}\right]$$

$$R_p = \left[\begin{array}{c|cc} i1_m & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 1_p & 0 \\ & 0 & -1_{n-p} \end{array}\right] \qquad I = \left[\begin{array}{c|c} -i1_m & 0 \\ \hline 0 & i1_n \end{array}\right]$$

b) The special unitary Lie superalgebras $\operatorname{su}(m, n; p, q)$:

$$\operatorname{su}(m, n; p, q)_s = \{a \in \operatorname{sl}(m, n; \mathbb{C})_s \mid S_{p,q}^{-1} \bar{a}^{\top} S_{p,q} = -i^s a\}$$

c) Orthosymplectic $\operatorname{osp}(m, n; p; \mathbb{R})$:

$$\operatorname{osp}(m, n; p; \mathbb{R})_s = \{a \in \operatorname{sl}(m, n; \mathbb{R})_s \mid T_p^{-1} a^{\top} T_p = -i^s a\}$$

d) Quaternionian orthosymplectic hosp $(m, n; p)$:

$$\operatorname{hosp}(m, n; p)_s = \{a \in \operatorname{sl}(m, n; \mathbb{H})_s \mid R_p^{-1} \bar{a}^{\top} R_p = -i^s a\}$$

e) $\boldsymbol{D}(2,1;\alpha;p)$. For every $p=0,1,2$ there is representation of

$$\mathrm{so}(4,4-p;\mathbb{R})\otimes\mathrm{sl}(2,\mathbb{R}) \quad \text{in} \quad \boldsymbol{D}(2,1;\alpha)_1$$

which defines forms of $\boldsymbol{D}(2,1;\alpha)$.

f) $\boldsymbol{F}(4;p)$. Each algebra $\mathrm{so}(p,7-p;\mathbb{R}), p=0,1,2,3$, has spinor representation $\mathrm{spin}_{p,7-p}$ that is real form of the B_3-module spin_7. For each of the four Lie algebras $\mathrm{spin}_{p,7-p}\otimes\mathrm{sl}_2$ there is a unique algebra $\boldsymbol{F}(4;p), p=0,1,2,3$

g) $\boldsymbol{G}(3;p)$. The standard representation of the real form $G_{2,p}, p=0,1$ of the complex Lie algebra G_2 is 7-dimensional. Therefore by Proposition 3.2b) there is the unique real Lie superalgebra $\boldsymbol{G}(3;p), p=0,1$, which is a real form of $\boldsymbol{G}(3)$.

h) $\boldsymbol{UQ}(n;p)$. Let $G_{\bar 0}$ and $G_{\bar 1}$ be duplicates of $su(p,n+1-p)$. Define a Lie superalgebra structure on $\boldsymbol{UQ}(n,p)=G_{\bar 0}\oplus G_{\bar 1}$ by

$$[a_1,a_2]=a_1a_2-a_2a_1,\, a_{\bar 1}\in G_{\bar 0},\, a_2\in G_s,\, s\in\mathbb{Z}_2$$

$$[b_1,b_2]=i(b_1b_2+b_2b_1)-2i/(n+1)\cdot\mathrm{tr}(b_1b_2)1_{n+1},\, b_1,b_2\in G_{\bar 1}$$

i) $\boldsymbol{HQ}(n)=G_{\bar 0}\oplus G_{\bar 1}$, the latter being duplicates of $\mathrm{sl}(n+1,\mathbb{H})(=\mathrm{sl}(n+1,0;\mathbb{H}))$. The structure of a Lie superalgebra is defined on $\boldsymbol{HQ}(n)$ by formulae

$$[a_1,a_2]=a_1a_2-a_2a_1,\, a_1\in G_{\bar 0},\, a_2\in G_s,\, s\in\mathbb{Z}_2$$

$$[b_1,b_2]=b_1b_2+b_2b_1-2/(n+1)\cdot\mathrm{Re}\,\mathrm{tr}(b_1b_2)1_{n+1},\, b_i\in G_{\bar 1}$$

k) definition of $\boldsymbol{P}(n;\mathbb{R}), \boldsymbol{Q}(n;\mathbb{R}), \boldsymbol{W}(n;\mathbb{R}), \boldsymbol{S}(n;\mathbb{R})$ and $\tilde{\boldsymbol{S}}(n;\mathbb{R})$ is evident.

