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Large Groups of Automorphisms of C*-Algebras

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Abstract. Groups of $*$-automorphisms of C^*-algebras and their invariant states are studied. We assume the groups satisfy a certain largeness condition and then obtain results which contain many of those known for asymptotically abelian C^*-algebras and for inner automorphisms and traces of C^*-algebras. Our key result is the construction in certain "finite" cases, where the automorphisms are spatial, of an invariant linear map of the C^*-algebra onto the fixed point algebra carrying with it most of the relevant information.

1. Introduction

In the works of Doplicher, Kastler, Robinson, and Ruelle [7, 13, 15] there is developed a theory for C^*-algebras acted upon by representations of the translation group R^n as $*$-automorphisms. It is assumed that the C^*-algebra in question is asymptotically abelian, which means roughly that the group of automorphisms obtained, is very large, or somewhat more precisely, that if given two operators in the algebra then large translations of one of them will commute with the other. Recently parts of this theory have been generalized to arbitrary groups by Lanford and Ruelle [14] in a paper which clarifies the underlying mathematical structure of this theory in some respects, but which leaves other aspects of it open. We shall therefore in the present paper develop a theory for representations of groups as "large" groups of automorphisms of C^*-algebras, thus obtaining a better understanding of the underlying mathematics. It will be shown that our situation is less general than that studied by Lanford and Ruelle. However, having more structure, we shall be able to obtain stronger results, see Theorems 3.1 and 3.7. In particular, in section 5 we shall recover more of the results known for C^*-algebras asymptotically abelian with respect to R^n. Technically we shall say a group G is represented as a large group of automorphisms of a C^*-algebra $\mathfrak{A}$ if there is a representation $g \to \tau_g$ of G as $*$-automorphisms of $\mathfrak{A}$ such that for A self-adjoint in $\mathfrak{A}$

$$\operatorname{conv}\big(\pi_\varrho(\tau_g(A)) : g \in G\big)^- \cap \pi_\varrho(\mathfrak{A})' \neq \emptyset$$

for all G-invariant states $\varrho = \omega_{x_\varrho} \circ \pi_\varrho$ of $\mathfrak{A}$, π_ϱ being the canonical cyclic representation of $\mathfrak{A}$ induced by ϱ. In addition to the asymptotically abelian case an example of a large group of automorphisms of a C^*-algebra with identity is the group of inner automorphisms. Hence, since for large groups the G-invariant states form a simplex, our results generalize a theorem of THOMA [17], in which he shows that the normalized traces of a C^*-algebra form a simplex, the extreme points of which are the factor traces.

We shall refer the reader to the two books of DIXMIER [4, 5] for the theory of von Neumann and C^*-algebras, to [1, 3] for the theory of simplexes, and to [7, 10, 13, 15] for the physical background of our work. We shall use the notation ω_x for positive linear functionals of the form $A \to (Ax, x)$ whenever x is a vector in the underlying Hilbert space (it should be remembered that our inner products are linear in the left variable and conjugate linear in the right). If $g \to \tau_g$ is a representation of the group G as $*$-automorphisms of the C^*-algebra $\mathfrak{A}$, the map $\varrho \to \varrho \circ \tau_g$ is an affine isomorphism of the state space $S(\mathfrak{A})$ of $\mathfrak{A}$. We shall say a state ϱ is *G-invariant* if $\varrho = \varrho \circ \tau_g$ for each $g \in G$. More generally, if Φ is a positive linear map of $\mathfrak{A}$ into another C^*-algebra $\mathfrak{B}$ we shall say Φ is G-invariant if $\Phi = \Phi \circ \tau_g$ for all $g \in G$. If $\mathfrak{U}$ is a group of unitary operators such that the maps $u: A \to UAU^{-1}$ are automorphisms of $\mathfrak{A}$ for all $U \in \mathfrak{U}$, we shall also say ϱ is *$\mathfrak{U}$-invariant* if it is invariant with respect to the group of automorphisms u. If it is clear which group we have in mind we denote by $I(\mathfrak{A})$ the G-invariant states of $\mathfrak{A}$. If G is a topological group we shall say the representation $g \to \tau_g$ is *norm continuous* (resp. *strongly continuous*), if

$$\lim \sup_{\|A\| \leq 1} \|\tau_g(A) - A\| = 0 \quad (\text{resp. } \lim \|\tau_g(A) - A\| = 0 \text{ for all } A \in \mathfrak{A})$$

as g converges to the identity element e in G. If $\mathfrak{M}$ is a set of operators we shall denote by $\mathfrak{M}^-$ or $\overline{\mathfrak{M}}$ the weak closure of $\mathfrak{M}$, and by $\mathfrak{M}'$ its commutant. Since it is no restriction to assume $\mathfrak{A}$ has an identity we shall always assume our C^*-algebras have identities denoted by I. If $\{A_\alpha : \alpha \in J\}$ is a set of operators we shall denote by $\operatorname{conv}(A_\alpha : \alpha \in J)$ the set of all finite convex combinations of the A_α.

We shall not in the present paper discuss the problem of existence of G-invariant states. It should, however, be noted that when G is abelian the Markov-Kakutani Theorem [8,V.10.6] yields the existence of G-invariant states, see also [6].

The main ideas of our proof are based on those of THOMA [17] together with those behind DIXMIER's construction of the center trace in finite von Neumann algebras [4, Ch. III, §§ 4, 5]. Two examples will describe the ideas and some of the difficulties of our approach. The essential point is that we want a positive linear G-invariant map Φ of $\mathfrak{A}$ onto the fixed point algebra of the automorphisms carrying with it enough structure. We shall as an illustration do it for G compact.

Example 1.1. Let $\mathfrak{A}$ be a C^*-algebra, G a compact group, and $g \to \tau_g$ a strongly continuous representation of G as $*$-automorphisms of $\mathfrak{A}$. Let $\mathfrak{B}$ denote the sub-C^*-algebra of $\mathfrak{A}$ of those operators B such that $\tau_g(B) = B$ for all $g \in G$. Then there exists a positive linear G-invariant map Φ of $\mathfrak{A}$ onto $\mathfrak{B}$ such that Φ restricted to $\mathfrak{B}$ is the identity map, and such that the map $\Phi^* : S(\mathfrak{B}) \to S(\mathfrak{A})$ defined by $\Phi^*(\varrho) = \varrho \circ \Phi$ is an affine isomorphism of $S(\mathfrak{B})$ onto $I(\mathfrak{A})$. In particular, $I(\mathfrak{A})$ is a simplex if and only if $\mathfrak{B}$ is abelian.

In fact let $\Phi(A) = \int_G \tau_g(A)\, dg$, where dg is the normalized Haar measure. If $h \in G$ then $\tau_h(\Phi(A)) = \int \tau_h \tau_g(A)\, dg = \int \tau_{hg}(A)\, dg = \int \tau_g(A)\, dg = \Phi(A)$, so $\Phi(A) \in \mathfrak{B}$. Clearly, Φ is positive linear, $\Phi | \mathfrak{B}$ is the identity. If $h \in G$, then since G is unimodular,

$$\Phi(\tau_h(A)) = \int \tau_g \tau_h(A)\, dg = \int \tau_{gh}(A)\, dg = \int \tau_g(A)\, dg = \Phi(A)\,,$$

so Φ is G-invariant. Let $\varrho \in S(\mathfrak{B})$. Then $\Phi^*(\varrho)$ is clearly in $I(\mathfrak{A})$. Since Φ is onto, Φ^* is one-to-one. If $\omega \in I(\mathfrak{A})$ then $\omega | \mathfrak{B} \in S(\mathfrak{B})$ since $I \in \mathfrak{B}$. Thus, for $A \in \mathfrak{A}$,

$$\begin{aligned}\omega(A) &= \int \omega(A)\, dg = \int \omega(\tau_g(A))\, dg = \omega\left(\int \tau_g(A)\, dg\right) \\ &= \omega(\Phi(A)) = (\omega | \mathfrak{B}) \circ \Phi(A)\,.\end{aligned}$$

Thus $\omega = \Phi^*(\omega | \mathfrak{B})$, so Φ^* is onto, hence an affine isomorphism of $S(\mathfrak{B})$ onto $I(\mathfrak{A})$. Now $S(\mathfrak{B})$ is a simplex if and only if $\mathfrak{B}$ is abelian. Hence the last statement follows.

If G is non compact the results in the above example are in general false; there may not even exist invariant states. But even with invariant states around the map Φ with all properties in Example 1.1 may not necessarily exist.

Example 1.2. Let $\mathscr{H}$ be a Hilbert space. Let $\mathfrak{C}(\mathscr{H})$ denote the completely continuous operators on $\mathscr{H}$. Let $\mathfrak{A}$ be an abelian C^*-algebra on $\mathscr{H}$ such that $\mathfrak{A} \cap \mathfrak{C}(\mathscr{H}) = \{0\}$. Then $\mathfrak{A} + \mathfrak{C}(\mathscr{H})$ is an irreducible C^*-algebra, and if G is the set of unitary operators in $\mathfrak{A} + \mathfrak{C}(\mathscr{H})$ then the fixed point algebra of the automorphisms $A \to UAU^{-1}$, $U \in G$, is the scalars, while $I(\mathfrak{A} + \mathfrak{C}(\mathscr{H})) = S(\mathfrak{A})$.

In fact, since the irreducible C^*-algebra $\mathfrak{A} + \mathfrak{C}(\mathscr{H})$ is generated by its unitary operators, the fixed point algebra is the scalars. If ϱ is a state of $\mathfrak{A} + \mathfrak{C}(\mathscr{H})$ which annihilates $\mathfrak{C}(\mathscr{H})$ then ϱ is a trace. Since every state of $\mathfrak{A}$ has such an extension to $\mathfrak{A} + \mathfrak{C}(\mathscr{H})$, we have obtained the desired identification of $I(\mathfrak{A} + \mathfrak{C}(\mathscr{H}))$ and $S(\mathfrak{A})$.

In our treatment of the problem we shall be able to construct a map Φ_ϱ like Φ in Example 1.1 of $\pi_\varrho(\mathfrak{A})$ for each G-invariant state ϱ, and then use similar arguments in order to obtain the desired results.

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2. Abelian von Neumann Algebras

If α is a $*$-automorphism of a von Neumann algebra $\mathfrak{A}$ then α maps the center $\mathfrak{C}$ of $\mathfrak{A}$ onto itself, hence $\alpha|\mathfrak{C}$ is a $*$-automorphism of $\mathfrak{C}$. It is therefore necessary to study $*$-automorphisms of abelian von Neumann algebras. This will be done in the present section. We shall throughout the section assume $\mathfrak{A}$ is an abelian von Neumann algebra acting on a Hilbert space $\mathscr{H}$, $\mathfrak{U}$ a group of unitary operators on $\mathscr{H}$ such that $U\mathfrak{A}U^{-1} = \mathfrak{A}$ for all $U \in \mathfrak{U}$. $\mathfrak{B}$ shall denote the fixed points in $\mathfrak{A}$ of the automorphisms. Hence $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{U}'$ is an abelian von Neumann algebra. We shall develop a comparison theory for projections in $\mathfrak{A}$ with respect to $\mathfrak{U}$, and then use this to prove a variant of DIXMIER's „Théorème d'approximation'' [4, Ch. III, § 5].

