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Spectra of Liouville Operators

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Abstract. Spectra of the generators of time translations ("Liouville operators") on representation spaces determined by thermodynamic equilibrium states are compared and their nature is investigated.

1. Introduction

For macroscopic systems the density of energy levels is approximately $(\Delta E)^N$ where ΔE is the energy above the groundstate and N the number of degrees of freedom. Because of this enormous growth of level density in the thermodynamic limit one often says that the energy spectrum becomes continuous in that limit.

It is the purpose of this paper to study spectral properties of relevant objects, that govern the dynamics of quantum systems. In the quantum theory of a finite number of particles the above mentioned questions are discussed in terms of the spectral properties of the Hamiltonian, i.e. the generator of time-translations, of the system. In a quantum mechanical treatment of a thermodynamic system, i.e. of a system consisting of an infinite number of particles in infinite space with a finite density, the generator of time-translations is not unambiguously defined, let alone its spectrum.

We shall assume that we have a C^*-algebra $\mathfrak{A}$ of quasi-local observables with local algebras isomorphic to $\mathscr{B}(\mathfrak{h}_V)$, i.e. the local algebras consist of all bounded operators on the Hilbert-space $\mathfrak{h}_V$ that is pertinent to the description of a quantum system inside a volume V. The dynamics is assumed to be given by a one-parameter group of automorphisms α_t of $\mathfrak{A}$ that admits of a K.M.S.-state and satisfies some regularity conditions to be specified in section 2b. As we shall see in section 2b these regularity conditions permit us to construct a separable C^*-algebra $\mathfrak{A}_0$ inside $\mathfrak{A}$, that is $\sigma(\mathfrak{A}, N)$ dense in $\mathfrak{A}$. (Here N is the set of locally normal states on $\mathfrak{A}$ and the $\sigma(\mathfrak{A}, N)$ topology on $\mathfrak{A}$ is the weak topology defined by N on $\mathfrak{A}$.) The construction of $\mathfrak{A}_0$ depends on α_t and is such that α_t acts strongly continuous on $\mathfrak{A}_0$, i.e. $\|\alpha_t(A)-A\| \xrightarrow[t\to 0]{} 0$ for $A \in \mathfrak{A}_0$.

For a quantum lattice system $\mathfrak{A}_0$ coincides with $\mathfrak{A}$ and the regularity conditions we impose imply then strong continuity of α_t and also that α_t is obtained as a limit of local automorphisms. Hence α_t, in the terminology of Powers and Sakai [1], is approximately inner. In this case, i.e. the case of a quantum lattice system, one might ask whether or not, for a one-parameter automorphism group that acts strongly continuous and is approximately inner, the requirement that it also admits of a K.M.S.-state is any further restriction at all on the automorphism group. That this is no further restriction follows from the work of Powers and Sakai [1] theorem 3.2. If one accepts their conjecture that every strongly continuous one-parameter automorphism group of a U.H.F. algebra is approximately inner one can prove the following statement: "A one-parameter group of automorphisms on a U.H.F. algebra admits of a K.M.S.-state if and only if it is strongly continuous". Indeed the if part follows from the conjecture and [1] theorem 3.2, whereas the only if part follows from the conjecture and the fact that every one-parameter automorphism group that admits of a K.M.S.-state on a simple and norm-separable C^*-algebra like a U.H.F. algebra, acts strongly continuous [2].

Let ω_1 and ω_2 be K.M.S.-states with respect to an evolution α_t of $\mathfrak{A}$ at inverse temperatures β_1 and β_2 (possibly $\beta_1=\beta_2$). Since every α_t K.M.S.-state on $\mathfrak{A}$ is invariant under the action of α_t we have generators H_{ω_1} and H_{ω_2} of the unitary groups $U_t^{\omega_1}$ and $U_t^{\omega_2}$ implementing α_t on the Gel'fand-Segal representation spaces $\mathfrak{h}_{\omega_1}$ and $\mathfrak{h}_{\omega_2}$. The question we are going to investigate in this paper is the relation between the spectra of H_{ω_1} and H_{ω_2}. We will find (theorem A) that, with our assumptions, the spectral sets of H_{ω_1} and H_{ω_2} coincide, as sets. Let us apply this to the situation where we have a non-primary K.M.S.-state ω at an inverse temperature β that admits a decomposition into extremal K.M.S.-states, i.e.

$$\omega=\int d\mu(\gamma)\omega_\gamma,$$

where ω_γ is extremal K.M.S..

Theorem A implies that the spectral sets of H_ω and H_{ω_γ} coincide as sets. These results generalize similar statements that could be made on the basis of work by Kastler [3] in the case where the automorphism group α_t acts strongly continuous on $\mathfrak{A}$. More detailed information is obtained from the pointwise comparison between these spectral sets. We shall prove (theorem B) that a discrete point, different from zero, in the spectrum of H_ω appears as a discrete point ("survives the decomposition") in the spectra of H_{ω_γ} for γ in some set with nonzero μ-measure. The statement is trivially true for the point zero in the spectrum of H_ω; the set for which the theorem is true has μ-measure one in this case.

Using results of Størmer [4] we are able to show that for a separating state (i.e. a state with the property that its cyclic vector is also separating for the von Neumann algebra on the representation space) that is invariant under the action of a one-parameter group of automorphisms (not necessarily the dynamical automorphism group), extremal invariance of the state and no discrete points except zero in the spectrum of the generator of the unitary group implementing the automorphism group, imply that the spectrum of the generator equals $\mathbb{R}^1$.

Applying the foregoing results to a state ω that is α_t K.M.S., where α_t in addition to satisfying the regularity criteria also acts asymptotically abelian on $\mathfrak{A}$, and where the state ω admits a decomposition into extremal α_t K.M.S.-states ω_γ, we obtain:

i) $\mathrm{Sp}\, H_\omega = \mathrm{Sp}\, H_{\omega_\gamma} = \mathbb{R}^1$;

ii) zero is the only discrete point in $\mathrm{Sp}\, H_\omega$ with multiplicity determined by the centre of $\pi_\omega(\mathfrak{A})''$.

These are precisely the spectral properties one encounters in the free Bose gas below the critical temperature, as is easily seen by direct verification from the results given in [5].

2. Definitions and Results

a) Results Pertaining to the Finite Quantum System

A K.M.S.-state ω at inverse temperature β with respect to a one-parameter group of automorphisms α_t of a C^*-algebra $\mathfrak{A}$ is defined by the following:

i) $t \to \omega(A\alpha_t B)$ is a continuous function of t; A, $B \in \mathfrak{A}$.

ii) $\int \omega(A\alpha_t B) f(t - i\beta) dt = \int \omega(\alpha_t(B)A) f(t) dt$ for f with Fourier-transform in D; $A, B \in \mathfrak{A}$.

For a while we shall concentrate on a quantum system in a box of volume V. As usual the algebra of observables is the algebra of all bounded operators on a separable, infinite dimensional Hilbert space $\mathfrak{h}_V$. This algebra is denoted by $\mathfrak{A}(V)$ or sometimes as $\mathscr{B}(\mathfrak{h}_V)$. We consider the usual faithful representation of $\mathscr{B}(\mathfrak{h}_V)$ on the Hilbert space of Hilbert-Schmidt operators $\mathfrak{h}_S$, i.e. $A \in \mathscr{B}(\mathfrak{h}_V) \to \pi(A) \in \mathscr{B}(\mathfrak{h}_S)$: $\pi(A)K = AK$, $\forall K \in \mathfrak{h}_S$. (We shall also consider $\pi'(A) \in \mathscr{B}(\mathfrak{h}_S)$ defined by $\pi'(A)K = KA$, $\forall K \in \mathfrak{h}_S$ and $A \in \mathscr{B}(\mathfrak{h}_V)$.) Suppose that the dynamics of the finite system under consideration is given by a Hamiltonian H on $\mathfrak{h}_V$, giving rise to a one-parameter unitary group $U_t = \exp iHt$ which induces the automorphism group α_t of $\mathscr{B}(\mathfrak{h}_V)$ given by $\alpha_t(A) = U_t A U_{-t}$, $A \in \mathscr{B}(\mathfrak{h}_V)$. On $\mathfrak{h}_S$ the automorphism α_t from $\mathscr{B}(\mathfrak{h}_V)$ is implemented by $W_t K = \pi(U_t) \cdot \pi'(U_{-t})K$, $K \in \mathfrak{h}_S$. Indeed $\pi(\alpha_t(A)) = W_t \pi(A) W_{-t}$, $\forall A \in \mathscr{B}(\mathfrak{h}_V)$. As one easily shows W_t is a strongly continuous one-parameter group of unitaries on $\mathfrak{h}_S$ (i.e. $(K_1, W_t K_2)_S = Tr(K_1^* W_t K_2)$ is a continuous function of t).