1) $\boldsymbol{H}(n;p)=\{D\in\boldsymbol{W}(n;\mathbb{R})|D(\delta\xi_1+\ldots+\delta\xi_p-\delta\xi_{p+1}-\ldots-\delta\xi_n)=0\}$.

Assume that $G=G_{\bar 0}\oplus G_{\bar 1}$ is a real Lie superalgebra. There corresponds a Lie superalgebra G' that is the form of $G\otimes\mathbb{C}$. We put $[a,b]'=-[a,b]$ for $a,b\in G_{\bar 1}$ and $[a,b]'=[a,b]$ in other cases. Lie superalgebras G and G' are called dual.

The classification of simple real Lie algebras implies the following theorem.

Theorem 9. A simple finite dimensional real Lie superalgebra is isomorphic up to duality either to one of complex simple Lie superalgebra considered as real (of double dimension) or to one of following ones:

A) $\mathrm{sl}(m,n;\mathbb{R}), \mathrm{su}(m,n;p,q)$ for $m,n\geqq 1, n+m\geqq 2, \mathrm{sl}(m,n;\mathbb{H})$ for $m,n\geqq 1$ (when $m=n$ algebras of this series have 1-dimensional centre, we must pass to the quotient); $\boldsymbol{H}(4,p;\mathbb{R})$
B) $\mathrm{osp}(m,n;p;\mathbb{R})$ for m being odd, $m\geqq 1, n\geqq 2$
C) $\mathrm{osp}(2,n;p;\mathbb{R}), \mathrm{hosp}(1,n;p)$ for $n\geqq 2$
D) $\mathrm{osp}(m,n;p;\mathbb{R})$ for m being even, $m\geqq 4, n\geqq 2$ $\mathrm{hosp}(m,n;p)$ for $m\geqq 2, \boldsymbol{D}(2,1;\alpha;p)$ for $p=0,1,2$
F) $\boldsymbol{F}(4,p)$ for $p=0,1,2,3$
G) $\boldsymbol{G}(3,p)$ for $p=0,1$
P, Q) $\boldsymbol{P}(n;\mathbb{R})$ for $n\geqq 3$; $\boldsymbol{Q}(n;\mathbb{R}), \boldsymbol{UQ}(n;p)$ for $n\geqq 3$; $\boldsymbol{HQ}(n)$ for $n\geqq 2$
W, S, H) $\boldsymbol{W}(n;\mathbb{R}), n\geqq 3$; $\boldsymbol{S}(n;\mathbb{R}), \tilde{\boldsymbol{S}}(n;\mathbb{R}), n\geqq 4$; $\boldsymbol{H}(n;p;\mathbb{R}), n\geqq 5$.

§4. On the Classification of Infinite Dimensional Primitive Lie Superalgebras

A Lie superalgebra L with maximal subalgebra L_0 is called *primitive*, if L_0 contains no nonzero ideals of L.

In this section some partial results are given on classification of infinite-dimensional primitive Lie superalgebras.

1. Two Algebras of Differential Forms. Denote by $\Omega(m)$ the superalgebra of differential forms with coefficients in the polynomial ring $K[x_1, \dots, x_m]$, i.e. $\Omega(m)$ is the associative superalgebra over $K[x_1, \dots, x_m]$ with generators $dx_1, \dots, dx_m$, defining relations being $dx_i \wedge dx_j = -dx_j \wedge dx_i$, $\deg dx_i = \bar{1}$, $i, j = 1, \dots, m$.

On $\Omega(m)$, there is a well known differential of degree $\bar{1}, d{:}d(x_i) = dx_i, d^2 = 0$.