Definition 2.1. Let A and B be self-adjoint operators in $\mathfrak{A}$. We say A and B are $\mathfrak{U}$-*equivalent,* written $A \sim B$, if there exist operators A_α (resp. B_α) $\alpha \in J$, in $\mathfrak{A}$ such that $A_\alpha A_\beta = 0$ (resp. $B_\alpha B_\beta = 0$) if $\alpha \neq \beta$, $A = \sum_{\alpha\in J} A_\alpha$, $B = \sum_{\alpha\in J} B_\alpha$, and $U_\alpha \in \mathfrak{U}$ such that $B_\alpha = U_\alpha A_\alpha U_\alpha^{-1}$ for all $\alpha \in J$. If A is a projection so is B, and the A_α and B_α form orthogonal families of subprojections of A and B respectively. If E and F are projections in $\mathfrak{A}$ we write $E \prec F$ (or $F \succ E$) if $E \sim F_1 \leqq F$. By the $\mathfrak{U}$-*carrier* of E, denoted by $\mathfrak{U}_E$, we shall mean the least projection in $\mathfrak{B}$ greater than or equal to E. Note that since $I \in \mathfrak{B}$ and $\mathfrak{B}$ is a von Neumann algebra, this definition makes sense. If E and F are projections in $\mathfrak{A}$ we say they are $\mathfrak{U}$-*disjoint* if there exist no non zero $\mathfrak{U}$-equivalent subprojections E_1 and F_1 of E and F respectively.

Lemma 2.2. $\sim$ *is an equivalence relation.*

Proof. Since $I \in \mathfrak{U}$, $A \sim A$ for A self-adjoint in $\mathfrak{A}$. Suppose A, B and C are self-adjoint operators in $\mathfrak{A}$ such that $A \sim B$ and $B \sim C$. Then there exist orthogonal families $\{A_\alpha\}_{\alpha\in J}$, $\{B_\alpha\}_{\alpha\in J}$, $\{B'_\beta\}_{\beta\in J'}$, and $\{C_\beta\}_{\beta\in J'}$ such that $A = \Sigma A_\alpha$, $B = \Sigma B_\alpha = \Sigma B'_\beta$, $C = \Sigma C_\beta$, and unitary operators U_α, V_β in $\mathfrak{U}$ such that $U_\alpha A_\alpha U_\alpha^{-1} = B_\alpha$, $V_\beta B'_\beta V_\beta^{-1} = C_\beta$. Since $\mathfrak{A}$ is an abelian von Neumann algebra the self-adjoint operators in $\mathfrak{A}$ form a lattice. Let $B_{\alpha\beta} = B_\alpha \wedge B'_\beta$, for $\alpha \in J$, $\beta \in J'$. Then $B_\alpha = \sum_{\beta\in J'} B_{\alpha\beta}$, $B'_\beta = \sum_{\alpha\in J} B_{\alpha\beta}$, the sums being orthogonal. Let $A_{\alpha\beta} = U_\alpha^{-1} B_{\alpha\beta} U_\alpha$ and $C_{\alpha\beta} = V_\beta B_{\alpha\beta} V_\beta^{-1}$. Then it is immediate that $(\alpha, \beta) \neq (\alpha', \beta')$ implies $A_{\alpha\beta} A_{\alpha'\beta'} = 0 = C_{\alpha\beta} C_{\alpha'\beta'}$, so the families $\{A_{\alpha\beta}\}_{(\alpha,\beta)\in J\times J'}$ and $\{C_{\alpha\beta}\}_{(\alpha,\beta)\in J\times J'}$ are orthogonal. Moreover,

$$\sum_{\alpha\beta} A_{\alpha\beta} = \sum_\alpha U_\alpha^{-1} \sum_\beta B_{\alpha\beta} U_\alpha = \sum_\alpha U_\alpha^{-1} B_\alpha U_\alpha = \sum_\alpha A_\alpha = A ,$$

and similarly $\sum_{\alpha\beta} C_{\alpha\beta} = C$. Since $\mathfrak{U}$ is a group, and

$$(V_\beta U_\alpha)\, A_{\alpha\beta} (V_\beta U_\alpha)^{-1} = V_\beta B_{\alpha\beta} V_\beta^{-1} = C_{\alpha\beta}\,,$$

$A \sim C$. The proof is complete.

Lemma 2.3. *If E is a projection in $\mathfrak{A}$, $\mathfrak{U}_E$ is the maximal projection F in $\mathfrak{A}$ such that $F = \Sigma F_\alpha$, F_α orthogonal projections in $\mathfrak{A}$, and for each α there exist a projection $E_\alpha \leqq E$ in $\mathfrak{A}$ and $U_\alpha \in \mathfrak{U}$ such that $U_\alpha E_\alpha U_\alpha^{-1} = F_\alpha$.*

Proof. If $F = \Sigma F_\alpha$ is such a sum then $\mathfrak{U}_E F = \Sigma \mathfrak{U}_E F_\alpha = \Sigma \mathfrak{U}_E U_\alpha E_\alpha U_\alpha^{-1} = \Sigma U_\alpha \mathfrak{U}_E E_\alpha U_\alpha^{-1} = \Sigma U_\alpha E_\alpha U_\alpha^{-1} = F$, and $\mathfrak{U}_E \geqq F$, hence $\mathfrak{U}_E$ majorizes any maximal such F. The projections $F = \Sigma F_\alpha$ as above form a partially ordered set by inclusion, and each totally ordered subset has an upper bound, namely the union. Let $F = \Sigma F_\alpha$ be a maximal element. Then $F = UFU^{-1}$ for all $U \in \mathfrak{U}$. For otherwise there exists $U \in \mathfrak{U}$ such that $F_1 = (UFU^{-1})\,(I - F) \neq 0$. Then F_1 is a projection in $\mathfrak{A}$, and $G_1 = U^{-1} F_1 U \leqq F$ is of the same form as F, hence so is F_1, and therefore $F_1 + F$, contradicting the maximality of F. Thus $F = UFU^{-1}$ for all U in $\mathfrak{U}$, $F \in \mathfrak{B}$. Now $F \geqq E$, for if not the projection $E - FE$ could be adjoined to F, again contradicting the maximality of F. Thus $F \geqq \mathfrak{U}_E$, and they are equal.

Lemma 2.4. *Let E and F be projections in $\mathfrak{A}$. Then they are $\mathfrak{U}$-disjoint if and only if $\mathfrak{U}_E \mathfrak{U}_F = 0$.*

Proof. If E and F are not $\mathfrak{U}$-disjoint let E_1 and F_1 be non zero subprojections of E and F respectively such that there exists U in $\mathfrak{U}$ with $F_1 = U E_1 U^{-1}$. By Lemma 2.3 $E_1 \leqq \mathfrak{U}_E \mathfrak{U}_F$, so the latter projection is non zero. Conversely assume $G = \mathfrak{U}_E \mathfrak{U}_F \neq 0$. By Lemma 2.3 there exist non zero projections $E_\alpha \leqq E$, $F_\beta \leqq F$ and U_α, V_β in $\mathfrak{U}$ such that $G = \Sigma U_\alpha E_\alpha U_\alpha^{-1} = \Sigma V_\beta F_\beta V_\beta^{-1}$. Choose E_α and F_β such that

$$G_1 = U_\alpha E_\alpha U_\alpha^{-1} V_\beta F_\beta V_\beta^{-1} \neq 0\,.$$

Choose non zero projections E_1 and F_1 in $\mathfrak{A}$, $E_1 \leqq E$, $F_1 \leqq F$ such that $U_\alpha E_1 U_\alpha^{-1} = G_1 = V_\beta F_1 V_\beta^{-1}$. Then $F_1 = (V_\beta^{-1} U_\alpha)\, E_1 (V_\beta^{-1} U_\alpha)^{-1}$, so E and F are not $\mathfrak{U}$-disjoint.

From this point on the proof of the Comparison Theorem is a straightforward modification of the proof of the usual Comparison Theorem for von Neumann algebras. For completeness we shall include a proof based on one given by Kadison during lectures at Columbia University, Spring 1963. We remark that it is also easy to show that the ordering $\prec$ is a partial ordering of the projections in $\mathfrak{A}$. As we shall not need this the proof is omitted. The next lemma is immediate from the definition of $\sim$.

Lemma 2.5. *If $\{E_\alpha\}$ and $\{F_\alpha\}$ are each families of orthogonal projections in $\mathfrak{A}$ and $E_\alpha \sim F_\alpha$* (resp. $E_\alpha \prec F_\alpha$) *for all α, then $\Sigma E_\alpha \sim \Sigma F_\alpha$* (resp. $\Sigma E_\alpha \prec \Sigma F_\alpha$). *If G is a projection in $\mathfrak{B}$ and $E \sim F$ (resp. $E \prec F$) for E and F projections in $\mathfrak{A}$ then $GE \sim GF$ (resp. $GE \prec GF$).*

Lemma 2.6. *Let E and F be projections in $\mathfrak{A}$. Then there exists a non zero projection G in $\mathfrak{B}$ such that either $GE \prec GF$ or $GE \succ GF$.*

Proof. If $E \prec F$ or $F \prec E$ take $G = I$. We assume that $E \not\prec F$ and $F \not\prec E$. In particular $E \neq 0 \neq F$. If $\mathfrak{U}_E \mathfrak{U}_F = 0$ then $G = \mathfrak{U}_E \neq 0$ will work, for $0 \neq E = \mathfrak{U}_E E \succ \mathfrak{U}_E F = 0$. We may therefore assume $\mathfrak{U}_E \mathfrak{U}_F \neq 0$. Let $\mathscr{S}$ be the set of ordered pairs $\langle \{E_\alpha\}, \{F_\alpha\} \rangle$ with first term a family of mutually orthogonal subprojections of E contained in $\mathfrak{A}$ and $\{F_\alpha\}$ same for F, such that $E_\alpha \sim F_\alpha$. $\mathscr{S}$ is non null since E and F are not $\mathfrak{U}$-disjoint (Lemma 2.4). Partially order $\mathscr{S}$ by inclusion termwise (i.e. $\langle \{E_\alpha\}, \{F_\alpha\} \rangle \leqq \langle \{N_\beta\}, \{M_\beta\} \rangle$ if and only if $\{E_\alpha\} \subset \{N_\beta\}$ and $\{F_\alpha\} \subset \{M_\beta\}$). Let $\langle \{E_\alpha\}, \{F_\alpha\} \rangle$ be a maximal element. Let $E_0 = \Sigma E_\alpha$, $F_0 = \Sigma F_\alpha$, $E_1 = E - E_0$, and $F_1 = F - F_0$. By Lemma 2.5, $E_0 \sim F_0$. If $E_1 = 0$ then $E = E_0 \sim F_0 \leqq F$, so $E \prec F$, a case which is ruled out. Thus $E_1 \neq 0$, hence $\mathfrak{U}_{E_1} \neq 0$. Now $\mathfrak{U}_{E_1} \mathfrak{U}_{F_1} = 0$, for if not then by Lemma 2.4, E_1 and F_1 have non zero equivalent subprojections which could be adjoined to $\{E_\alpha\}$ and $\{F_\alpha\}$ respectively, contradicting the maximality of $\langle \{E_\alpha\}, \{F_\alpha\} \rangle$. Thus $\mathfrak{U}_{E_1} F_1 = 0$, and by Lemma 2.5 $\mathfrak{U}_{E_1} F = \mathfrak{U}_{E_1} F_0 \sim \mathfrak{U}_{E_1} E_0 \leqq \mathfrak{U}_{E_1} E$. The proof is complete.