For every $f \in L^1(\mathbb{R})$ we define on $\mathscr{B}(\mathfrak{h}_V)$ the operator $\pi_1(f) : \pi_1(f)A = \int \alpha_t(A) f(t) dt$, where the right-hand side is obtained by the Riesz-theorem as the unique operator defined by: $\int (\phi, \alpha_t(A)\psi) f(t) dt = (\phi, \int \alpha_t(A) f(t) dt \psi)$, $\phi, \psi \in \mathfrak{h}_V$. We also define for every $f \in L^1(\mathbb{R})$ an operator $\pi_2(f)$ on $\mathfrak{h}_S$ by $\pi_2(f)K = \int W_t K f(t) dt$. Here the right-hand side exists as a Bochner-integral on $\mathfrak{h}_S$ due to the strong continuity of the group $\{W_t\}$ on $\mathfrak{h}_S$.

Giving $L^1(\mathbb{R})$ its usual algebraic structure (i.e. considering $L^1(\mathbb{R})$ as the convolution group algebra of the additive group of the real numbers) it can be shown that the map $f \in L^1(\mathbb{R}) \to \pi_i(f)$ is a continuous representation of $L^1(\mathbb{R})$ into the bounded linear operators on $\mathscr{B}(\mathfrak{h}_V)$ for $i = 1$ and on $\mathfrak{h}_S$ for $i = 2$.

Following Arveson [6] one can define a spectrum for the homomorphism $t \in \mathbb{R}^1 \to \alpha_t$ denoted by $\mathrm{Sp}\, \alpha$ which is defined as $\mathrm{Sp}\, \alpha = \mathrm{hull}\, \ker \pi_1$, where $\ker \pi_1$ denotes the kernel of the representation π_1 of $L^1(\mathbb{R})$. Similarly one defines a spectrum for the homomorphism $t \in \mathbb{R}^1 \to W_t$ denoted by $\mathrm{Sp}\, W$ as $\mathrm{Sp}\, W = \mathrm{hull}\, \ker \pi_2$.

It follows from this definition of spectrum that we have

$$\mathrm{Sp}\,\alpha=\{\gamma\in\mathbb{R}^1:\hat{f}(\gamma)=0\,\forall f\in\ker\pi_1\}$$

and similarly

$$\mathrm{Sp}\,W=\{\gamma\in\mathbb{R}^1:\hat{f}(\gamma)=0\,\forall f\in\ker\pi_2\}\,.$$

A more familiar notion of spectrum, when we talk about the unitary group W_t, is the spectrum of its generator $\hat{H}$. There is then the following well known [4] lemma:

$$\mathrm{Sp}\,W=\mathrm{Sp}\,\hat{H}\,.$$

The proof of this lemma is accomplished by realizing that for the matrix elements of $\pi_1(f)$ we have

$$(\phi,\pi_1(f)\psi)=\int\hat{f}(\lambda)d(E_\lambda\phi,\psi)\,,$$

where $\{E_\lambda\}$ is the spectral resolution of $\hat{H}$, and by realizing that the measure $(E_\lambda\phi,\psi)$ varies on exactly Sp W and is constant on $\mathbb{R}^1\backslash\mathrm{Sp}\,W$. Another lemma that holds is the following:

$$\mathrm{Sp}\,\alpha=\mathrm{Sp}\,W=\mathrm{Sp}\,\hat{H}\,.$$

Clearly this statement is proven as soon as we have established that $\ker\pi_1=\ker\pi_2$.

That this statement is indeed true can be seen as follows: suppose $f\in\ker\pi_1$, then $\pi_1(f)A=0\,\forall A\in\mathscr{B}(\mathfrak{h}_V)$; in particular this holds for all Hilbert-Schmidt operators and hence $\pi_1(f)K=0\,\forall K\in\mathfrak{h}_S$. This implies however that $\pi_2(f)K=0\,\forall K\in\mathfrak{h}_S$ (as vectors in $\mathfrak{h}_S$ this time). Therefore $\ker\pi_1\subseteq\ker\pi_2$. Suppose conversely that $f\in\ker\pi_2$, i.e. $\pi_2(f)K=0$ as vectors on $\mathfrak{h}_S$, for all $K\in\mathfrak{h}_S$. It then follows as a result of a simple computation that $\pi_1(f)K=0$, considered as an operator on $\mathfrak{h}_V$, $\forall K\in\mathfrak{h}_S$. All we need in order to conclude that $\ker\pi_1\supseteq\ker\pi_2$ and hence that $\ker\pi_1=\ker\pi_2$ is that $\pi_1(f)K=0\,\forall K\in\mathfrak{h}_S$ implies $\pi_1(f)A=0\,\forall A\in\mathscr{B}(\mathfrak{h}_V)$. The latter fact follows from proposition 1.4 [6] and the fact that the Hilbert-Schmidt operators are $\sigma(\mathscr{B}(\mathfrak{h}_V),\mathscr{B}(\mathfrak{h}_V)_*)$ dense in $\mathscr{B}(\mathfrak{h}_V)$.

Let ω be a K.M.S.-state on the algebra $\mathscr{B}(\mathfrak{h}_V)$. Then we know that the Hamiltonian H should be such that $\mathrm{Tr}\,(e^{-\beta H})<\infty$ [7]. The cyclic representation one considers is the one on $\mathfrak{h}_S$ with cyclic and separating vector $e^{-\frac{1}{2}\beta H}$ [8]. From what we have seen above we conclude that the evolution α_t gives rise to a spectrum that equals, as a set, the spectrum of the generator of time-translations on the representation space for every state that is a K.M.S.-state for the evolution α_t, regardless of β. From this it follows that Sp H_{ω_1} and Sp H_{ω_2} are equal as sets for two states that are α_t K.M.S. (of course at different temperatures).

From explicit construction of the representation for two α_t K.M.S.-states ω_1 and ω_2 [8] we know that Sp H_{ω_1} is identical with Sp H_{ω_2} because $H_{\omega_1}=H_{\omega_2}$.

b) Regularity Conditions on the Thermodynamical Evolution α_t

Most of the interesting thermodynamic evolutions are not strongly continuous, i.e. for the one-parameter group of automorphisms α_t we do not have $\|\alpha_t(A)-A\|\underset{t\to 0}{\longrightarrow}0$ for all A in the quasi-local algebra $\mathfrak{A}$. Rather than assuming strong continuity of α_t we shall assume regularity conditions on α_t that permit us to draw still a good deal of the conclusions concerning spectra, that we could have drawn if α_t were strongly continuous. Part of the regularity conditions resembles, as we shall see, the properties α_t would have if it were approximately inner [1].

Given a thermodynamical evolution α_t of the quasi-local algebra $\mathfrak{A}$ that satisfies the regularity conditions (to be specified later) we are able to construct a separable α_t-invariant sub C^*-algebra $\mathfrak{A}_0$ of $\mathfrak{A}$, that depends on α_t, with the properties that it is $\sigma(\mathfrak{A}, N)$ dense in $\mathfrak{A}$ and α_t acts strongly continuous on it. N denotes here the set of locally normal states on $\mathfrak{A}$.