Put $\Delta(m, n) = \Omega(m) \otimes \Delta(n)$ and $\Theta(m, n) = \Omega(m) \otimes \Theta(n)$. Differentials δ and θ could be extended to $\Delta(m, n)$ and $\Theta(m, n)$ via $\delta = d \otimes 1 + 1 \otimes \delta, \theta = d \otimes 1 + 1 \otimes \theta$.

The properties of δ and θ analogous to those of Section 6.2 and are easy to prove. Put $\Lambda(m, n) = K[x_1, \dots, x_m] \otimes \Lambda(n)$. If we put deg $x_i = \deg \xi_i = 1$, $\Lambda(m, n)$ becomes $\mathbb{Z}$-graded (the grading does not agree with $\mathbb{Z}_2$-grading).

Any derivation D of degree s of $\Lambda(m, n)$ is uniquely extended to a derivation of $\Delta(m, n)$ or $\Theta(m, n)$ by the formula $[D, \theta]f = [D, \delta]f = 0$, $f \in \Lambda(m, n)$.

2. Six Series of Infinite-dimensional Lie Superalgebras. We introduce the following forms:

$$v = dx_1 \wedge \dots \wedge dx_m \wedge \theta\xi_1 \dots \wedge \theta\xi_n \in \Theta(m, n),$$

$$h = 2 \sum_{i=1}^{k} dx_i \wedge dx_{k+i} + \sum_{i=1}^{n} (\delta\xi_i)^2 \in \Delta(m, n), m = 2k,$$

$$k = dx_{2k+1} + \sum_{i=1}^{k} (x_i dx_{k+i} - x_{k+i} dx_i) + \sum_{i=1}^{n} \xi_i \delta\xi_i \in \Delta(m, n),$$

$$m = 2k + 1.$$

Denote six series of infinite dimensional Lie superalgebras ($m > 0$):

I. $\boldsymbol{W}(m, n) = \operatorname{der} \Lambda(m, n)$
II. $\boldsymbol{S}(m, n) = \{D \in \boldsymbol{W}(m, n) \mid Dv = 0\}$
II′. $\boldsymbol{CS}(m, n) = \{D \in \boldsymbol{W}(m, n) \mid Dv = \lambda v, \lambda \in K\}$
III. $\boldsymbol{H}(m, n) = \{D \in \boldsymbol{W}(m, n) \mid Dh = 0\}$
III′. $\boldsymbol{CH}(m, n) = \{D \in \boldsymbol{W}(m, n) \mid Dh = \lambda h, \lambda \in K\}$
IV. $\boldsymbol{K}(m, n) = \{D \in \boldsymbol{W}(m, n) \mid Dk = uk, u \in \Lambda(m, n)\}$.

Note, that for $m = 0$ the well-known Cartan series are obtained.

$\mathbb{Z}$-grading of $\Lambda(m, n)$ induces $\mathbb{Z}$-grading in $\boldsymbol{W}(m, n)$ (that does not agree). In more detail it could be described as follows. Each element $D \in \boldsymbol{W}(m, n)$ has the form

$$D = \sum_{i=1}^{m} P_i \partial/\partial x_i + \sum_{j=1}^{n} Q_j \partial/\partial \xi_j, P_i, Q_j \in \Lambda(m, n) \tag{4.1}$$

$\mathbb{Z}$-grading is defined by $\deg x_i = \deg \xi_i = 1, \deg \partial/\partial x_i = \deg \partial/\partial \xi_i = -1$, so $\boldsymbol{W}(m, n) = \bigoplus_{i \geqq -1} \boldsymbol{W}(m, n)_i$.

There is a canonical filtration corresponding to this grading with distinguished subalgebra $\bigoplus_{i \geqq 0} \boldsymbol{W}(m, n)_i$. The filtration and the distinguished subalgebra induce a filtration on each subalgebra $L \subset \boldsymbol{W}(m, n)$ and give a distinguished subalgebra $L_0 = L \cap \bigoplus_{i \geqq 0} \boldsymbol{W}(m, n)_i$.