Lemma 2.7. *(The Comparison Theorem.) Let E and F be two projections in $\mathfrak{A}$. Then there exist two projections G and G' in $\mathfrak{B}$ such that $G + G' = I$ and such that $GE \prec GF$ and $G'F \prec G'E$.*

Proof. Let $\{G_\alpha\}$ be a maximal orthogonal family of projections in $\mathfrak{B}$ such that $G_\alpha E \prec G_\alpha F$, and let $\{G'_\beta\}$ be a similar such family with $G'_\beta E \succ G'_\beta F$ for all α, β. Let $G_0 = \Sigma G_\alpha$, $G'_0 = \Sigma G'_\beta$. By Lemma 2.5, $G_0 E \prec G_0 F$ and $G'_0 F \prec G'_0 E$. Let $S = I - G_0 - G'_0 + G_0 G'_0$. If $S \neq 0$ there exists by Lemma 2.6 some non zero projection P in $\mathfrak{B}$, $P \leqq S$, such that $PE \prec PF$ or $PE \succ PF$. In the first case $\{P, G_\alpha\}$ contradicts the maximality of $\{G_\alpha\}$, and in the second $\{P, G'_\beta\}$ that of $\{G'_\beta\}$. Thus $S = 0$, $I = G_0 + G'_0 - G_0 G'_0$. Let $G = G_0$, $G' = G'_0 - G_0 G'_0$. Then $G, G' \in \mathfrak{B}$, $G + G' = I$, and by Lemma 2.5, $GE \prec GF$, $G'F \prec G'E$.

We shall next begin the proof of an analogue of DIXMIER's «Théorème d'approximation», and complete it in the next section. Our proof will be modelled on that of DIXMIER [4, Ch. III, § 5], but will, due to our more complicated ordering, be more technical. If T is a self-adjoint operator in $\mathfrak{A}$, E a non zero projection in $\mathfrak{A}$ then we put

$$M_E(T) = \sup\{\sum_\alpha \omega_{x_\alpha}(T) : E x_\alpha = x_\alpha \in \mathscr{H}, \text{ and } \sum_\alpha \|x_\alpha\|^2 = 1\} .$$

$$m_E(T) = \inf\{\sum_\alpha \omega_{x_\alpha}(T) : E x_\alpha = x_\alpha \in \mathscr{H}, \text{ and } \sum_\alpha \|x_\alpha\|^2 = 1\} .$$

$$\omega_E(T) = M_E(T) - m_E(T) .$$

If $E = I$ we drop the subscripts and write $M(T)$, $m(T)$, and $\omega(T)$. If $\mathfrak{F}$ is a family of orthogonal projections in $\mathfrak{A}$ we put $\omega_{\mathfrak{F}}(T) = \sup_{E \in \mathfrak{F}} \omega_E(T)$.

Lemma 2.8. *Let T be a self-adjoint operator in $\mathfrak{A}$. Then there exist two orthogonal projections G and G' in $\mathfrak{B}$ with sum I and a self-adjoint operator $S \in \mathfrak{A}$, $S \sim T$, such that*

1) $$\omega_G(\tfrac{1}{2}(T+S)) \leqq \tfrac{3}{4}\,\omega(T)$$

2) $$\omega_{G'}(\tfrac{1}{2}(T+S)) \leqq \tfrac{3}{4}\,\omega(T)\,.$$

Proof. Let $n(T) = \frac{1}{2}\,(M(T) + m(T))$. There exist spectral projections E and F of T such that $M_E(T) \leqq n(T)$, $m_F(T) \geqq n(T)$. By Lemma 2.7 there exist two orthogonal projections G and G' in $\mathfrak{B}$ with sum I such that $EG \prec FG$ and $FG' \prec EG'$. Thus there exist orthogonal projections E_α in $\mathfrak{A}$ such that $GE = \Sigma E_\alpha$, and there exist U_α in $\mathfrak{U}$ such that the projections $U_\alpha E_\alpha U_\alpha^{-1}$ are all orthogonal, and $G_1 = \Sigma U_\alpha E_\alpha U_\alpha^{-1} \leqq$ $\leqq FG$. Similarly there exist orthogonal projections F_β in $\mathfrak{A}$ such that $\Sigma F_\beta = G'F$, and $W_\beta \in \mathfrak{U}$ such that the projections $W_\beta F_\beta W_\beta^{-1}$ are orthogonal, and $G_1' = \Sigma W_\beta F_\beta W_\beta^{-1} \leqq EG'$. Let $F_\alpha' = U_\alpha E_\alpha U_\alpha^{-1}$, so $G_1 = \Sigma F_\alpha'$. Let $T_\alpha' = TF_\alpha'$, so $TG_1 = \Sigma T_\alpha'$. Let $E_\beta' = W_\beta F_\beta W_\beta^{-1}$, so $G_1' = \Sigma E_\beta'$. Let $T_\beta' = TE_\beta'$, so $TG_1' = \Sigma T_\beta'$. Let $T_\alpha = TE_\alpha$, $T_\beta = TF_\beta$. Thus $TGE = \Sigma T_\alpha$, $TG'F = \Sigma T_\beta$. T can thus be written as an orthogonal sum as follows:

$$\begin{aligned} T &= TEG + TG_1 + T(FG - G_1) + TFG' + TG_1' + T(EG' - G_1') \\ &= \Sigma T_\alpha + \Sigma T_\alpha' + T(FG - G_1) + \Sigma T_\beta + \Sigma T_\beta' + T(EG' - G_1')\,. \end{aligned}$$

Let

$$\begin{aligned} S = \Sigma U_\alpha T_\alpha U_\alpha^{-1} &+ \Sigma U_\alpha^{-1} T_\alpha' U_\alpha + T(FG - G_1) + \\ &+ \Sigma W_\beta T_\beta W_\beta^{-1} + \Sigma W_\beta^{-1} T_\beta' W_\beta + T(EG' - G_1')\,. \end{aligned}$$

Then all summands in sums for S and T are orthogonal, so $S \sim T$. Note that

$$T_\alpha' U_\alpha E = TF_\alpha' U_\alpha E = TU_\alpha E_\alpha U_\alpha^{-1} U_\alpha E = TU_\alpha E_\alpha\,.$$

Therefore

$$\begin{aligned} &m_{EG}(\Sigma U_\alpha^{-1} T_\alpha' U_\alpha) \\ &\quad = \inf\{\sum_\varrho \omega_{x_\varrho}(\sum_\alpha U_\alpha^{-1} T_\alpha' U_\alpha) : GEx_\varrho = x_\varrho \in \mathscr{H}, \Sigma\|x_\varrho\|^2 = 1\} \\ &\quad = \inf\{\sum_{\varrho\,\alpha} (TU_\alpha E_\alpha x_\varrho, U_\alpha E_\alpha x_\varrho) : GEx_\varrho = x_\varrho \in \mathscr{H}, \Sigma\|x_\varrho\|^2 = 1\}\,. \\ &\quad = \inf\{\sum_{\varrho\,\alpha} \omega_{U_\alpha E_\alpha x_\varrho}(T) : GEx_\varrho = x_\varrho \in \mathscr{H}, \Sigma\|x_\varrho\|^2 = 1\}\,. \end{aligned}$$

Now,

$$\sum_{\varrho\,\alpha} \|U_\alpha E_\alpha x_\varrho\|^2 = \sum_\varrho \sum_\alpha \|E_\alpha x_\varrho\|^2 = \sum_\varrho \|x_\varrho\|^2 = 1,$$

since

$$\Sigma E_\alpha = EG,$$

and

$$EGx_\varrho = x_\varrho.$$

Since also

$$U_\alpha E_\alpha x_\varrho \in F'_\alpha \leqq G_1,$$

we have

$$m_{EG}(\Sigma U_\alpha^{-1} T'_\alpha U_\alpha) \geqq m_{G_1}(T) \geqq m_F(T) \geqq n(T)\,.$$

Consequently,

$$\begin{aligned} TG &= TEG + TG_1 + T(FG - G_1) \geqq \\ &\geqq m(T)\,EG + n(T)\,G_1 + n(T)\,(FG - G_1) \\ SG &= (\Sigma U_\alpha T_\alpha U_\alpha^{-1})\,G_1 + (\Sigma U_\alpha^{-1} T'_\alpha U_\alpha)\,EG + T(FG - G_1) \geqq \\ &\geqq m(T)\,G_1 + n(T)\,EG + n(T)\,(FG - G_1)\,. \end{aligned}$$

Adding these two inequalities we obtain,

$$\begin{aligned} &\tfrac{1}{2}(T+S)\,G \geqq \tfrac{1}{2}(m(T)+n(T))\,(G_1 + EG) + n(T)\,(FG - G_1) \geqq \\ &\qquad \geqq \tfrac{1}{2}(m(T)+n(T))\,(G_1 + EG + FG - G_1) = (M(T) - \tfrac{3}{4}\omega(T))G\,. \end{aligned}$$

Thus

$$m_G(\tfrac{1}{2}\,(T+S)G) \geqq M(T) - \tfrac{3}{4}\,\omega(T).$$

Since clearly,

$$M_G(\tfrac{1}{2}(T+S)) \leqq M(T)\,,$$

(1) follows.

In order to show (2), a computation like the one above shows $M_{FG'}(\Sigma W_\beta^{-1} T'_\beta W_\beta) \leqq n(T)$. Thus

$$\begin{aligned} TG' &\leqq M(T)\,FG' + n(T)\,G'_1 + n(T)\,(EG' - G'_1) \\ SG' &\leqq M(T)\,G'_1 + n(T)\,FG' + n(T)\,(EG' - G'_1)\,. \end{aligned}$$

Now continue as above to show (2).

Lemma 2.9. *Let T be a self-adjoint operator in $\mathfrak{A}$, $\mathfrak{F}$ a finite family of orthogonal projections in $\mathfrak{B}$ with sum I. Then there exists a finite family $\mathfrak{F}'$ of orthogonal projections in $\mathfrak{B}$ with sum I and $S \sim T$ in $\mathfrak{A}$ such that*

$$\omega_{\mathfrak{F}'}(\tfrac{1}{2}\,(T+S)) \leqq \tfrac{3}{4}\,\omega(T)\,.$$

Proof. If the lemma is true for a finite family of von Neumann algebras, it also follows for their product. The proof is therefore reduced to the case when $\mathfrak{F}$ consists of the projection I, hence the lemma is a consequence of Lemma 2.8, with $\mathfrak{F}'$ consisting of two operators.

With T self-adjoint in $\mathfrak{A}$ let

$$\mathfrak{L}_1 = \operatorname{conv}(S : S \in \mathfrak{A},\, S \sim T)\,.$$

Inductively we let for $n \geqq 2$,

$$\mathfrak{L}_n = \operatorname{conv}(S : S \in \mathfrak{A},\, S \sim S',\, S' \in \mathfrak{L}_{n-1})\,.$$

Then $\mathfrak{L}_n \supset \mathfrak{L}_{n-1}$. We put $\mathfrak{L}_T$ equal to the norm closure of $\bigcup_{n=1}^{\infty} \mathfrak{L}_n$.

Lemma 2.10. *Let T be self-adjoint in $\mathfrak{A}$, and let $\varepsilon > 0$ be given. Then there exist a positive integer n, $S \in \mathfrak{L}_n$, and an operator B in $\mathfrak{B}$ such that $\|S - B\| < \varepsilon$.*

Proof. For each integer $p > 0$ there exist a family $\mathfrak{F} = \{E_1, \ldots, E_m\}$ of orthogonal projections in $\mathfrak{B}$ such that $\Sigma E_j = I$, an integer n, and $S \in \mathfrak{L}_n$ such that

$$\omega_{\mathfrak{F}}(S) < (\tfrac{3}{4})^p \omega(T) .$$

This follows from Lemma 2.9 by iterations of p. If we choose the real numbers α_j properly, we have

$$\|S - \sum_{j=1}^{m} \alpha_j E_j\| < (\tfrac{3}{4})^p \omega(T) .$$

Let $B = \Sigma \alpha_j E_j$. Then $B \in \mathfrak{B}$, and if p is sufficiently large $\|S - B\| < \varepsilon$.