Suppose we subdivide $\mathbb{R}^3$ into disjoint finite volumes $\{V_n, n\in\mathbb{N}\}$. Suppose V_1 and V_2 are two such volumes. The algebra of observables for the finite system in V_n is $\mathscr{B}(\mathfrak{h}_{V_n})$. Then we have $\mathscr{B}(\mathfrak{h}_V)\simeq\mathscr{B}(\mathfrak{h}_{V_1}\otimes\mathfrak{h}_{V_2})$, where $V=V_1\cup V_2$. $\mathscr{B}(\mathfrak{h}_{V_1})$ is considered as the sub-algebra $\mathscr{B}(\mathfrak{h}_{V_1})\otimes\mathbb{1}$ of $\mathscr{B}(\mathfrak{h}_{V_1}\otimes\mathfrak{h}_{V_2})$ whereas $\mathscr{B}(\mathfrak{h}_{V_2})$ is considered as the sub-algebra $\mathbb{1}\otimes\mathscr{B}(\mathfrak{h}_{V_2})$ of $\mathscr{B}(\mathfrak{h}_{V_1}\otimes\mathfrak{h}_{V_2})$. Due to isotony, the quasi-local algebra $\mathfrak{A}$ is given by

$$\mathfrak{A}=\overline{\bigcup_{V\subset\mathbb{R}^3}\mathscr{B}(\mathfrak{h}_V)}^{n}$$

where every V is the union of a finite number of V_i's and every finite subvolume of $\mathbb{R}^3$ is contained in some V.

Definition 2.1. $\mathscr{C}(\mathfrak{h})$ denotes the C^*-algebra generated by the compacts and the scalars on the Hilbert space $\mathfrak{h}$. In short

$$\mathscr{C}(\mathfrak{h})=\overline{\{C+\lambda\mathbb{1};\ C\text{ compact},\ \lambda\text{ complex}\}}^{n}.$$

For a given partition of $\mathbb{R}^3$ into disjoint finite volumes $\{V_n, n\in\mathbb{N}\}$ consider, for a finite subset I of $\mathbb{N}$ with p members $\{n_1, \ldots, n_p\}$ say, the C^*-algebra $\mathscr{C}^I$ which is defined as the following C^*-tensor product

$$\mathscr{C}^I=\mathscr{C}(\mathfrak{h}_{n_1})\otimes\mathscr{C}(\mathfrak{h}_{n_2})\otimes\ \ldots\ \otimes\mathscr{C}(\mathfrak{h}_{n_p}).$$

We shall denote by $\mathscr{C}_0$ the C^*-inductive limit defined as

$$\mathscr{C}_0=\overline{\bigcup_{I\subset\mathbb{N}}\mathscr{C}^I}^{n}$$

Clearly $\mathscr{C}_0$ is a separable sub C^*-algebra of $\mathfrak{A}$.

Remark. If we would specialize to a quantum lattice algebra then every operator on $\mathfrak{h}_{V_n}$ is compact (including the unit operator) and then $\mathscr{C}_0$ coincides with $\mathfrak{A}$ (which is a separable C^*-algebra for a quantum lattice). In the case of a continuous system $\mathfrak{A}$ is not separable in the norm topology because none of the algebras $\mathscr{B}(\mathfrak{h}_V)$ is and therefore $\mathscr{C}_0\subsetneqq\mathfrak{A}$.

Lemma 2.2. *N is the set of all states of $\mathfrak{A}$ that can be obtained as projective limits of normal states on the elements $\mathscr{B}(\mathfrak{h}_V)$ that make up the quasi-local algebra $\mathfrak{A}$.*

Proof. See Z. Takeda [9].

Lemma 2.3. *$\mathscr{C}_0$ is $\sigma(\mathfrak{A}, N)$ dense in $\mathfrak{A}$.*

Proof. Let A be arbitrary in $\mathfrak{A}$, then we can find for every $\varepsilon>0$ $A_n\in\mathscr{B}(\mathfrak{h}_{V_n})$ for some V_n with $\|A-A_n\|<\varepsilon$. Denote by $U_{\varrho_i,\varepsilon_i;i=1,\dots K}(A)$ a $\sigma(\mathfrak{A}, N)$ neighbourhood of A, i.e.

$$B\in U_{\varrho_i,\varepsilon_i;i=1,\dots K}(A)\Rightarrow|\varrho_i(A)-\varrho_i(B)|<\varepsilon_i,\qquad i=1,\dots K,\ \varrho_i\in N\,.$$

Because $\mathscr{C}(\mathfrak{h}_{V_n})$ is $\sigma(\mathscr{B}(\mathfrak{h}_{V_n}),\mathscr{B}(\mathfrak{h}_{V_n})_*)$ dense in $\mathscr{B}(\mathfrak{h}_{V_n})$ we can find C_n such that

$$|\varrho_i(A_n)-\varrho_i(C_n)|<\varepsilon_i/2,\qquad i=1,\dots K,\ \varrho_i\in N$$

with $C_n\in\mathscr{C}(\mathfrak{h}_{V_n})$. Putting everything together we have

$$|\varrho_i(A)-\varrho_i(C_n)|\leqq|\varrho_i(A)-\varrho_i(A_n)|+|\varrho_i(A_n)-\varrho_i(C_n)|<\varepsilon+\varepsilon_i/2\,.$$

If we choose $\varepsilon=\frac{1}{2}\inf\varepsilon_i$ we find $|\varrho_i(A)-\varrho_i(C_n)|<\varepsilon_i$, $i=1,\dots K$ and hence

$$U_{\varrho_i,\varepsilon_i;i=1,\dots K}(A)\cap\mathscr{C}_0\neq\emptyset\,.$$ Q.E.D.

Let $(\Omega_\omega,\mathfrak{h}_\omega,\pi_\omega)$ be the cyclic vector, the Hilbert space and the representation of $\mathfrak{A}$ on $\mathfrak{h}_\omega$ respectively as obtained from the Gel'fand-Segal construction from a state ω.

Lemma 2.4. *Ω_ω is cyclic for $\pi_\omega(\mathscr{C}_0)$ for $\omega\in N$* [7].

Proof. Suppose $(\chi,\pi_\omega(\mathscr{C}_0)\Omega_\omega)=0$. Because $\omega\in N$ we have that $\pi_\omega|_{\mathscr{B}(\mathfrak{h}_{V_n})}$ is normal, hence $\omega_{\chi,\Omega_\omega}\circ\pi_\omega$ is $\sigma(\mathfrak{A},N)$ continuous on $\mathfrak{A}$, therefore $(\omega_{\chi,\Omega_\omega}\circ\pi_\omega)(A)=0\,\forall A\in\mathfrak{A}$ because of lemma 2.3. Q.E.D.

Let us now formulate the *first regularity condition on α_t*. For every given partition of $\mathbb{R}^3$ into disjoint finite volumes $\{V_n, n\in\mathbb{N}\}$ there exists a sequence of local Hamiltonians H_n on $\mathfrak{h}_{V_n}$ inducing automorphisms α_t^n on $\mathscr{B}(\mathfrak{h}_{V_n})$ given by

$$\alpha_t^n(A)=\exp iH_ntA\exp -iH_nt,\qquad A\in\mathscr{B}(\mathfrak{h}_{V_n})\,.$$

Consider the sequence of volumes $\{V_N\}$ whose elements have the properties

i) Every V_N is a union of a finite number of volumes V_n;
ii) $V_N\subset V_{N'}$, $N\leqq N'$;
iii) Every V_n is contained in some V_N for N sufficiently large.

In $\mathscr{B}(\mathfrak{h}_{V_N})$ there exists a local Hamiltonian H_N inducing automorphisms

$$\alpha_t^N(A)=\exp iH_NtA\exp -iH_Nt$$

with the properties that

$$\lim_{N\to\infty}\alpha_t^N(A)=\alpha_t(A)\,,\qquad A\in\bigcup_{I\subset\mathbb{N}}\mathscr{C}^I\,,$$

where the limit is i) in the norm topology on $\mathscr{C}_0$ and ii) uniformly in t on a neighbourhood of zero.