The Lie superalgebra $\boldsymbol{S}(m, n)$ consists of all operators (4.1) that satisfy

$$\operatorname{div} D = \sum_{i=1}^{m} \partial P_i/\partial x_i - (-1)^{\deg D} \sum_{j=1}^{n} \partial Q_j/\partial \xi_j = 0 .$$

Hence, $\boldsymbol{S}(m, n)$ is the linear span of

$$\frac{\partial a}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial a}{\partial \xi_j}\frac{\partial}{\partial \xi_i}, \quad \frac{\partial a}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial}{\partial x_i},$$

$$\frac{\partial a}{\partial \xi_j}\frac{\partial}{\partial x_i} + (-1)^{\deg a}\frac{\partial a}{\partial x_i}\frac{\partial}{\partial \xi_j}, \quad a \in \Lambda(m, n) .$$

Lie superalgebra $\boldsymbol{H}(m, n)$ consists of the following elements

$$D_a = \sum_{i=1}^{n} \frac{\partial a}{\partial \xi_i}\frac{\partial}{\partial \xi_i} + \sum_{i=1}^{k} \left(\frac{\partial a}{\partial x_i}\frac{\partial}{\partial x_{k+i}} - \frac{\partial a}{\partial x_{k+i}}\frac{\partial}{\partial x_i}\right), \quad a \in \Lambda(m, n) .$$

$[D_a, D_b] = D_{\{a, b\}}$, where

$$\{a, b\} = (-1)^{\deg a} \sum_{i=1}^{n} \frac{\partial a}{\partial \xi_i}\frac{\partial b}{\partial \xi_i} + \sum_{i=1}^{k} \left(\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_{k+i}} - \frac{\partial a}{\partial x_{k+i}}\frac{\partial b}{\partial x_i}\right).$$

Furthermore, $\boldsymbol{CS}(m, n) = \boldsymbol{S}(m, n) \oplus \left\langle \sum_{i=1}^{n} \xi_i \partial/\partial \xi_i + \sum_{i=1}^{m} x_i \partial/\partial x_i \right\rangle$

$\boldsymbol{CH}(m, n) = \boldsymbol{H}(m, n) \oplus \langle \sum \xi_i \partial/\partial \xi_i + 2 \sum x_i \partial/\partial x_i \rangle$.

Finally, $\boldsymbol{K}(m, n)$ consists of the elements D'_a,

$$D'_a = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_m}\xi_i - \frac{\partial a}{\partial \xi_i}\right)\frac{\partial}{\partial \xi_i} + \sum_{i=1}^{k} \left(\left(\frac{\partial a}{\partial x_m}x_i + \frac{\partial a}{\partial x_{k+i}}\right)\frac{\partial}{\partial x_i}\right.$$

$$\left. + \left(\frac{\partial a}{\partial x_m}x_{k+i} - \frac{\partial a}{\partial x_i}\right)\frac{\partial}{\partial x_{k+i}}\right) + \left((-1)^{\deg a} 2a\right.$$

$$\left. + \sum_{i=1}^{n} \frac{\partial a}{\partial \xi_i}\xi_i - \sum_{i=1}^{m-1} \frac{\partial a}{\partial x_i}x_i\right)\frac{\partial}{\partial x_m}, \quad a \in \Lambda(m, n) .$$

$\mathbb{Z}$-grading of $\boldsymbol{W}(m, n)$ induces a $\mathbb{Z}$-grading of the form $G = \bigoplus_{i \geqq -1} G_i$. For Lie superalgebras of series I–III.

This is not so in the $\boldsymbol{K}(m, n)$ case. We obtain $\mathbb{Z}$-grading though if we put

$$\deg x_i = \deg \xi_j = -\deg \partial/\partial x_i = -\deg \partial/\partial \xi_j = 1 , \quad 1 \leqq i \leqq m-1 , \quad 1 \leqq j \leqq n ,$$

$$\deg x_m = -\deg \partial/\partial x_m = 2 .$$

Such $\mathbb{Z}$-grading of $\boldsymbol{W}(m, n)$ induces a $\mathbb{Z}$-grading on $\boldsymbol{K}(m, n)$ of the form: $G = \bigoplus_{i \geqq -2} G_i$.