3. Automorphisms of C^*-algebras

The main results of the paper appear in this section. We first study large groups of spatial automorphisms, in which case our treatment will be related to Dixmier's study of the center trace in finite von Neumann algebras [4, Ch. III, §§ 4, 5], and should also be compared with Example 1.1. Then this theorem is applied to representations of groups as $*$-automorphisms of C^*-algebras.

Theorem 3.1. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space $\mathscr{H}$. Let $\mathfrak{U}$ be a group of unitary operators on $\mathscr{H}$ such that $U \mathfrak{A} U^{-1} = \mathfrak{A}$ for all $U \in \mathfrak{U}$. Let $\mathfrak{C}$ denote the center of $\mathfrak{A}^-$, let $\mathfrak{B} = \mathfrak{C} \cap \mathfrak{U}'$, and let $(\mathfrak{A}, \mathfrak{U})$ denote the von Neumann algebra generated by $\mathfrak{A}$ and $\mathfrak{U}$. Assume there exists a normal $\mathfrak{U}$-invariant state ω which is faithful on $\mathfrak{B}$, and that for each self-adjoint operator A in $\mathfrak{A}$*

$$\operatorname{conv}(U A U^{-1} : U \in \mathfrak{U})^- \cap \mathfrak{A}' \neq \emptyset .$$

Then there exists a unique normal $\mathfrak{U}$-invariant positive linear map Φ of $\mathfrak{A}^-$ onto $\mathfrak{B}$ such that

1) *$\Phi(AB) = \Phi(A) B$ and $\Phi(I) = I$ whenever $A \in \mathfrak{A}^-$, $B \in \mathfrak{B}$.*

2) *If ϱ is a normal $\mathfrak{U}$-invariant state of $\mathfrak{A}$ then $\varrho = (\varrho | \mathfrak{B}) \circ \Phi$.*

If moreover we assume ω is of the form ω_x with x a unit vector in $\mathscr{H}$ cyclic under $\mathfrak{A}$ such that $Ux = x$ for all U in $\mathfrak{U}$ then the following properties hold:

3) *$(\mathfrak{A}, \mathfrak{U})' = \mathfrak{B}$.*

4) *If P denotes the projection onto the set of vectors y in $\mathscr{H}$ such that $Uy = y$ for all U in $\mathfrak{U}$, then P is an abelian projection in $(\mathfrak{A}, \mathfrak{U})$ with central carrier I, hence $P(\mathfrak{A}, \mathfrak{U}) P = \mathfrak{B} P$.*

5) *If ϱ is a $\mathfrak{U}$-invariant positive linear functional of $\mathfrak{A}$ such that $\varrho \leqq \lambda \omega_x$ on $\mathfrak{A}$ for some $\lambda > 0$, then there exists a unique positive operator B_ϱ in $\mathfrak{B}$ such that $\varrho = \omega_x(B_\varrho \cdot)$ The mapping $\varrho \to B_\varrho$ is an order-isomorphism between such functionals and $\mathfrak{B}^+$.*

Proof. Notice that by weak continuity, $U \mathfrak{A}^- U^{-1} = \mathfrak{A}^-$, and $U \mathfrak{C} U^{-1} = \mathfrak{C}$ for all $U \in \mathfrak{U}$. Furthermore, $\mathfrak{A} \cap \mathfrak{U}' \subset \mathfrak{B}$, for if $B \in \mathfrak{A} \cap \mathfrak{U}'$ then $\operatorname{conv}(UBU^{-1} : U \in \mathfrak{U}) = \{B\}$, so by hypothesis $B \in \mathfrak{A} \cap \mathfrak{A}' \subset \mathfrak{C}$.

By [4, Théorème 1, p. 117] there exist a locally compact Hausdorff space Z, a positive measure ν on Z, and an isomorphism of the normed $*$-algebra $L_c^\infty(Z, \nu)$ onto $\mathfrak{B}$, and we identify $\mathfrak{B}$ and $L_c^\infty(Z, \nu)$ by this isomorphism. Then the positive part $\mathfrak{B}^+$ of $\mathfrak{B}$ is imbedded in the set $\widehat{\mathfrak{B}}^+$ of positive measurable functions, finite or not, on Z. As is shown on [4, p. 262] ω extends uniquely to a faithful normal trace on $\widehat{\mathfrak{B}}^+$. Now follow [4, Prop. 3, p. 264]. Let $S \in \overline{\mathfrak{A}}^+$. The map $T \to \omega(ST)$, $T \in \mathfrak{B}$, is a normal trace on $\mathfrak{B}$. By [4, Lemme 1, p. 262] there exists a unique element $\Phi(S) \in \widehat{\mathfrak{B}}^+$ such that

$$\omega(ST) = \omega(\Phi(S)T)\,, \tag{$*$}$$

for all T in $\mathfrak{B}^+$. Let $S, S_1 \in \overline{\mathfrak{A}}^+$, $U \in \mathfrak{U}$, $T, T_1 \in \mathfrak{B}^+$. Then

$$\begin{aligned}
\omega(\Phi(S+S_1)T) &= \omega((S+S_1)T) = \omega(ST) + \omega(S_1T)\\
&= \omega\big((\Phi(S) + \Phi(S_1))T\big)\,,\\
\omega(\Phi(ST_1)T) &= \omega(ST_1T) = \omega(\Phi(S)T_1T)\,,\\
\omega(\Phi(USU^{-1})T) &= \omega(USU^{-1}T) = \omega(USTU^{-1}) = \omega(ST)\\
&= \omega(\Phi(S)T)\,.
\end{aligned}$$

By uniqueness of $\Phi(S)$ such that $(*)$ holds, Φ is linear $\mathfrak{U}$-invariant, and $\Phi(ST_1) = \Phi(S)T_1$. Moreover, by $(*)$ $\Phi(I) = I$. The argument in [4, Lemme 1, p. 262] shows $\Phi(S)$ is the unique operator in $\widehat{\mathfrak{B}}$ such that $(*)$ holds and that Φ can be extended by linearity to a positive linear $\mathfrak{U}$-invariant map of norm 1 of $\mathfrak{A}^-$ into $\widehat{\mathfrak{B}}$ such that $\Phi|\mathfrak{B}$ is the identity map. Φ is normal. In fact, let $\{S_\alpha\}$ be a monotone increasing generalized sequence in $\overline{\mathfrak{A}}^+$ with least upper bound S in $\overline{\mathfrak{A}}^+$. Then, since Φ is positive and of norm 1, $\{\Phi(S_\alpha)\}$ is a monotone, increasing generalized sequence in $\widehat{\mathfrak{B}}^+$ with a least upper bound Q. Since $\omega(S_\alpha T) = \omega(\Phi(S_\alpha)T)$ for $T \in \mathfrak{B}^+$, $\omega(ST) = \lim_\alpha \omega(S_\alpha T) = \lim_\alpha \omega(\Phi(S_\alpha)T) = \omega(QT)$, so $Q = \Phi(S)$ by uniqueness. Φ is normal.

Now $\Phi : \mathfrak{A}^- \to \mathfrak{B}$. In fact, since $\mathfrak{B} = \mathfrak{C} \cap \mathfrak{U}'$ and $U\mathfrak{C}U^{-1} = \mathfrak{C}$ for all U in $\mathfrak{U}$ the results of section 2 are applicable to $\mathfrak{C}$, $\mathfrak{B}$ and $\mathfrak{U}$. Let A be a self-adjoint operator in $\mathfrak{A}$. Let $\mathfrak{K}_A = \operatorname{conv}(UAU^{-1} : U \in \mathfrak{U})$. By hypothesis $\mathfrak{K}_A^- \cap \mathfrak{C} \neq \emptyset$. Say $B \in \mathfrak{K}_A^- \cap \mathfrak{C}$. Then for each positive integer j there exist by Lemma 2.10 operators $S_j \in \mathfrak{L}_B$, $B_j \in \mathfrak{B}$ such that $\|S_j - B_j\| < 2^{-j}$. Now Φ is normal, hence weakly continuous on the bounded sets $\mathfrak{K}_A^-$ and $\mathfrak{L}_B^-$. Since also Φ is $\mathfrak{U}$-invariant $\Phi(\mathfrak{K}_A^-) = \Phi(\mathfrak{L}_B)$ $= \{\Phi(A)\}$. Thus $\Phi\ (S_j) = \Phi(B) = \Phi(A)$ for all j, and $\Phi(B_j) = B_j$. Thus

$$\|\Phi(A) - B_j\| = \|\Phi(S_j - B_j)\| \leqq \|S_j - B_j\| < 2^{-j}\,.$$

In particular, $\{B_j\}$ is a Cauchy sequence converging uniformly to $\Phi(A)$. Thus $\Phi(A) \in \mathfrak{B}$, $\Phi : \mathfrak{A} \to \mathfrak{B}$, and by continuity $\Phi : \mathfrak{A}^- \to \mathfrak{B}$ as asserted. Furthermore,

$$\|\Phi(A) - S_j\| \leqq \|\Phi(A) - B_j\| + \|B_j - S_j\| < 2^{-j} + 2^{-j} = 2^{-j+1}\,,$$

so $S_j \to \Phi(A)$ uniformly; in particular $\Phi(A) \in \mathfrak{L}_B \cap \mathfrak{B}$. Let now $\mathfrak{M}_A = \operatorname{conv}(\mathfrak{L}_B : B \in \mathfrak{K}_A^- \cap \mathfrak{C})^-$. Then by the weak continuity of Φ on bounded sets, $\Phi(\mathfrak{M}_A) = \{\Phi(A)\}$, and $\Phi(A) \in \mathfrak{M}_A \cap \mathfrak{B}$. Let Φ_1 be another map with the same properties as Φ. Then by the arguments above applied to Φ_1, $\Phi_1(A) \in \mathfrak{M}_A \cap \mathfrak{B}$. However, if $T \in \mathfrak{M}_A \cap \mathfrak{B}$, then $T = \Phi(T) = \Phi(A)$, so $\mathfrak{M}_A \cap \mathfrak{B} = \{\Phi(A)\}$, and $\Phi_1 = \Phi$, Φ is unique. This completes (1) and assertions of Φ preceding (1).

In order to prove (2) let ϱ be a normal $\mathfrak{U}$-invariant state of $\mathfrak{A}$. Then by the arguments above, $\{\varrho(A)\} = \varrho(\mathfrak{M}_A)$ for each self-adjoint operator A in $\mathfrak{A}$. Thus $\varrho(A) = \varrho(\mathfrak{M}_A \cap \mathfrak{B}) = \varrho(\Phi(A)) = (\varrho|\mathfrak{B}) \circ \Phi(A)$, and (2) is proved.