This regularity condition implies that α_t acts strongly continuous on $\mathscr{C}_0$ provided we can show that α_t^N acts strongly continuous on every $\mathscr{C}^I$ with I such that $V_N\supset\bigcup_{k\in I}V_k$. Kallman [10] has shown that α_t^n acts strongly continuous on $\mathscr{C}(\mathfrak{h}_{V_n})$. The goal of the second regularity condition will be to make α_t^N act continuously on $\mathscr{C}_0$. The problem with the latter is that it contains for instance elements of the form $\mathscr{C}(\mathfrak{h}_{V_n})\otimes\mathscr{C}(\mathfrak{h}_{V_{n'}})$ which do not necessarily belong to $\mathscr{C}(\mathfrak{h})$ for some suitable $\mathfrak{h}$.

(Take $C\otimes\mathbb{1}_{V_{n'}}$, for instance with C compact on $\mathfrak{h}_{V_n}$.) These problems do not exist for a quantum lattice system; there we have that the first regularity condition implies that α_t acts strongly continuous on $\mathfrak{A}$.

Let us now prepare the ground for the formulation of the second regularity condition. Let γ_t be a one-parameter automorphism group of some $\mathscr{B}(\mathfrak{h})$ with the property that $\varphi(\gamma_t(A))$ is a continuous function of t for all $A\in\mathscr{B}(\mathfrak{h})$ and $\phi\in\mathscr{B}(\mathfrak{h})_*$. It follows from Robinson and Bratteli [11] that there exists an unbounded derivation δ of $\mathscr{B}(\mathfrak{h})$ with domain $D(\delta)$ that is a strongly dense sub-*algebra of $\mathscr{B}(\mathfrak{h})$. $D(\delta)$ is defined as follows

$$D(\delta)=\left\{A\in\mathscr{B}(\mathfrak{h}):\exists B\in\mathscr{B}(\mathfrak{h}) \text{ with } \phi(B)=\lim_{t\to 0}\frac{\phi(\gamma_t(A)-A)}{t},\quad \phi\in\mathscr{B}(\mathfrak{h})_*\right\}.$$

Lemma 2.5. *Let $A\in D(\delta)$, then it follows that $\|\gamma_t(A)-A\|\underset{t\to 0}{\longrightarrow}0$ (cf. [12]).*

Proof. Since $A\in D(\delta)$ we have that

$$\lim_{t\to 0}\frac{\phi(\gamma_t(\mathrm{A})-\mathrm{A})}{t}$$

exists for all $\phi\in\mathscr{B}(\mathfrak{h})_*$. Hence for $\phi\in\mathscr{B}(\mathfrak{h})_*$ we have

$$\sup_t\left|\frac{\phi(\gamma_t(A)-A)}{t}\right|<\infty\,.$$

Because $\mathscr{B}(\mathfrak{h})_*$ is a determining manifold we have [13]

$$\sup_t\left\|\frac{\gamma_t(A)-A}{t}\right\|<\infty$$

implying

$$\|\gamma_t(A)-A\|\underset{t\to 0}{\longrightarrow}0\,. \qquad \text{Q.E.D.}$$

Consider two finite disjoint volumes belonging to the partition $\{V_n, n\in\mathbb{N}\}$ of $\mathbb{R}^3$, V_1 and V_2 say. Consider on $\mathfrak{h}_{V_1}\otimes\mathfrak{h}_{V_2}$ the unitary group $U_t^0=U_t^{(1)}\otimes U_t^{(2)}$ where $U_t^{(1)}=\exp iH_1t$ and $U_t^{(2)}=\exp iH_2t$. Consider $U_t=\exp iHt$ with H the local Hamiltonian on $\mathfrak{h}_{V_1}\otimes\mathfrak{h}_{V_2}$. Clearly U_t^0 and U_t give both rise to automorphisms, β_t^0 and β_t say, which in turn give rise to derivations on $\mathscr{B}(\mathfrak{h}_{V_1}\otimes\mathfrak{h}_{V_2})$ denoted by δ^0 and δ.

Suppose now that

$$D(\delta)\cap D(\delta^0)\cap\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2}) \text{ is uniformly dense in } D(\delta^0)\cap\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2}),$$

then we have the following

Theorem 2.6. $\|\beta_t(B)-B\|\underset{t\to 0}{\longrightarrow}0\ \forall B\in\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2})$.

Proof. From Kallman [10] we know that $\alpha_t^i(A_i)=U_t^{(i)}A_iU_{-t}^{(i)}$ is strongly continuous for $A_i\in\mathscr{C}(\mathfrak{h}_{V_i})$, $i=1,2$. β_t^0 implemented by U_t^0 acts strongly continuous on $\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2})$. Therefore since β_t^0 leaves $\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2})$ invariant, we have that $D(\delta^0)\cap\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2})$ is uniformly dense in $\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2})$. A fortiori

$D(\delta)\cap\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2})$ is uniformly dense in $\mathscr{C}(\mathfrak{h}_{V_1})\otimes\mathscr{C}(\mathfrak{h}_{V_2})$. Using lemma 2.5 the theorem is proved.

If we now want to impose a condition that makes α_t^N act strongly continuous on $\mathscr{C}_0$, it suffices to show that it acts strongly continuous on algebras that we denoted by $\mathscr{C}^I$ for all finite subsets $I\subset\mathbb{N}$. This will be done by transfinite induction. Let $V_{N(I)}$ be a finite volume in $\mathbb{R}^3$ with

$$V_{N(I)}=\bigcup_{k\in I} V_k,\ I \text{ some finite subset of } \mathbb{N}.$$

Consider $\mathscr{C}^I=\bigotimes_{k\in I}\mathscr{C}(\mathfrak{h}_{V_k})$ and denote as before by $\alpha_t^{N(I)}$ the automorphism on $\mathscr{B}(\mathfrak{h}_{V_{N(I)}})$ induced by the local Hamiltonian on $\mathfrak{h}_{V_{N(I)}}$; its derivation is denoted by $\Delta_{N(I)}$. If $V_p\cap V_{N(I)}=\emptyset$, we denote by γ_t^0 the automorphism on $\mathscr{B}(\mathfrak{h}_{V_{N(I)}}\otimes\mathfrak{h}_{V_p})$ given by $\alpha_t^{N(I)}\otimes\alpha_t^p$. Let $\alpha_t^{N(I)+1}$ be the automorphism on $\mathscr{B}(\mathfrak{h}_{V_{N(I)}}\otimes\mathfrak{h}_{V_p})$ generated by the local Hamiltonian on $\mathfrak{h}_{V_{N(I)}}\otimes\mathfrak{h}_{V_p}$. Γ_0, $\Delta_{N(I)+1}$, δ_p stand for the derivations associated with γ_t^0, $\alpha_t^{N(I)+1}$ and α_t^p respectively.

We now impose the following: *second regularity condition on* α_t $D(\Delta_{N(I)+1})\cap D(\Gamma_0)\cap\mathscr{C}^I\otimes\mathscr{C}(\mathfrak{h}_{V_p})$ is uniformly dense in $D(\Gamma_0)\cap\mathscr{C}^I\otimes\mathscr{C}(\mathfrak{h}_{V_p})$,

$$\forall N(I),\ V_p\cap V_{N(I)}=\emptyset\,.$$

Theorem 2.7. *Let $D(\Delta_{N(I)})\cap\mathscr{C}^I$ be uniformly dense in $\mathscr{C}^I$ and let furthermore the second regularity condition on α_t be satisfied, then it follows that $\alpha_t^{N(I)+1}$ acts strongly continuous on $\mathscr{C}^I\otimes\mathscr{C}(\mathfrak{h}_{V_p})$.*