Note, that the G_0-modules G_{-1} are isomorphic to the following standard representations of linear Lie superalgebras: $l(m, n)$ for $\boldsymbol{W}(m, n)$ and $\boldsymbol{CS}(m, n)$, $\mathrm{sl}(m, n)$ for $\boldsymbol{S}(m, n)$, $\mathrm{osp}(n, m)$ for $\boldsymbol{H}(m, n)$, $\mathrm{csp}(n, m)$ for $\boldsymbol{CH}(m, n)$, $\mathrm{csp}(n, m-1)$ for $\boldsymbol{K}(m, n)$.

3. On the Classification of Primitive Lie Superalgebras. Suppose that L is an infinite-dimensional primitive Lie superalgebra, L_0 being the distinguished subalgebra.

Let L_{-1} be the minimal ($\mathbb{Z}_2$-graded) subspace of L, which contains L_0, is not L_0 itself and is ad L_0-invariant.
$L=L_{-d}\supset L_{-d+1}\supset \dots \supset L_{-1}\supset L_0\supset \dots$ by putting (cf. [6]):

$$L_{-(s+1)}=[L_{-1}, L_s]+L_{-s}, L_s=\{a\in L_{s-1}|[a, L_{-1}]\subset L_{s-1}\}, s>0.$$

The corresponding associated $\mathbb{Z}$-graded Lie superalgebra $\operatorname{Gr}L=\bigoplus_{i\geqq -d} G_i$ has the following properties:

1°. $\operatorname{Gr}L$ is transitive and irreducible,

2°. $G_{-s}=G_{-1}^s$, $s>0$.

It is also possible to believe, that

3°. $\bigoplus_{i<0} G_i$ does not contain any nonzero ideals of Lie superalgebra $\operatorname{Gr}L$ (by taking a quotient if there is a nontrivial ideal). If $\mathbb{Z}$-grading does agree then the following holds:

4°. $[G_0, G_0]$ is a contragredient Lie superalgebra.

It seems that 4° holds in general case, too, but I can not prove it.

The remain result of this section is as follows.

Theorem 10. *Let* $G=\bigoplus_{i\geqq -d} G_i$ *be infinite-dimensional Lie superalgebra, satisfying* 1°–4°. *Then* G *is isomorphic as a* $\mathbb{Z}$*-graded superalgebra to one of* $W(m,n), \dots, K(m,n)$, $m>0$.

The proof uses the same methods as the proof of Theorem 4 and is based on Theorem 3.

A primitive Lie superalgebra L with marked subalgebra L_0 is called *complete* if L is complete in the topology defined by subspaces of transitive filtration of the pair (L, L_0) (see 1.2.3). Note that $\bar{\Lambda}(m,n)=K[[x_1, \dots, x_m]]\otimes\Lambda(m)$ is supplied by the topology defined by the filtration (and is complete in such a topology). Denote by $\bar{W}(m,n)$ the Lie superalgebra of all continuous derivations of $\bar{\Lambda}(m,n)$. $\bar{W}(m,n)$ is a complete primitive Lie superalgebra with the natural marked subalgebra. The Lie superalgebras $\bar{S}(m,n), \dots, \bar{K}(m,n)$ are complete and primitive, too.

A wellknown result by Cartan is that $\bar{W}(m,0), \dots, \bar{K}(m,0)$ exhaust all complete infinite-dimensional primitive Lie algebras.

Hypothesis 1. *Complete infinite-dimensional Lie superalgebra are isomorphic to one of* $\bar{W}(m,n), \dots, \bar{K}(m,n)$ *for* $m>0$.