Assume now $\omega = \omega_x$ with x a unit vector in $\mathscr{H}$ cyclic under $\mathfrak{A}$ and such that $Ux = x$ for all U in $\mathfrak{U}$. Then x is a separating vector for $\mathfrak{A}'$, so in particular for $\mathfrak{C}$, and hence for $\mathfrak{B}$. Thus ω_x is faithful on $\mathfrak{B}$. By definition $\mathfrak{B} \subset \mathfrak{A}' \cap \mathfrak{U}' = (\mathfrak{A}, \mathfrak{U})'$. Let B' be a positive operator in the von Neumann algebra $\mathfrak{A}' \cap \mathfrak{U}'$. Then $\omega_{B'x}$ is a normal $\mathfrak{U}$-invariant positive linear functional on $\mathfrak{A}$. In fact, if $U \in \mathfrak{U}$ and $A \in \mathfrak{A}$ then

$$\omega_{B'x}(U^{-1}AU) = (AUB'x, UB'x) = (AB'x, B'x) = \omega_{B'x}(A)\,.$$

Now $\omega_{B'x} \leq \|B'\|^2 \omega_x$ on $\mathfrak{A}$. By the Radon-Nikodym Theorem, see e.g. [4, Théorème 3, p. 89], there exists a unique positive operator B in $\mathfrak{B}$ such that $\omega_{B'x} = \omega_{Bx}$ on $\mathfrak{B}$. But ω_{Bx} is $\mathfrak{U}$-invariant, so by (2) $\omega_{B'x}(A) = \omega_{B'x}(\Phi(A)) = \omega_{Bx}(\Phi(A)) = \omega_{Bx}(A)$ for all A in $\mathfrak{A}$. Thus for S, T in $\mathfrak{A}$,

$$(B'^2 Sx, Tx) = (T^* S B'x, B'x) = (T^* S Bx, Bx) = (B^2 Sx, Tx)\,.$$

Since x is cyclic under $\mathfrak{A}$, $B'^2 = B^2$. Since B' and B are both positive, $B' = B \in \mathfrak{B}$, and $\mathfrak{A}' \cap \mathfrak{U}' = \mathfrak{B}$, (3) is proved.

Let P denote the projection onto the subspace of vectors y such that $Uy = y$ for all U in $\mathfrak{U}$. Then $Px = x$ so $P \neq 0$. If $y \in \mathscr{H}$ then $UPy = Py$ for all U in $\mathfrak{U}$, so $UP = P = (UP)^* = PU^*$, and $UP = P = PU$. If $B \in \mathfrak{B}$ is self-adjoint then $UBP = BUP = BP$, so for $y \in \mathscr{H}$, $PBPy = BPy$, and $BP = PBP = (PBP)^* = PB$. Thus $P \in \mathfrak{B}' = (\mathfrak{A}, \mathfrak{U})$, by (3) and the Double Commutant Theorem. Let $A \in \mathfrak{A}$. Then $PAP = PUAU^{-1}P$ by the above for all U in $\mathfrak{U}$. Let $S \in \mathfrak{M}_A$. Then a trivial argument shows $PAP = PSP$. Hence $PAP = P\Phi(A)P = \Phi(A)\,P \in \mathfrak{B}P$. Operators of the form $\Sigma \prod_{j=1}^{n} A_j U_j$, the sum being finite with $A_j \in \mathfrak{A}$, $U_j \in \mathfrak{U}$, are weakly dense in $(\mathfrak{A}, \mathfrak{U})$. Therefore, in order to show $P(\mathfrak{A}, \mathfrak{U})P = \mathfrak{B}P$ it suffices to show $P \prod_{j=1}^{n} A_j U_j P \in \mathfrak{B}P$ for each positive integer n. If $n = 1$, $PA_1U_1P = PA_1P \in \mathfrak{B}P$ by the preceding argument. Assume it true for $n-1$. Then

$$P\prod_{j=1}^{n} A_j U_j P = P\left(\prod_{j=1}^{n-2} A_j U_j\right) A_{n-1} U_{n-1} A_n U_n P$$
$$= P\left(\prod_{j=1}^{n-2} A_j U_j\right) (A_{n-1} U_{n-1} A_n U_{n-1}^{-1}) IP \in \mathfrak{B}P$$

by the induction assumption. Hence $P(\mathfrak{A}, \mathfrak{U})P = \mathfrak{B}P$. Since x is a separating vector for $\mathfrak{B}$, and $Px = x$, P is separating for $\mathfrak{B}$. Since $\mathfrak{B}$ equals the center of $(\mathfrak{A}, \mathfrak{U})$ the central carrier of P in $(\mathfrak{A}, \mathfrak{U})$ equals I, (4) is proved.

Let ϱ be a $\mathfrak{U}$-invariant positive linear functional of $\mathfrak{A}$ such that $\varrho \leqq \lambda\omega_x$ for some positive real number λ. Then there exists by [4, Lemme 1, p. 50] $y \in \mathscr{H}$ such that $\varrho = \omega_y$, hence ϱ is normal. As was pointed out in the proof of (3) there exists a unique positive operator B_ϱ in $\mathfrak{B}$ such that $\varrho(A) = \omega_x(B_\varrho A)$ for all A in $\mathfrak{A}$. Thus the mapping $\varrho \to B_\varrho$ is positive, and by uniqueness linear. If $B \in \mathfrak{B}$ and the functional $A \to \omega_x(BA)$ is positive on $\mathfrak{A}$ then, since it is $\mathfrak{U}$-invariant and majorized by a multiple of ω_x, there exists a positive operator B' in $\mathfrak{B}$ such that $\omega_x(AB) = \omega_x(AB')$ for all $A \in \mathfrak{A}$. The argument used in proving (3) shows $B = B'$, hence $B \geqq 0$, and the mapping $\varrho \to B_\varrho$ is an order-isomorphism, i.e. if ϱ_1 and ϱ_2 are $\mathfrak{U}$-invariant positive linear functionals of $\mathfrak{A}$ majorized by multiples of ω_x then $B_{\varrho_1+\varrho_2} = B_{\varrho_1} + B_{\varrho_2}$, and $\varrho_1 \leqq \varrho_2$ on $\mathfrak{A}$ if and only if $B_{\varrho_1} \leqq B_{\varrho_2}$. The proof is complete.

In section 2 we promised to prove a variant of DIXMIER's «Théorème d'approximation» when $\mathfrak{A}$ is abelian. This can now be done if $\mathfrak{A}$ is assumed to be "finite".

Corollary 3.2. *Let $\mathfrak{A}$ be an abelian von Neumann algebra, $\mathfrak{U}$ a group of unitary operators such that $U\mathfrak{A}U^{-1} = \mathfrak{A}$ for all U in $\mathfrak{U}$. Let $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{U}'$ and assume there exists a normal $\mathfrak{U}$-invariant state which is faithful on $\mathfrak{B}$. Let A be a self-adjoint operator in $\mathfrak{A}$, and let $\mathfrak{L}_A$ be defined as in section 2. Then $\mathfrak{L}_A \cap \mathfrak{B}$ consists of exactly one operator.*

Proof. With Φ the mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ constructed in the above theorem we showed $\mathfrak{L}_A \cap \mathfrak{B} = \{\Phi(A)\}$ if A belonged to the center.

We shall now apply Theorem 3.1 to representations of groups as $*$-automorphisms of C^*-algebras. If ϱ is a state of a C^*-algebra $\mathfrak{A}$ then ϱ has a unique decomposition $\varrho = \omega_{x_\varrho} \circ \pi_\varrho$ as a composition of a vector state and a cyclic representation. We shall mainly be concerned with large groups.

Definition 3.3. Let $\mathfrak{A}$ be a C^*-algebra and G a group. Let $g \to \tau_g$ be a representation of G as $*$-automorphisms of $\mathfrak{A}$. We say G is represented by τ as a *large group of automorphisms of* $\mathfrak{A}$ if for all G-invariant states ϱ and all self-adjoint A in $\mathfrak{A}$

$$\operatorname{conv}\big(\pi_\varrho(\tau_g(A)) : g \in G\big)^- \cap \pi_\varrho(\mathfrak{A})' \neq \emptyset .$$

Once and for all we introduce some notation.

Notation 3.4. Let $\mathfrak{A}$ be a C^*-algebra, G a group, and $g \to \tau_g$ a representation of G as $*$-automorphisms of $\mathfrak{A}$. $I(\mathfrak{A})$ denotes the G-invariant states of $\mathfrak{A}$, i.e. the convex compact set of states ϱ of $\mathfrak{A}$ such that $\varrho \circ \tau_g = \varrho$ for all $g \in G$. If $\varrho \in I(\mathfrak{A})$, $\varrho = \omega_{x_\varrho} \circ \pi_\varrho$, where π_ϱ is a representation of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_\varrho$, and x_ϱ is a unit vector in $\mathscr{H}_\varrho$ cyclic under $\pi_\varrho(\mathfrak{A})$. There is a unitary representation $g \to U_\varrho(g)$ of G on $\mathscr{H}_\varrho$, strongly continuous if G is a topological group and $g \to \tau_g$ is strongly continuous, such that $U_\varrho(g) x_\varrho = x_\varrho$ and $\pi_\varrho(\tau_g(A)) = U_\varrho(g)\, \pi_\varrho(A)\, U_\varrho(g)^{-1}$ for all $g \in G$ and A in $\mathfrak{A}$, see [16]. We denote by $\mathfrak{U}(\varrho)$ the group of the unitary operators $U_\varrho(g)$, by $\mathfrak{C}(\varrho)$ the center of $\pi_\varrho(\mathfrak{A})^-$, and by $\mathfrak{B}(\varrho)$ the von Neumann algebra $\mathfrak{C}(\varrho) \cap \mathfrak{U}(\varrho)'$. We shall denote by P_ϱ the projection onto the set of vectors y in $\mathscr{H}_\varrho$ for which $U_\varrho(g) y = y$ for all $g \in G$. Then $P_\varrho x_\varrho = x_\varrho$ so $P_\varrho \neq 0$. We denote by $\mathfrak{K}_\varrho(A)$ the set

$$\operatorname{conv}(\pi_\varrho(\tau_g(A)) : g \in G)$$

for each self-adjoint A in $\mathfrak{A}$. Thus

$$\mathfrak{K}_\varrho(A) = \operatorname{conv}(U_\varrho(g)\, \pi_\varrho(A)\, U_\varrho(g)^{-1} : g \in G)\,.$$

In this notation τ represents G as a large group of automorphisms if and only if $\mathfrak{K}_\varrho(A)^- \cap \pi_\varrho(\mathfrak{A})' \neq \emptyset$ for all $\varrho \in I(\mathfrak{A})$. Thus Theorem 3.1 is applicable to $\pi_\varrho(\mathfrak{A})$ and $\mathfrak{U}(\varrho)$. We denote by Φ_ϱ the map Φ constructed in Theorem 3.1 of $\pi_\varrho(\mathfrak{A})^-$ onto $\mathfrak{B}(\varrho)$.

It should be remarked that if G is represented as a large group of automorphisms of $\mathfrak{A}$ then by Theorem 3.1 (4), $P_\varrho\, \pi_\varrho(\mathfrak{A}) P_\varrho$ is an abelian family of operators for each $\varrho \in I(\mathfrak{A})$. This means that $\mathfrak{A}$ is G-abelian in the sense of Lanford and Ruelle [14]. They show [14, Theorem 2.3] that $\mathfrak{A}$ is G-abelian if and only if for all self-adjoint A, B in $\mathfrak{A}$, and all $\varrho \in I(\mathfrak{A})$

$$\inf |\varrho([A', B])| = 0\,,$$

as A' runs over $\operatorname{conv}(\tau_g(A) : g \in G)$, where $[\ ,\ ]$ denotes the commutator. We shall now characterize large groups in a similar fashion, thus pointing out how our structure is stricter than theirs. If ϱ is a state of $\mathfrak{A}$ and $S \in \mathfrak{A}$ we denote by ϱ_S the positive linear functional $A \to \varrho(S^* A S)$ of $\mathfrak{A}$.