Proof. Take $A\in D(\Delta_{N(I)})\cap\mathscr{C}^I$ and $B\in D(\delta_p)\cap\mathscr{C}(\mathfrak{h}_{V_p})$. Then we have

$$\begin{aligned}
&\left\|\frac{\alpha_t^{N(I)}(A)\otimes\alpha_t^p(B)-A\otimes B}{t}-\Delta_{N(I)}(A)\otimes B-A\otimes\delta_p(B)\right\|\leqq\\
&\leqq\left\|\frac{\alpha_t^{N(I)}(A)\otimes\alpha_t^p(B)-\alpha_t^{N(I)}(A)\otimes B}{t}-\alpha_t^{N(I)}(A)\otimes\delta_p(B)\right\|+\\
&\quad+\|\alpha_t^{N(I)}(A)\otimes\delta_p(B)-A\otimes\delta_p(B)\|+\left\|\frac{\alpha_t^{N(I)}(A)\otimes B-A\otimes B}{t}-\Delta_{N(I)}(A)\otimes B\right\|\leqq\\
&\leqq\|A\|\left\|\frac{\alpha_t^p(B)-B}{t}-\delta_p(B)\right\|+\|\delta_p(B)\|\ \|\alpha_t^{N(I)}(A)-A\|+\\
&\quad+\left\|\frac{\alpha_t^{N(I)}(A)-A}{t}-\Delta_{N(I)}(A)\right\|\ \|B\|\underset{t\to 0}{\longrightarrow}0\,,
\end{aligned}$$

due to the assumptions on A,B and the strong continuity of $\alpha_t^{N(I)}$ on $\mathscr{C}^I$. What the above estimate shows is that $A\otimes B$ belongs to $D(\Gamma_0)$ for $\forall A\in D(\Delta_{N(I)})\cap\mathscr{C}^I$ and $\forall B\in D(\delta_p)\cap\mathscr{C}(\mathfrak{h}_{V_p})$. By assumption $D(\Delta_{N(I)})\cap\mathscr{C}^I$ is norm dense in $\mathscr{C}^I$ and $\mathscr{C}(\mathfrak{h}_{V_p})\cap D(\delta_p)$ is norm dense in $\mathscr{C}(\mathfrak{h}_{V_p})$ by the same argument as used in theorem 2.6. Therefore we conclude that $D(\Gamma_0)$ has dense intersection with $\mathscr{C}^I\otimes\mathscr{C}(\mathfrak{h}_{V_p})$.

The same, by our second regularity condition holds for $D(\Delta_{N(I)+1})$. Lemma 2.5 then guarantees that $\alpha_t^{N(I)+1}$ acts strongly continuous on this dense set and therefore acts strongly continuous on all of $\mathscr{C}^I\otimes\mathscr{C}(\mathfrak{h}_{V_p})$. Q.E.D.

By combination of the regularity conditions we have

Theorem 2.8. *Let α_t be a one-parameter group of automorphisms of $\mathfrak{A}$ satisfying the regularity conditions, then α_t acts strongly continuous on $\mathscr{C}_0$.*

Proof. Take arbitrary $V_N \equiv V_{N(I)} = \bigcup_{k\in I} V_k$. Take V_{k_1} and V_{k_2} from $\bigcup_{k\in I} V_k$. Theorem 2.6 gives the strong continuity on $\mathscr{C}(\mathfrak{h}_{V_{k_1}}) \otimes \mathscr{C}(\mathfrak{h}_{V_{k_2}})$ of the automorphism group induced by the local Hamiltonian belonging to $V_{k_1} \cup V_{k_2}$. Now by successively applying theorem 2.7 on the remaining elements of $\bigcup_{k\in I} V_k$, we obtain the strong continuity of α_t^N on $\mathscr{C}^I$. From the first regularity condition it follows that α_t acts strongly continuous on $\mathscr{C}_0$. Q.E.D.

Nowhere it is guaranteed that $\mathscr{C}_0$ is α_t invariant as a set.

Definition 2.9. $\mathfrak{A}_0$ is the C^*-algebra generated by all elements A of $\mathscr{C}_0$ and the translates $\alpha_{t_i}(A)$ thereof with t_i rational.

Due to the strong continuity of α_t on $\mathscr{C}_0$ we can easily show that $\mathfrak{A}_0$ contains all elements $\alpha_t(A)$ and furthermore is invariant under the action of α_t. Clearly by construction $\mathfrak{A}_0$ is a norm separable C^*-algebra contained in $\mathfrak{A}$. Because $\mathfrak{A}$ is not separable in its norm topology we have that $\mathfrak{A}_0 \subsetneqq \mathfrak{A}$. However $\mathfrak{A}_0$ is $\sigma(\mathfrak{A}, N)$ dense in $\mathfrak{A}$ by lemma 2.3.

It follows from its construction that $\mathfrak{A}_0$ will in general depend on the automorphism group α_t. Again when we specialize to a quantum lattice algebra, there is no such dependence on the dynamics because $\mathscr{C}_0$ on its own coincides already with $\mathfrak{A}$. Also in the case of a quantum lattice algebra, the first regularity condition on α_t implies already strong continuity of α_t. The second regularity condition is trivially satisfied because all local algebras are finite-dimensional matrices and the derivations appearing in this condition are everywhere defined. (The local Hamiltonians are bounded operators.) Furthermore the second regularity condition is easily verifiable for non-interacting Fermi and Bose systems. In these latter cases the local Hamiltonians are double differentiation operators and dense subsets of $D(\delta) \cap \mathscr{C}(\mathfrak{h}_{V_1}) \otimes \mathscr{C}(\mathfrak{h}_{V_2})$ and $D(\delta^0) \cap \mathscr{C}(\mathfrak{h}_{V_1}) \otimes \mathscr{C}(\mathfrak{h}_{V_2})$ can be constructed by combination of finite rank operators which are formed from suitable C^∞-functions with compact support.

c) Results for Thermodynamic Systems

We assume that there exists a partition of $\mathbb{R}^3$ which we consider as fixed for our further reasoning. We assume that our dynamics α_t is given by a one-parameter group of automorphisms of the quasi-local algebra $\mathfrak{A}$ that satisfies the regularity conditions as described in section 2b. Let ω be an α_t K.M.S.-state at inverse temperature β and $\mathfrak{h}_\omega$ the Gel'fand-Segal representation space carrying the representation π_ω of $\mathfrak{A}$ with a cyclic vector Ω_ω that is also separating for $\pi_\omega(\mathfrak{A})''$. U_t^ω denotes the unitary one-parameter group that implements α_t, H_ω denotes the generator of U_t^ω. On $\mathfrak{h}_\omega$ one has the following representation of the convolution algebra $L^1(\mathbb{R})$:

$$f \in L^1(\mathbb{R}) \xrightarrow{\pi} \pi(f) \in \mathscr{B}(\mathfrak{h}_\omega),$$

where $\pi(f)\chi = \int f(t) U_t^\omega \chi dt$, $\forall \chi \in \mathfrak{h}_\omega$ and the integral is in the Bochner sense. Following Arveson we define like we did in section 2a

$$\mathrm{Sp}\, U = \{\gamma \in \mathbb{R}^1 : \hat{f}(\gamma) = 0 \,\forall f \in \ker \pi\},$$

which by a lemma in section 2a equals the spectral set of H_ω. We now state

Theorem A. *Let ω be any α_t K.M.S.-state at inverse temperature β, then* $\mathrm{Sp}\, H_\omega$ *is independent of ω and β.*

Proof. Since ω as an α_t K.M.S.-state is locally normal [7] we have by lemma 2.4 that Ω_ω is cyclic for $\pi_\omega(\mathfrak{A}_0)$. Furthermore observe that, since α_t acts strongly continuous on $\mathfrak{A}_0$ and $\mathfrak{A}_0$ is invariant under the action of α_t, we can Bochner integrate on $\mathfrak{A}_0$. In particular we observe the existence in the Bochner sense of objects like

$$\int \alpha_t(A) f(t) dt \quad \text{for} \quad f \in L^1(\mathbb{R}) \quad \text{and} \quad A \in \mathfrak{A}_0 .$$

Also we then know for representations π of $\mathfrak{A}_0$ that

$$\pi\left(\int \alpha_t(A) f(t) dt\right) = \int \pi(\alpha_t(A)) f(t) dt .$$

The proof of our theorem now proceeds by proving that for any K.M.S.-state at any inverse temperature $\beta \neq 0$

$$\begin{aligned}&\{\gamma \in \mathbb{R}^1 : \hat{f}(\gamma) = 0 \,\forall f \text{ with } \int f(t) U_t^\omega \chi dt = 0 \,\forall \chi \in \mathfrak{h}_\omega\} \\ &\qquad = \{\gamma \in \mathbb{R}^1 : \hat{f}(\gamma) = 0 \,\forall f \text{ with } \int f(t) \alpha_t(A) dt = 0 \,\forall A \in \mathfrak{A}_0\} .\end{aligned}$$