§ 5. Some Unsolved Problems

1. Classification of Infinite-dimensional Primitive Lie Superalgebras. See hypothesis 1 and Theorem 10.

2. Characters and the Dimension Formula for Irreducible Representations. The most interesting case is the contragredient Lie superalgebra $G=G(A,\tau)$. Let V be a G-module with highest weight Λ, $V(\Lambda)=\bigoplus_{\lambda} V_\lambda$ be the weight decomposition relative to $H=\langle h_1, \dots, h_r\rangle$. Let

$$\operatorname{ch} V(\Lambda)=\sum_{\lambda}(\dim V_\lambda)e^\lambda, \operatorname{sch} V(\Lambda)=\sum_{\lambda}(-1)^{\deg\lambda}(\dim V_\lambda)e^\lambda.$$

Functions ch and sch are called the character and supercharacter of $V(\Lambda)$. The method of [14] fits only for $\boldsymbol{B}(0, n)$.

Let $\Delta_{\bar{0}}^{+}$ (resp. $\Delta_{\bar{1}}^{+}$) be a set of even (resp. odd) positive roots, $\bar{\Delta}_{\bar{0}}^{+} = \{\alpha \in \Delta_{\bar{0}}^{+} | \alpha/2 \notin \Delta_{\bar{1}}^{+}\}$, $d = \# \Delta_{\bar{1}}^{+}$.

Let ϱ_0 (resp. ϱ_1) be the halfsum of even (resp. odd) positive roots, $\varrho = \varrho_0 - \varrho_1$. For $w \in W$ (the Weil group of $G_{\bar{0}}$) let $\varepsilon(w) = (-1)^{l(w)}$, $\varepsilon'(w) = (-1)^{l'(w)}$, where $l(w)$ is the number of reflections $s_\alpha, \alpha \in \Delta_{\bar{0}}^{+}$, in some expression of $wl'(w)$—the number of those $s_\alpha, \alpha \in \bar{\Delta}_{\bar{0}}^{+}$. Then for $\boldsymbol{B}(0, n)$ we have:

$$\operatorname{ch} V(\Lambda) = \sum_{w} \varepsilon(w) e^{w(\Lambda + \varrho)} / \sum_{w} \varepsilon(w) e^{w(\varrho)},$$

$$\operatorname{sch} V(\Lambda) = \sum_{w} \varepsilon'(w) e^{w(\Lambda + \varrho)} / \sum_{w} \varepsilon'(w) e^{w(\varrho)},$$

$$\dim V(\Lambda) = 2^{d} \prod_{\alpha \in \Delta_{\bar{0}}^{+}} (\Lambda + \varrho, \alpha)/(\varrho_0, \alpha),$$

$$\dim V_{\bar{0}}(\Lambda) - \dim V_{\bar{1}}(\Lambda) = \prod_{\alpha \in \bar{\Delta}_{\bar{0}}^{+}} (\Lambda + \varrho, \alpha)/(\varrho_0, \alpha).$$

3. Cohomology. For definition of cohomology for Lie superalgebras see [17]. If V is a finite-dimensional irreducible G-module, Γ is the Kazimir operator (provided there is an invariant bilinear form) and $\Gamma(V) \neq 0$, then $H^n(G, V) = 0, n > 0$. This is equivalent to $(\Lambda, \Lambda + 2\varrho) \neq 0$ for the highest weight Λ and holds for every representation only in $\boldsymbol{B}(0, n)$ case. Furthermore, cohomology for simple finite-dimensional Lie superalgebras with trivial coefficients as well as cohomology for complete infinite-dimensional primitive Lie superalgebras are of interest.

The trivialness of $H^k(G, V)$ is in close connection to the complete reducibility and the Levi-Malcev theorems. $\mathrm{sl}(n, n)$, serves as counterexample for the Levi theorem. Its adjoint representation is the counter-example to complete reducibility. As was noted, representations of $\boldsymbol{B}(0, n)$ are completely reducible. It is not difficult to demonstrate that if G is a classical type Lie superalgebra then for any irreducible representation of G, save a finite number of them, $H^1(G, V) = 0$.

4. Infinite-dimensional Representations. Evidently, the Kirillov method of orbits is extended to Lie superalgebras (see Theorem 7). There is almost nothing known about infinite-dimensional representations of simple Lie superalgebras. The first step on the way is surely di-spin algebra $\boldsymbol{B}(0, 1)$.

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