Theorem 3.5. *Let $\mathfrak{A}$ be a C^*-algebra, G a group, and $g \to \tau_g$ a representation of G as $*$-automorphisms of $\mathfrak{A}$. Then τ represents G as a large group of automorphisms of $\mathfrak{A}$ if and only if for each $\varrho \in I(\mathfrak{A})$, and each finite family $\{A, B_1, \ldots, B_n\}$ of self-adjoint operators in $\mathfrak{A}$*

$$\inf |\varrho_S([A', B_j])| = 0, \quad j = 1, \ldots, n\,,$$

for all S in $\mathfrak{A}$, where A' ranges through $\operatorname{conv}(\tau_g(A) : g \in G)$.

Proof. If $I(\mathfrak{A}) = \emptyset$ the theorem is trivial, so assume $\varrho \in I(\mathfrak{A})$. Assume G is large. Choose a net $\{\pi_\varrho(A_\alpha)\}$ in $\mathfrak{K}_\varrho(A)$ such that $\pi_\varrho(A_\alpha)$ converges weakly to an operator $D \in \mathfrak{K}_\varrho(A)^- \cap \pi_\varrho(\mathfrak{A})'$. Then

$$[\pi_\varrho(A_\alpha), \pi_\varrho(B_j)] \to [D, \pi_\varrho(B_j)] = 0, \quad j = 1, \ldots, n,$$

weakly, whenever $B_1, \ldots, B_n$ is a finite set of self-adjoint operators in $\mathfrak{A}$. Therefore, for given $\varepsilon > 0$ and S in $\mathfrak{A}$ there exists α such that

$$|\varrho_S([A_\alpha, B_j])| = |([\pi_\varrho(A_\alpha), \pi_\varrho(B_j)]\,\pi_\varrho(S)x_\varrho, \pi_\varrho(S)x_\varrho)| < \varepsilon .$$

Thus $\inf|\varrho_S([A', B_j])| = 0$ for all S in $\mathfrak{A}$. Conversely assume this condition holds. Then, since x_ϱ is cyclic,

$$\inf|\omega_y([\pi_\varrho(A'), \pi_\varrho(B_j)])| = 0 \quad \text{for all} \quad y \in \mathscr{H} .$$

Thus

$$\inf|\omega([\pi_\varrho(A'), \pi_\varrho(B_j)])| = 0, \quad j = 1, \ldots, n,$$

for all weakly continuous states ω. Let $\mathscr{H}^n = \mathscr{H}_\varrho \oplus \cdots \oplus \mathscr{H}_\varrho$, the sum taken n times. Let $\mathfrak{B}(\mathscr{H}^n)$ denote the bounded operators on $\mathscr{H}^n$. The map $\chi : \pi_\varrho(\mathfrak{A})^- \to \mathfrak{B}(\mathscr{H}^n)$ by

$$S \to \sum_{j=1}^{n} \oplus\ [S, \pi_\varrho(B_j)]$$

is weakly continuous. Now $\inf|\omega(\chi(A''))| = 0$ for all weakly continuous states ω of $\mathfrak{B}(\mathscr{H}^n)$, as A'' runs through $\mathfrak{K}_\varrho(A)$. Since $\mathfrak{K}_\varrho(A)^-$ is weakly compact, $\chi(\mathfrak{K}_\varrho(A)^-)$ is weakly compact, hence there exists $D \in \mathfrak{K}_\varrho(A)^-$ such that $\chi(D) = 0$, i.e. $[D, \pi_\varrho(B_j)] = 0, j = 1, \ldots, n$.

Let $\mathfrak{A}(B_1, \ldots, B_n) = \mathfrak{K}_\varrho(A)^- \cap \{\pi_\varrho(B_j) : j = 1, \ldots, n\}'$. By the above $\mathfrak{A}(B_1, \ldots, B_n) \neq \emptyset$. Clearly, if $C_1, \ldots, C_m$ is another finite family of self-adjoint operators in $\mathfrak{A}$

$$\mathfrak{A}(B_1, \ldots, B_n) \cap \mathfrak{A}(C_1, \ldots, C_m) = \mathfrak{A}(B_1, \ldots, B_n, C_1, \ldots, C_m) \neq \emptyset .$$

Hence the sets $\mathfrak{A}(B_1, \ldots, B_n)$ have the finite intersection property. As they are all weakly closed subsets of the weakly compact set $\mathfrak{K}_\varrho(A)^-$ they have a non empty intersection. Thus $\mathfrak{K}_\varrho(A)^- \cap \pi_\varrho(\mathfrak{A})' \neq \emptyset$, and G is represented as a large group of automorphisms of $\mathfrak{A}$. The proof is complete.

If G is represented by τ as a large group of automorphisms of $\mathfrak{A}$ it follows from the remarks preceding Theorem 3.5 and [14, Corollary 3.2] that $I(\mathfrak{A})$ is a simplex. We shall include another proof of this more in the spirit of our treatment.

Theorem 3.6. *Let $\mathfrak{A}$ be a C^*-algebra. Let G be a group represented as a large group of $*$-automorphisms of $\mathfrak{A}$. If $I(\mathfrak{A}) \neq \emptyset$ then it is a simplex.*

Proof. Let $L(\mathfrak{A})$ denote the cone in $\mathfrak{A}^*$ of positive linear G-invariant functionals. Let $\varrho_1, \varrho_2 \in L(\mathfrak{A})$. Let $\varrho = \varrho_1 + \varrho_2$, and assume $\varrho(I) = 1$. Since $\varrho_j \leqq \varrho$, $\varrho_j = \omega_j \circ \pi_\varrho$ with ω_j $\mathfrak{U}(\varrho)$-invariant and $0 \leqq \omega_j \leqq \omega_{x_\varrho}$. By Theorem 3.1 (5) there exist unique positive operators B_j, $j = 1, 2$, in $\mathfrak{B}(\varrho)$ such that $\omega_j(\pi_\varrho(A)) = \omega_{x_\varrho}(B_j \pi_\varrho(A))$ for all A in $\mathfrak{A}$. Now $\mathfrak{B}(\varrho)$

is an abelian von Neumann algebra, hence its self-adjoint part is a lattice. Let $B = B_1 \wedge B_2$. Let $\omega = \omega_{x_\varrho}(B\cdot)$. Since the map $\varrho \to B_\varrho$ in Theorem 3.1 (5) is an order-isomorphism ω is the greatest positive linear $\mathfrak{U}(\varrho)$-invariant functional smaller than or equal to ω_j, $j = 1, 2$. Hence $\omega \circ \pi_\varrho = \varrho_1 \wedge \varrho_2$ in $L(\mathfrak{A})$, so the latter is a lattice; $I(\mathfrak{A}) = L(\mathfrak{A}) \cap S(\mathfrak{A})$ is a simplex.

Since $I(\mathfrak{A})$ is a simplex each G-invariant state can in a unique way be written as an average over the extreme boundary of $I(\mathfrak{A})$. Hence a knowledge of the extremal invariant states is very important.

Theorem 3.7. *Let $\mathfrak{A}$ be a C^*-algebra. Let G be a group represented as a large group of automorphisms of $\mathfrak{A}$. Use the notation in* 3.4. *Let $\varrho \in I(\mathfrak{A})$. Then the following five conditions are equivalent.*

1) *ϱ is extreme in $I(\mathfrak{A})$.*
2) *$\pi_\varrho(\mathfrak{A}) \cup \mathfrak{U}(\varrho)$ is an irreducible set of operators.*
3) *$\mathfrak{B}(\varrho)$ is the scalars.*
4) *P_ϱ is one-dimensional.*
5) *$\omega_{x_\varrho}(\Phi_\varrho(A)B) = \omega_{x_\varrho}(A)\,\omega_{x_\varrho}(B)$ for all $A, B \in \pi_\varrho(\mathfrak{A})$.*

Proof. By Theorem 3.1 (3) 2) $\Leftrightarrow$ 3). Hence by Theorem 3.1 (4) 2) $\Leftrightarrow$ 4). By Theorem 3.1 (5) 1) $\Leftrightarrow$ 3). Finally 3) $\Leftrightarrow$ 5), for if 5) holds then by continuity of Φ_ϱ it holds for $A, B \in \pi_\varrho(\mathfrak{A})^-$. Hence $\omega_{x_\varrho} | \mathfrak{B}(\varrho)$ is a homomorphism. Since x_ϱ is a separating vector for $\mathfrak{B}(\varrho)$, 3) holds. Conversely, if 3) holds then $\Phi_\varrho(A) = \omega_{x_\varrho}(A)I$, hence

$$\omega_{x_\varrho}(\Phi_\varrho(A)B) = \omega_{x_\varrho}(A)\,\omega_{x_\varrho}(B)\,,$$

and 5) holds.

The equivalence 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 4) is also a consequence of [14, Proposition 4.1] and Theorem 3.1 (4). Notice that 5) is a generalized clustering property. In a number of cases $\mathfrak{B}(\varrho) = \mathfrak{C}(\varrho)$, in which case condition 3) above means π_ϱ is a factor representation. We note the following situation where this equality prevails.

Lemma 3.8. *Let $\mathfrak{A}$ be a C^*-algebra. Let G be a connected topological group. Let $g \to \tau_g$ be a norm continuous representation of G as a large group of automorphisms of $\mathfrak{A}$. Let ϱ be a G-invariant state and use notation in* 3.4. *Then $\mathfrak{B}(\varrho) = \mathfrak{C}(\varrho)$.*

Proof. If g is in G then for all A in $\mathfrak{A}$,

$$\|U_\varrho(g)\,\pi_\varrho(A)\,U_\varrho(g)^{-1} - \pi_\varrho(A)\| = \|\pi_\varrho(\tau_g(A) - A)\| \leqq \\ \leqq \|\tau_g(A) - A\|\,.$$

Hence the representation $g \to U_\varrho(g)\cdot U_\varrho(g)^{-1}$ is a norm continuous representation of G as $*$-automorphisms of $\pi_\varrho(\mathfrak{A})$. By [12, Corollary 8] there exist unitary operators U_g in $\pi_\varrho(\mathfrak{A})^-$ such that $U_\varrho(g)\,\pi_\varrho(A)\,U_g(g)^{-1} = U_g\pi_\varrho(A)U_g^{-1}$ for all A in $\mathfrak{A}$. If $S \in \mathfrak{C}(\varrho)$ then $U_\varrho(g)\,S\,U_\varrho(g)^{-1} = U_g S U_g^{-1} = S$ for all $g \in G$, so $S \in \mathfrak{U}(\varrho)' \cap \mathfrak{C} = \mathfrak{B}(\varrho)$, and $\mathfrak{B}(\varrho) = \mathfrak{C}(\varrho)$.

The nice simplexes are those with closed extreme boundary, hence those for which the extreme points form closed sets. For completeness it is desirable to know when the extremal G-invariant states form a closed subset of $I(\mathfrak{A})$. As the theorem to follow is of general nature, we do not assume G large. The reader should only keep in mind Theorem 3.6.