Indeed

$$\begin{aligned}&\int f(t) U_t^\omega \chi dt = 0 \,\forall \chi \in \mathfrak{h}_\omega \Rightarrow \int f(t) U_t^\omega \pi_\omega(A) \Omega_\omega dt = 0 \,\forall A \in \mathfrak{A}_0 \Rightarrow \\ &\qquad \Rightarrow \int f(t) \pi_\omega(\alpha_t(A)) \Omega_\omega dt = 0 \,\forall A \in \mathfrak{A}_0 \Rightarrow \\ &\qquad \Rightarrow \pi_\omega\left[\int f(t) \alpha_t(A) dt\right] \Omega_\omega = 0 \,\forall A \in \mathfrak{A}_0 \Rightarrow \\ &\qquad \Rightarrow \pi_\omega\left[\int f(t) \alpha_t(A) dt\right] = 0 \,\forall A \in \mathfrak{A}_0 .\end{aligned}$$

The latter step is due to the fact that Ω_ω is separating for $\pi_\omega(\mathfrak{A})''$ and hence for $\pi_\omega(\mathfrak{A}_0)$. Since $\mathfrak{A}$ is simple [14] we have that

$$\pi_\omega\left[\int f(t) \alpha_t(A) dt\right] = 0 \Leftrightarrow \int f(t) \alpha_t(A) dt = 0 , \ \forall A \in \mathfrak{A}_0 .$$

Conversely let $\int f(t) \alpha_t(A) dt = 0 \,\forall A \in \mathfrak{A}_0$. Hence

$$\begin{aligned}&\pi_\omega\left[\int f(t) \alpha_t(A) dt\right] = 0 \,\forall A \in \mathfrak{A}_0 \Rightarrow \int f(t) \pi_\omega(\alpha_t(A)) \Omega_\omega dt = 0 \,\forall A \in \mathfrak{A}_0 \Rightarrow \\ &\qquad \Rightarrow \int f(t) U_t^\omega \pi_\omega(A) \Omega_\omega dt = 0 \,\forall A \in \mathfrak{A}_0 .\end{aligned}$$

For arbitrary $\chi \in \mathfrak{h}_\omega$ we can find a suitable $A \in \mathfrak{A}_0$ such that for every $\varepsilon > 0$

$$\left\| \int f(t) U_t^\omega (\pi_\omega(A) \Omega_\omega - \chi) dt \right\| \leqq \varepsilon \| f \|_1 ,$$

where $\| f \|_1$ is the L^1 norm of f. Hence we have that

$$\int f(t) \alpha_t(A) dt = 0 \,\forall A \in \mathfrak{A}_0 \Leftrightarrow \int f(t) U_t^\omega \chi dt = 0 \,\forall \chi \in \mathfrak{h}_\omega . \qquad \text{Q.E.D.}$$

Remark. In any system where the dynamics is given by a strongly continuous one-parameter group of automorphisms theorem A holds (cf. [3]). In particular this holds for a quantum lattice gas.

The content of theorem A is a global statement about the spectra of generators of time translations as a whole in representations of states that are α_t K.M.S.. In the case of a finite quantum system (cf. section 2a) one is able to make a much more detailed comparison, namely the spectra are identical in nature. For thermodynamic systems this is no longer true. The rest of this section will be devoted to a pointwise comparison of spectra of generators in the case where we can decompose a given α_t K.M.S.-state into extremal K.M.S.-states at a given temperature.

In order to compare locally the spectra of generators of time translations, we restrict ourselves to a comparison that involves only α_t K.M.S.-states at a fixed temperature. Moreover we assume that the set of extremal points of the simplex K_β of α_t K.M.S.-states at a temperature β is a Borel set. (As pointed out in [7] one way to assure the latter fact is by assuming that K_β is compact.) Under these assumptions we have a unique decomposition for elements $\omega \in K_\beta$ into extremal points ω_γ, i.e. ω gives rise to a Borel measure on the set of states of $\mathfrak{A}$ with the property that it is concentrated on the extremal points of K_β:

$$\omega = \int_{\mathrm{Ext}\, K_\beta} d\mu_\omega(\gamma)\omega_\gamma .$$

We can now formulate[1]

Theorem B. *Let U_t^ω and $U_t^{\omega_\gamma}$ implement α_t in the representations given by ω and ω_γ respectively. Denote the appropriate generators of U_t^ω and $U_t^{\omega_\gamma}$ by H_ω and H_{ω_γ}. Denote the pointspectra of H_ω and H_{ω_γ} by $P\,\mathrm{Sp}\,H_\omega$ and $P\,\mathrm{Sp}\,H_{\omega_\gamma}$. The following is true:*

$$\lambda \in P\,\mathrm{Sp}\,H_\omega \to \lambda \in P\,\mathrm{Sp}\,H_{\omega_\gamma} \forall \omega_\gamma \in V \subseteq \mathrm{Ext}\,K_\beta \quad \text{where} \quad \mu_\omega(V) \neq 0 .$$

Proof. Suppose $\lambda \in P\,\mathrm{Sp}\,H_\omega$, then clearly by definition there exists $\chi \in \mathfrak{h}_\omega$ with

$$U_t^\omega \chi = e^{i\lambda t}\chi .$$

From [15] we conclude that there exists at least one element $A \in \pi_\omega(\mathfrak{A})''$ different from zero with the property that $\tilde{\alpha}_t(A) = e^{i\lambda t}A$, where $\tilde{\alpha}_t$ is the extension of α_t to $\pi_\omega(\mathfrak{A})''$.

We are discussing K.M.S.-states, therefore the states are separating on their associated von Neumann algebras and hence the von Neumann algebras are σ-finite.

From [16] page 31 corrolaire and Kaplansky's density theorem it follows that we can choose a sequence $\{A_n\} \in \mathfrak{A}$ such that

$$\pi_\omega(A_n) \xrightarrow[\text{strongly}]{} A \quad \text{with} \quad \|\pi_\omega(A_n)\| \leq \|A\| .$$

[1] For the following we do not have to assume that α_t satisfies the regularity conditions, nor do we have to assume that $\mathfrak{A}$ is quasi-local (or simple).

We shall prove that we can choose a subsequence $\{A_{n_k}\}$ from $\{A_n\}$ with the property that:

$$\pi_{\omega_\gamma}(A_{n_k})\Omega_{\omega_\gamma} \to \psi_\gamma \in \mathfrak{h}_{\omega_\gamma} \ \mu\text{-a.e.}$$

Furthermore we shall show that there exists a set $V \subseteq \text{Ext}\, K_\beta$ with $\mu_\omega(V) \neq 0$ such that for all γ with $\omega_\gamma \in V$ we have

$$\psi_\gamma \neq 0 \quad \text{and} \quad U_t^{\omega_\gamma}\psi_\gamma = e^{i\lambda t}\psi_\gamma .$$

Clearly the latter statement means that $\lambda \in P\,\text{Sp}\, H_{\omega_\gamma}$.

Let $\mathfrak{Z}_\omega$ be the centre of $\pi_\omega(\mathfrak{A})''$, we then have for $A \in \pi_\omega(\mathfrak{A})''$

$$(\Omega_\omega, A\Omega_\omega) = \int_\Gamma d\mu_\omega(\gamma)\varepsilon(A)(\gamma) ,$$

where Γ is the spectrum of $\mathfrak{Z}_\omega$; $\varepsilon(A) \in \mathfrak{Z}_\omega$ and is defined as $PAP = P\varepsilon(A)$ with $P = [\mathfrak{Z}_\omega \Omega_\omega]$, furthermore $\varepsilon(A)$ (γ) is the continuous function on Γ obtained from $\varepsilon(A) \in \mathfrak{Z}_\omega$ by the Gel'fand isomorphism. In particular we have:

$$\omega(B) = (\Omega_\omega, \pi_\omega(B)\Omega_\omega) = \int_\Gamma d\mu_\omega(\gamma)\varepsilon(\pi(B))(\gamma) = \int_\Gamma d\mu_\omega(\gamma)\omega_\gamma(B) ,$$

where $B \in \mathfrak{A}$, $\omega_\gamma \in S(\mathfrak{A}) \cap \text{Ext}\, K_\beta$ (cf. [7], [17]).