Theorem 3.9. *Let $\mathfrak{A}$ be a C^*-algebra. Let G be a group and $g \to \tau_g$ a representation of G as $*$-automorphisms of $\mathfrak{A}$. Assume $I(\mathfrak{A})$ is not empty. Then $I(\mathfrak{A})$ is a simplex with closed extreme boundary if and only if there exist an abelian C^*-algebra $\mathfrak{B}$ and a positive linear G-invariant map Φ of $\mathfrak{A}$ onto a norm dense subset of $\mathfrak{B}$ such that $\Phi(I) = I$ and such that the map $\Phi^* : \varrho \to \varrho \circ \Phi$ maps $S(\mathfrak{B})$ onto $I(\mathfrak{A})$. In this case $\mathfrak{B}$ is unique up to an isomorphism.*

Proof. If $\mathfrak{B}$ and Φ are as described then Φ^* is an affine isomorphism of $S(\mathfrak{B})$ onto $I(\mathfrak{A})$. Since $S(\mathfrak{B})$ is a simplex with closed extreme boundary, so is $I(\mathfrak{A})$. Conversely, assume $I(\mathfrak{A})$ is such a simplex. Let Z denote its extreme points. Then Z is a compact Hausdorff space. Let $\mathfrak{B} = C(Z)$. Define a map $\Phi : \mathfrak{A} \to \mathfrak{B}$ by $\Phi(A)(z) = z(A)$, $z \in Z$. Then Φ is positive, linear, $\Phi(I) = I$, and if $g \in G$ then $\Phi(\tau_g(A))(z) = z(\tau_g(A)) = z(A) = \Phi(A)(z)$, so Φ is G-invariant. In particular, if $\varrho \in S(\mathfrak{B})$ then $\Phi^*(\varrho) \in I(\mathfrak{A})$.

If A is self-adjoint in $\mathfrak{A}$ let $\hat{A}$ be the w^*-continuous real affine function on $S(\mathfrak{A})$ defined by $\hat{A}(\varrho) = \varrho(A)$. An application of the Hahn-Banach Theorem shows that the set of restrictions $\hat{A}\,|\,I(\mathfrak{A})$ is norm dense in $\mathrm{Aff}(I(\mathfrak{A}))$ — the w^*-continuous real affine functions on $I(\mathfrak{A})$. Since Z is closed, the map $b \to b\,|\,Z$ is an order-isomorphism of $\mathrm{Aff}(I(\mathfrak{A}))$ onto $C_R(Z)$ — the real continuous functions on Z, see e.g. [1, Satz 4.4.4]. Thus $\Phi(\mathfrak{A})$ is norm dense in $\mathfrak{B} = C(Z)$. If $z \in Z$ let e_z be the evaluation at z of functions in $C(Z)$. The pure states of $\mathfrak{B}$ are the states of this form. Let $\varrho \in I(\mathfrak{A})$. Then there exists a unique measure μ on $I(\mathfrak{A})$ with support in Z such that

$$\begin{aligned}\varrho &= \int_Z z\, d\mu(z) = \int_Z \Phi^*(e_z)\, d\mu(z)\\ &= \Phi^*\Big(\int_Z e_z\, d\mu(z)\Big) = \Phi^*(\tilde{\varrho})\,,\end{aligned}$$

where $\tilde{\varrho} = \int_Z e_z\, d\mu(z) \in S(\mathfrak{B})$. Thus Φ^* maps $S(\mathfrak{B})$ onto $I(\mathfrak{A})$. Let $\mathfrak{B}_1$ be another C^*-algebra, and Φ_1 a positive linear G-invariant map of $\mathfrak{A}$ onto a dense subset of $\mathfrak{B}_1$ such that $\Phi_1(I) = I$, and $\Phi_1^*(S(\mathfrak{B}_1)) = I(\mathfrak{A})$. Then $\Phi^{*-1} \circ \Phi_1^*$ is an affine isomorphism of $S(\mathfrak{B}_1)$ onto $S(\mathfrak{B})$. Thus $\mathfrak{B}_1$ is $*$-isomorphic to $\mathfrak{B}$, see e.g. [11, Corollary 4.7].

4. Traces of C^*-algebras

THOMA [17, p. 116] has shown that the normalized traces of a C^*-algebra $\mathfrak{A}$ form a simplex and that the extremal traces are exactly the factor traces, the latter being defined as those traces ϱ for which $\pi_\varrho(\mathfrak{A})^-$ is a factor. We shall show this by showing that the inner automorphisms of $\mathfrak{A}$ is a large group of automorphisms. If $\mathfrak{A}$ is a C^*-algebra we denote by $\mathfrak{U}(\mathfrak{A})$ the group of unitary operators in $\mathfrak{A}$.

Theorem 4.1. *Let $\mathfrak{A}$ be a C^*-algebra. Then $\mathfrak{U}(\mathfrak{A})$ is represented as a large group of automorphisms of $\mathfrak{A}$ via the representation $U \to U \cdot U^{-1}$. In particular, if there exists a normalized trace of $\mathfrak{A}$, then the normalized traces form a simplex the extreme points of which are the factor traces.*

Proof. Assume $\varrho = \omega_{x_\varrho} \circ \pi_\varrho$ is a normalized trace. Let $U \in \mathfrak{U}(\mathfrak{A})$, and let U_ϱ denote the unitary operator on $\mathscr{H}_\varrho$ such that

$$\pi_\varrho(U)\,\pi_\varrho(A)\,\pi_\varrho(U)^{-1} = \pi_\varrho(U A U^{-1}) = U_\varrho \pi_\varrho(A)\, U_\varrho^{-1}\,, \qquad (**)$$

for all $A \in \mathfrak{A}$, and such that $U_\varrho x_\varrho = x_\varrho$. Let $\mathfrak{U}(\varrho)$ denote the set of the U_ϱ. If $\mathfrak{B}$ is a C^*-algebra denote by $\mathfrak{U}(\mathfrak{B}, 2)$ the set of unitary operators in $\mathfrak{B}$ whose distance from I is less than 2. Then by [9, Theorem 2, Lemma 5] and their proofs,

$$\mathfrak{U}(\pi_\varrho(\mathfrak{A})^-) \subset \mathfrak{U}(\pi_\varrho(\mathfrak{A})^-, 2)^- = \mathfrak{U}(\pi_\varrho(\mathfrak{A}), 2)^- \subset \pi_\varrho(\mathfrak{U}(\mathfrak{A}))^-\,.$$

Hence by [4, Théorème 1, p. 272] and (**)

$$\operatorname{conv}(\pi_\varrho(U A U^{-1}) : U \in \mathfrak{U}(\mathfrak{A}))^- \cap \pi_\varrho(\mathfrak{A})' \neq \emptyset\,,$$

and $\mathfrak{U}(\mathfrak{A})$ is large. If $S \in \mathfrak{C}(\varrho)$ then $S = \pi_\varrho(U)\, S \pi_\varrho(U)^{-1} = U_\varrho S U_\varrho^{-1}$ for all U in $\mathfrak{U}(\mathfrak{A})$. Thus $S \in \mathfrak{B}(\varrho)$, and $\mathfrak{B}(\varrho) = \mathfrak{C}(\varrho)$. By Theorem 3.7 ϱ is extremal if and only if it is a factor trace.

5. Asymptotically Abelian Algebras

It is clear from Theorem 3.5 that the results on large groups of automorphisms obtained in section 3, make it possible to obtain considerable generalizations of the known theory for C^*-algebras asymptotically abelian with respect to the translation group R^n, as obtained in [7, 13, 15]. We shall in this section discuss one generalization.

Definition 5.1. Let $\mathfrak{A}$ be a C^*-algebra. Let G be a group. We say $\mathfrak{A}$ is *asymptotically abelian with respect to G* if there exists a representation $g \to \tau_g$ of G as $*$-automorphisms of $\mathfrak{A}$ such that for each self-adjoint operator A in $\mathfrak{A}$ there is a sequence $\{g_n(A)\}_{n=1,2,\ldots}$ of elements in G such that

$$\lim_n \|[\tau_{g_n(A)}(A), B]\| = 0$$

for all B in $\mathfrak{A}$.

In the usual definition one assumes $\lim\limits_{g \to \infty} \|[\tau_g(A), B]\| = 0$ whenever $g \to \infty$ in R^n, and that the representation $g \to \tau_g$ is strongly continuous. When $G = R^n$ our definition has the advantage that it also includes situations in which one only requires g to diverge to ∞ in some prescribed set. If $\mathfrak{A}$ is asymptotically abelian with respect to G then Theorem 3.5 shows τ represents G as a large group of automorphisms of $\mathfrak{A}$. We summarize the results in Theorems 3.1, 3.6, and 3.7 as follows.

Theorem 5.2. *Let $\mathfrak{A}$ be a C^*-algebra which is asymptotically abelian with respect to the group G. Let $g \to \tau_g$ be the corresponding representation of G. Use the notation introduced in* 3.4. *Suppose $\varrho \in I(\mathfrak{A})$. Then,*

1) *$I(\mathfrak{A})$ is a simplex.*

2) $(\pi_\varrho(\mathfrak{A}) \cup \mathfrak{U}(\varrho))' = \mathfrak{B}(\varrho)$.

3) *P_ϱ is an abelian projection with central carrier I in $(\pi_\varrho(\mathfrak{A}) \cup \mathfrak{U}(\varrho))''$.*

4) *The following five conditions are equivalent.*

i) *ϱ is extreme in $I(\mathfrak{A})$.*

ii) *$\pi_\varrho(\mathfrak{A}) \cup \mathfrak{U}(\varrho)$ is an irreducible set of operators.*

iii) *$\mathfrak{B}(\varrho)$ is the scalars.*

iv) *P_ϱ is one dimensional.*

v) *$\omega_{x_\varrho}(\Phi_\varrho(A)B) = \omega_{x_\varrho}(A)\,\omega_{x_\varrho}(B)$ for all $A, B \in \pi_\varrho(\mathfrak{A})$.*

For the rest of the section we shall be concerned with the structure of Φ_ϱ and the clustering property (4 v) in the above theorem.

Lemma 5.3. *Let $\mathfrak{A}$ be a C^*-algebra which is asymptotically abelian with respect to the group G. Let A be a self-adjoint operator in $\mathfrak{A}$. Let D be a weak limit point of the sequence $\{U_\varrho(g_n(A))\,\pi_\varrho(A)\,U_\varrho(g_n(A))^{-1}\}$. Then $D \in \mathfrak{C}(\varrho)$.*

Proof. Let $g_n = g_n(A)$. Since π_ϱ is norm continuous,

$$\lim_n \|[U_\varrho(g_n)\,\pi_\varrho(A)\,U_\varrho(g_n)^{-1}, \pi_\varrho(B)]\| = 0$$

for all $B \in \mathfrak{A}$. Let $B \in \mathfrak{A}$, let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be unit vectors in $\mathscr{H}_\varrho$, let $\varepsilon > 0$. Let n be so large that

$$|(([D, \pi_\varrho(B)] - [U_\varrho(g_n)\,\pi_\varrho(A)\,U_\varrho(g_n)^{-1}, \pi_\varrho(B)])x_j, y_j)| < \varepsilon/2$$

for $j = 1, \ldots, k$, and

$$\|[U_\varrho(g_n)\,\pi_\varrho(A)\,U_\varrho(g_n)^{-1}, \pi_\varrho(B)]\| < \varepsilon/2\,.$$

Then $|([D, \pi_\varrho(B)]x_j, y_j)| < \varepsilon$ for $j = 1, \ldots, k$. Since ε, x_j, y_j are arbitrary, $[D, \pi_\varrho(B)] = 0$, and $D \in \pi_\varrho(\mathfrak{A})^- \cap \pi_\varrho(\mathfrak{A})' = \mathfrak{C}(\varrho)$.

Using the terminology of [13] we say ϱ in $I(\mathfrak{A})$ is *strongly clustering* if

$$\lim_n \varrho(\tau_{g_n(A)}(A)B) = \varrho(A)\,\varrho(B)$$

whenever A and B are self-adjoint in $\mathfrak{A}$. This condition is very strong, as our next theorem shows.