As we have seen above $\pi_\omega(A_n) \xrightarrow{s} A$ and hence

$$(\Omega_\omega, (A - \pi_\omega(A_n))^*(A - \pi_\omega(A_n))\Omega_\omega) \xrightarrow[n\to\infty]{} 0 .$$

From this we conclude

$$\int d\mu_\omega(\gamma)\varepsilon[(\pi_\omega(A_n) - A)^*(\pi_\omega(A_n) - A)](\gamma) \xrightarrow[n\to\infty]{} 0 .$$

Since the Gel'fand isomorphism is orderpreserving we know that

$$f_n(\gamma) \equiv \varepsilon[(\pi_\omega(A_n) - A)^*(\pi_\omega(A_n) - A)](\gamma) \geqq 0 .$$

In short we can say that together with the choice of the sequence $\{A_n\}$ we have obtained a sequence of positive functions f_n that tends to zero in mean. We can therefore [18] choose a subsequence of f_n that tends to zero μ-a.e. and hence we can find a subsequence A_{n_k} such that

$$\varepsilon[(\pi_\omega(A_{n_k}) - A)^*(\pi_\omega(A_{n_k}) - A)](\gamma) \xrightarrow[n\to\infty]{} 0 \ \ \mu\text{-a.e.}$$

$\varepsilon(A)(\gamma)$ is, for γ fixed, a positive linear functional on $\pi_\omega(\mathfrak{A})''$. As such it satisfies the Schwartz inequality:

$$|\varepsilon(A^*B)(\gamma)|^2 \leqq \varepsilon(A^*A)(\gamma)\varepsilon(B^*B)(\gamma) .$$

From this one easily sees that the following holds true:

$$\varepsilon[(A-B)^*(A-B)](\gamma) \leqq [\varepsilon(A^*A)(\gamma)^{\frac{1}{2}} + \varepsilon(B^*B)(\gamma)^{\frac{1}{2}}]^2 \quad A,B \in \pi_\omega(\mathfrak{A})'' .$$

Denoting $[\varepsilon(A^*A)(\gamma)]^{\frac{1}{2}} = \|A\|_\gamma$ we have in fact that the map $A \in \pi_\omega(\mathfrak{A})'' \to \|A\|_\gamma$ is a semi-norm on $\pi_\omega(\mathfrak{A})''$. Let us now consider

$$\begin{aligned}\|\pi_\omega(A_n - A_m)\|_\gamma &= \|(\pi_\omega(A_n) - A) + (A - \pi_\omega(A_m))\|_\gamma \leqq \\ &\leqq \|\pi_\omega(A_n) - A\|_\gamma + \|A - \pi_\omega(A_m)\|_\gamma .\end{aligned}$$

For A_n and A_m from the above chosen subsequence of $\{A_n\}$ we find μ-a.e.:

$$\|\pi_\omega(A_n)-\pi_\omega(A_m)\|_\gamma^2=\varepsilon[(\pi_\omega(A_n)-\pi_\omega(A_m))^*(\pi_\omega(A_n)-\pi_\omega(A_m))](\gamma)\xrightarrow[n,m\to\infty]{}0\,.$$

This however means that $\omega_\gamma((A_n-A_m)^*(A_n-A_m))\xrightarrow[n,m]{}0$, which in turn means that in the representation space $\mathfrak{h}_{\omega_\gamma}$, carrying the representation π_{ω_γ} of $\mathfrak{A}$ with cyclic vector Ω_{ω_γ}, we have μ-a.e.

$$\|(\pi_{\omega_\gamma}(A_n)-\pi_{\omega_\gamma}(A_m))\Omega_{\omega_\gamma}\|\xrightarrow[n,m]{}0\,,$$

which means that $\pi_{\omega_\gamma}(A_n)\Omega_{\omega_\gamma}$ converges to a vector which we denote by ψ_γ.

We shall now show that ψ_γ is different from zero for all γ such that ω_γ belongs to a set V_0 with nonzero μ-measure. Indeed $(\Omega_\omega, A^*A\Omega_\omega)\neq 0$ because $A\neq 0$. (Ω_ω is separating for $\pi_\omega(\mathfrak{A})''$!) Since the sequence $\{\pi_\omega(A_n)\}$ converges strongly to A it follows that

$$(\Omega_\omega, A^*A\Omega_\omega)=\lim_n(\Omega_\omega,\pi_\omega(A_n^*)\pi_\omega(A_n)\Omega_\omega)=\lim_n\int_\Gamma\omega_\gamma(A_n^*A_n)d\mu_\omega(\gamma)\,.$$

Furthermore

$$\begin{aligned}\omega_\gamma(A_n^*A_n)&=\varepsilon(\pi_\omega(A_n^*)\pi_\omega(A_n))(\gamma)\leqq\sup_{\gamma\in\Gamma}\varepsilon(\pi_\omega(A_n^*)\pi_\omega(A_n))(\gamma)=\\&=\|\varepsilon(\pi_\omega(A_n^*)\pi_\omega(A_n))\|=\|P\varepsilon(\pi_\omega(A_n^*)\pi_\omega(A_n))\|=\\&=\|P\pi_\omega(A_n^*)\pi_\omega(A_n)P\|\leqq\|\pi_\omega(A_n)\|^2\leqq\|A\|^2\,.\end{aligned}$$

In these estimates we used successively that the representation theorem for commutative C^*-algebras is an isomorphism, the map $\mathfrak{Z}\to P\mathfrak{Z}$ is an isomorphic map and that we may choose for $\pi_\omega(A_n)$ a bounded sequence. (For a simple algebra $\|A_n\|=\|\pi_\omega(A_n)\|$ and hence the estimate is immediate because $\omega_\gamma(A_n^*A_n)\leqq\|A_n\|^2\leqq\|A\|^2$.) If we restrict our attention to the above chosen subsequence of $\pi_\omega(A_n)$ we can say that $\omega_\gamma(A_n^*A_n)^{\frac{1}{2}}=\|\pi_{\omega_\gamma}(A_n)\Omega_{\omega_\gamma}\|$ is a μ-a.e. convergent sequence that is uniformly bounded and therefore the Lebesgue dominated convergence theorem gives then for γ with $\omega_\gamma\in V_0\subset\Gamma$ with $\mu_\omega(V_0)\neq 0$ (because $(\Omega_\omega, A^*A\Omega_\omega)\neq 0$):

$$\|\psi_\gamma\|=\lim\|\pi_{\omega_\gamma}(A_n)\Omega_{\omega_\gamma}\|=\lim\omega_\gamma(A_n^*A_n)^{\frac{1}{2}}\neq 0\,.$$

From the fact that $\tilde{\alpha}_t(A)=e^{i\lambda t}A$ we conclude that

$$\begin{aligned}\|[\pi_\omega(\alpha_t(A_n))-e^{i\lambda t}\pi_\omega(A_n)]\Omega_\omega\|&=\|[\tilde{\alpha}_t(\pi_\omega(A_n))-e^{i\lambda t}\pi_\omega(A_n)]\Omega_\omega\|\leqq\\&\leqq\|\tilde{\alpha}_t(\pi_\omega(A_n)-A)\Omega_\omega\|+\|[\tilde{\alpha}_t(A)-e^{i\lambda t}\pi_\omega(A_n)]\Omega_\omega\|=\\&=2\|(\pi_\omega(A_n)-A)\Omega_\omega\|\xrightarrow[n\to\infty]{}0\,.\end{aligned}$$