Theorem 5.4. *Let $\mathfrak{A}$ be a C^*-algebra which is asymptotically abelian with respect to the group G. Let ϱ be a G-invariant state of $\mathfrak{A}$. Then ϱ is strongly clustering if and only if ϱ is extreme and*

$$\Phi_\varrho(\pi_\varrho(A)) = \text{weak}\lim_n U_\varrho(g_n(A))\,\pi_\varrho(A)\,U_\varrho(g_n(A))^{-1}$$

for all self-adjoint A in $\mathfrak{A}$.

Proof. Let $g_n = g_n(A)$. Assume ϱ is extreme in $I(\mathfrak{A})$ and that Φ_ϱ is given by the formula above. Then by Theorem 5.2 (4 v),

$$\begin{aligned}\lim_n \varrho(\tau_{g_n}(A)B) &= \lim_n \omega_{x_\varrho}(U_\varrho(g_n)\,\pi_\varrho(A)\;U_\varrho(g_n)^{-1}\,\pi_\varrho(B))\\ &= \omega_{x_\varrho}\big(\Phi_\varrho(\pi_\varrho(A))\,\pi_\varrho(B)\big)\\ &= \varrho(A)\,\varrho(B)\,.\end{aligned}$$

Conversely, assume ϱ is strongly clustering. Then ϱ is extreme. In fact, if ϱ is not extreme there exists by Theorem 5.2 (4 iv) a unit vector y in P_ϱ such that y is orthogonal to x_ϱ. Let $\varepsilon > 0$ be given. Choose $B \in \pi_\varrho(\mathfrak{A})$ such that $\|y - Bx_\varrho\| < \varepsilon/3$. Let $A \in \mathfrak{A}$ be self-adjoint, and $\|\pi_\varrho(A)\| \leqq 1$. Then for n sufficiently large,

$$\begin{aligned}|(y, \pi_\varrho(A)x_\varrho)| &= |(U_\varrho(g_n)\,\pi_\varrho(A)\;U_\varrho(g_n)^{-1}y, x_\varrho)| \leqq\\ &\leqq |(U_\varrho(g_n)\,\pi_\varrho(A)\;U_\varrho(g_n)^{-1}\,Bx_\varrho, x_\varrho)| + \varepsilon/3 \leqq\\ &\leqq |\omega_{x_\varrho}(\pi_\varrho(A))\;\omega_{x_\varrho}(B)| + \varepsilon/3 + \varepsilon/3 \leqq\\ &\leqq |(Bx_\varrho, x_\varrho)| + 2\varepsilon/3 \leqq\\ &\leqq \varepsilon\,.\end{aligned}$$

Since ε is arbitrary, $(y, \pi_\varrho(A)x_\varrho) = 0$ for all A in $\mathfrak{A}$. But x_ϱ is cyclic, hence $y = 0$, a contradiction. Thus P_ϱ is one-dimensional, ϱ is extreme.

Let D be a weak limit point of the sequence $\{U_\varrho(g_n)\,\pi_\varrho(A)\;U_\varrho(g_n)^{-1}\}$. By Lemma 5.3 $D \in \mathfrak{C}(\varrho)$. Choose a subsequence $\{U_\varrho(g_{n_j})\;\pi_\varrho(A)\,U_\varrho(g_{n_j})^{-1}\}$ which converges weakly to D. Since $\varrho(\tau_{g_n}(A)B) \to \varrho(A)\,\varrho(B)$ the same convergence holds for the subsequence $\varrho(\tau_{g_{n_j}}(A)B)$. Hence, by the above paragraph and Theorem 5.2 (4 v),

$$\begin{aligned}\omega_{x_\varrho}\big(\Phi_\varrho(\pi_\varrho(A))\;\pi_\varrho(B)\big) &= \omega_{x_\varrho}(\pi_\varrho(A))\;\omega_{x_\varrho}(\pi_\varrho(B))\\ &= \lim_{n_j} \omega_{x_\varrho}(U_\varrho(g_{n_j})\;\pi_\varrho(A)\;U_\varrho(g_{n_j})^{-1}\;\pi_\varrho(B))\\ &= \omega_{x_\varrho}(D\pi_\varrho(B))\,,\end{aligned}$$

for all $B \in \mathfrak{A}$. Thus

$$\big(\pi_\varrho(B)x_\varrho, \big(\Phi_\varrho(\pi_\varrho(A)) - D\big)x_\varrho\big) = 0$$

for all $B \in \mathfrak{A}$. Since x_ϱ is cyclic, $\big(\Phi_\varrho(\pi_\varrho(A)) - D\big)x_\varrho = 0$. Since x_ϱ is separating for $\mathfrak{C}(\varrho)$, $\Phi_\varrho(\pi_\varrho(A)) = D$. Thus $\Phi_\varrho(\pi_\varrho(A))$ is the unique weak limit point of the sequence $\{U_\varrho(g_n)\,\pi_\varrho(A)\;U_\varrho(g_n)^{-1}\}$, hence this sequence converges to $\Phi_\varrho(\pi_\varrho(A))$. The proof is complete.

Kadison and Ringrose remark in the introduction of [12] that the assumption of the spectrum condition is closely related to the assumption that G is a topological group and the representation $g \to \tau_g$ is norm continuous. Our next result should therefore be compared with [2, Theorem 1], and Lemma 3.8

Corollary 5.5. *Assume $\mathfrak{A}$ is a C^*-algebra asymptotically abelian with respect to the group G. Let ϱ be a G-invariant state such that $\mathfrak{C}(\varrho) = \mathfrak{B}(\varrho)$. Then*

$$\Phi_\varrho(\pi_\varrho(A)) = \text{weak} \lim_n U_\varrho(g_n(A))\, \pi_\varrho(A)\, U_\varrho(g_n(A))^{-1}$$

for each self-adjoint operator A in $\mathfrak{A}$. Moreover, the following three conditions are equivalent.

1) *ϱ is extreme in $I(\mathfrak{A})$.*

2) *π_ϱ is a factor representation.*

3) *ϱ is strongly clustering.*

Proof. Let A be self-adjoint in $\mathfrak{A}$. Let D be a weak limit point of the sequence $\{U_\varrho(g_n(A))\pi_\varrho(A)\, U_\varrho(g_n(A))^{-1}\}$. By Lemma 5.3 $D \in \mathfrak{C}(\varrho) = \mathfrak{B}(\varrho)$. But from the proof of Theorem 3.1 there is at most one point in $\mathfrak{K}_\varrho(A)^- \cap \mathfrak{B}(\varrho)$, namely $\Phi_\varrho(\pi_\varrho(A))$. Thus $D = \Phi_\varrho(\pi_\varrho(A))$, and the sequence converges weakly to $\Phi_\varrho(\pi_\varrho(A))$. The equivalence of 1), 2), and 3) now follows from Theorems 5.2 and 5.4.

We conclude this section by showing that the condition (4 v) in Theorem 5.2 reduces to the weak clustering property of Ruelle, condition (2) in [15, Corollary 2], when $G = R^\nu$. Recall some of his notation. If $a = (a^1, \ldots, a^\nu) \in R^\nu$ with $a^j > 0$ we let

$$V(a) = \prod_{j=1}^{\nu} a^j,\quad \Lambda(a) = \{x \in R^\nu : 0 \leqq x^j < a^j, j = 1, \ldots, \nu\},$$

and

$$\mathfrak{M}_a A = V(a)^{-1} \int_{\Lambda(a)} \tau_b(A)\, db .$$

We assume $\lim\limits_{a\to\infty} \|[\tau_a A, B]\| = 0$, whenever $a \to \infty$ in R^ν. Let A be self-adjoint, and let D and D' be weak limit points of sequences in the bounded set $\{\pi_\varrho(\mathfrak{M}_a A)\} = \{\mathfrak{M}_a \pi_\varrho(A)\}$ as $a \to \infty$, $\varrho \in I(\mathfrak{A})$. By [15, Lemma 3]

$$t = \lim_{a_1, a_2 \to \infty} \varrho(\mathfrak{M}_{a_1} A\, \mathfrak{M}_{a_2} A)$$

exists independently of the order of convergence. Choose sequences $\{a_n\}$ and $\{b_m\}$ in R^ν converging to ∞ such that $\pi_\varrho(\mathfrak{M}_{a_n} A) \to D$ weakly, and $\pi_\varrho(\mathfrak{M}_{b_m} A) \to D'$ weakly. Then

$$t = \lim_m \omega_{x_\varrho}(D\pi_\varrho(\mathfrak{M}_{b_m} A)) = \omega_{x_\varrho}(DD') .$$

Let c_n be a subsequence of a_n so that still $\pi_\varrho(\mathfrak{M}_{c_n} A) \to D$ weakly. Then

$$\begin{aligned} t &= \lim_{a_n, c_n} \varrho(\mathfrak{M}_{a_n} A\, \mathfrak{M}_{c_n} A) = \lim_{c_n} \omega_{x_\varrho}(D\pi_\varrho(\mathfrak{M}_{c_n} A)) \\ &= \omega_{x_\varrho}(D^2) . \end{aligned}$$

Thus $\|Dx_\varrho\|^2 = (Dx_\varrho, D'x_\varrho) \leqq \|Dx_\varrho\| \, \|D'x_\varrho\|$. By symmetry $\|Dx_\varrho\| \times \times \|D'x_\varrho\| = (Dx_\varrho, D'x_\varrho)$. Therefore there exists a complex number λ such that $D'x_\varrho = \lambda D x_\varrho$. Since as in Lemma 5.3 D and D' belong to $\mathfrak{C}(\varrho)$, and x_ϱ is separating for $\mathfrak{C}(\varrho)$, $D' = \lambda D$. Since $(Dx_\varrho, D'x_\varrho) = \|Dx_\varrho\| \times \times \|D'x_\varrho\|$, $\lambda = 1$, and $D = D'$. Thus, for any sequence $a_n \to \infty$,

$$\pi_\varrho(\mathfrak{M}_{a_n} A) \to D \quad \text{weakly} .$$

Let $x \in R^\nu$. Then

$$\begin{aligned} U_\varrho(x)\, D\, U_\varrho(x)^{-1} &= U_\varrho(x)\, (\text{weak} \lim_{a_n} \mathfrak{M}_{a_n} \pi_\varrho(A))\, U_\varrho(x)^{-1} \\ &= \text{weak} \lim_{a_n} U_\varrho(x)\, \mathfrak{M}_{a_n} \pi_\varrho(A)\, U_\varrho(x)^{-1} \\ &= \text{weak} \lim_{a_n} \mathfrak{M}_{a_n} \pi_\varrho(A) = D . \end{aligned}$$

Thus $D \in \mathfrak{B}(\varrho)$. As argued in the proof of Corollary 5.5

$$\Phi_\varrho(\pi_\varrho(A)) = \text{weak} \lim_{a_n \to \infty} \pi_\varrho(\mathfrak{M}_{a_n} A) .$$

In particular, by Theorem 5.2 (4 v), ϱ is extreme if and only if

$$\omega_{x_\varrho}\big(\Phi_\varrho(\pi_\varrho(A))\, \Phi_\varrho(\pi_\varrho(B))\big) = \varrho(A)\, \varrho(B) ,$$

or by the above, if and only if

$$\lim_{a, b \to \infty} \varrho(\mathfrak{M}_a A\, \mathfrak{M}_b B) = \varrho(A)\, \varrho(B) ,$$

whenever A, B are in $\mathfrak{A}$. This is the same as condition (2) in [15, Corollary 2].

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