This means

$$\omega[(\alpha_t(A_n)-e^{i\lambda t}A_n)^*(\alpha_t(A_n)-e^{i\lambda t}A_n)]\xrightarrow[n\to\infty]{}0\,.$$

Like we did above we can again choose a subsequence to the effect that, with t and λ fixed,

$$\omega_\gamma[(\alpha_t(A_n)-e^{i\lambda t}A_n)^*(\alpha_t(A_n)-e^{i\lambda t}A_n)]\xrightarrow[n\to\infty]{}0\quad\mu\text{-a.e.}$$

This implies for a subset V_t of V_0 with $\mu(V_0)=\mu(V_t)$ that

$$\begin{aligned}\|U_t^{\omega_\gamma}\psi_\gamma - e^{i\lambda t}\psi_\gamma\| \leqq & \|U_t^{\omega_\gamma}\psi_\gamma - U_t^{\omega_\gamma}\pi_{\omega_\gamma}(A_n)\Omega_{\omega_\gamma}\| \\ & + \|U_t^{\omega_\gamma}\pi_{\omega_\gamma}(A_n)\Omega_{\omega_\gamma} - e^{i\lambda t}\pi_{\omega_\gamma}(A_n)\Omega_{\omega_\gamma}\| \\ & + \|e^{i\lambda t}\pi_{\omega_\gamma}(A_n)\Omega_{\omega_\gamma} - e^{i\lambda t}\psi_\gamma\| \xrightarrow[n\to\infty]{} 0 .\end{aligned}$$

We can now find a subset V of V_0 with $\mu(V)=\mu(V_0)$ such that $(U_t^{\omega_\gamma} - e^{i\lambda t})\psi_\gamma = 0$ for all rational t and $\omega_\gamma \in V$. Using strong continuity of $U_t^{\omega_\gamma}$ one extends this to all t. Q.E.D.

3. Some Miscellaneous Results

To conclude this paper we want to discuss some situations that are special in the sense that the spectra of the generators that we encounter coincide with all of $\mathbb{R}^1$. Some of the results we shall discuss here are somewhat disconnected from the ones discussed in the previous sections. We shall therefore state the conditions under which the results of this section are valid separately.

Let α_x be a one-parameter group of automorphisms of a C^*-algebra $\mathfrak{A}$. For every state on $\mathfrak{A}$ that we shall consider $\omega(A\alpha_x(B))$ is a continuous function of x. Let ω be an α_x-invariant state on $\mathfrak{A}$. Then it is known that the representation π of $L^1(\mathbb{R})$ in $\mathscr{B}(\mathfrak{h}_\omega)$, given by

$$f \in L^1(\mathbb{R}) \to \pi(f) : \pi(f)\chi = \int U_x \chi f(x) dx \quad \chi \in \mathfrak{h}_\omega ,$$

is faithful iff $\mathrm{Sp}\, U = \mathrm{Sp}\, P_\omega = \mathbb{R}^1 (U_x = \exp iP_\omega x)$. Let ω be α_x-invariant and let $\tilde{\alpha}_x$ denote the extension of α_x to $\pi_\omega(\mathfrak{A})''$. U_x denotes the continuous unitary group that implements $\tilde{\alpha}_x$. Then we have [4]

Theorem 3.1. $\mathrm{Sp}\, U = \mathbb{R}^1$ *if* $\pi_\omega(\mathfrak{A})''$ *is non-abelian and* $\pi_\omega(\mathfrak{A})'' \cap U'_x = \{\lambda 1\}$.

Let $\mathfrak{A}$ be a non-abelian C^*-algebra and ω an extremal α_x-invariant state that gives rise to a faithful representation and is furthermore separating then we have:

Theorem 3.2. $\mathrm{Sp}\, U = \mathbb{R}^1$.

Proof. Since $\mathfrak{A}$ is non-abelian and π_ω is faithful $\pi_\omega(\mathfrak{A})''$ is non-abelian. Extremal invariance implies that $\pi_\omega(\mathfrak{A})' \cap U'_x = \{\lambda 1\}$. The fact that ω is separating gives [19] that $\pi_\omega(\mathfrak{A})'' \cap U'_x = \pi_\omega(\mathfrak{A})' \cap U'_x = \{\lambda 1\}$. The theorem now follows from applying theorem 3.1. Q.E.D.

Suppose now that we have a simple C^*-algebra $\mathfrak{A}$, with α_t a one-parameter group of automorphisms, representing the dynamics, that acts asymptotically abelian on $\mathfrak{A}$. Let ω be α_t K.M.S. at an inverse temperature β admitting a decomposition into extremal α_t K.M.S. states at the inverse temperature β,

$$\omega = \int d\mu(\gamma)\omega_\gamma .$$

Since ω_γ is primary and α_t acts asymptotically abelian ω_γ is extremal invariant for α_t. Furthermore ω_γ is separating since it is a K.M.S.-state and hence by theorem 3.2 $\mathrm{Sp}\, H_{\omega_\gamma} = \mathbb{R}^1$. By theorem B we find that the only discrete point in $\mathrm{Sp}\, H_\omega$ is zero. The multiplicity of this eigenvalue is determined by the centre because the centre is pointwise invariant for a K.M.S.-state [19], and $\pi_\omega(\mathfrak{A})' \cap U'_t$ is contained in the centre because α_t acts asymptotically abelian [20].

If we would further specialize our situation by taking for $\mathfrak{A}$ the quasi-local algebra and for α_t not only an automorphism group that acts asymptotically abelian but also satisfies the regularity criteria, then we have $\mathrm{Sp}\, H_\omega = \mathrm{Sp}\, H_{\omega_\gamma} = \mathbb{R}^1$. This is true because we are allowed to use theorem A.

We consider next for a non-abelian C^*-algebra $\mathfrak{A}$ the following

Theorem 3.3. *Let ω be a primary, separating, α_x invariant state on a C^*-algebra $\mathfrak{A}$ (with $\omega(A\alpha_x B)$ continuous and π_ω faithful) then the existence of a sequence $x_n(x_n \xrightarrow[n\to\infty]{} \infty)$ with $\omega(C[A, \alpha_{x_n}(B)]D) \xrightarrow[n\to\infty]{} 0$ $\forall A,B,C,D \in \mathfrak{A}$, implies* $\mathrm{Sp}\, U = \mathbb{R}^1$.

Proof. $\omega(C[A, \alpha_{x_n}(B)]D) \to 0\, \forall C, D \in \mathfrak{A}$ implies that $\omega(A\alpha_{x_n}B) \to \omega(A)\omega(B)$ [21] and hence that Ω_ω is uniquely invariant for U_x, i.e. ω is extremal invariant. This in turn means, because ω is separating, by theorem 3.2, that $\mathrm{Sp}\, U = \mathrm{Sp}\, \mathbb{R}^1$. Q.E.D.

For a primary separating α_x-invariant state the fact that $\omega(C[A, \alpha_{x_n}(B)]D) \to 0$ is equivalent [21] with strong clustering, i.e. $\omega(A\alpha_{x_n}(B)) \to \omega(A)\omega(B)$, we can therefore reformulate theorem 3.3 as

Theorem 3.3′. *Let ω be a primary, separating, strongly clustering, α_x invariant state on $\mathfrak{A}$ (with $\omega(A\alpha_x B)$ continuous and π_ω faithful), then* $\mathrm{Sp}\, U = \mathbb{R}^1$.

A generalization of this theorem is

Theorem 3.4 *Let ω be a non-primary, separating, strongly clustering, α_x invariant state on $\mathfrak{A}(\omega(A\alpha_x B)$ continuous and π_ω faithful), then* $\mathrm{Sp}\, U = \mathbb{R}^1$.

The fact that we took separating states in theorem 3.3′ and 3.4 permits us to exclude the otherwise still existing possibility that $\mathrm{Sp}\, U = \mathbb{R}^{1+}$ or $\mathbb{R}^{1-}$, cf. [3].

Remark. As will be clear from theorem 3.2 the faithfulness of π_ω in theorems 3.3, 3.3′ and 3.4 could be replaced by non-abelianness of $\pi_\omega(\mathfrak{A})''$.

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