Representations of the CAR Generated by Representations of the CCR. Fock Case

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Abstract. We present a method of constructing the Fock representation of the canonical anti-commutation relations in the Fock representation of the canonical commutation relations. An explicit formula for Fermi creation and annihilation operators in terms of Bose ones is given.

1. Introduction

The spinor theory of Heisenberg [1] is an example of the philosophy that a fundamental theory of elementary particles must involve Fermi rather than Bose fields in the basic formalism. Quite the contrary, there were many attempts [2, 3] and references there-in to describe fermions in terms of bosons. Streater and Wilde [2] have shown that in two-dimensional space-time fermion states of a boson field do exist.

Kalnay, MacCotrina and Kademova [3] succeeded to show that in the case of Fock representations, free Fermi field can be expanded into a sum of pairs of Bose operators.

We want to present an independent investigation showing that certain homomorphisms of the n-th power space $K^{\otimes n}$ can be utilized to produce the Fock representations of the CAR algebra expressed by infinite series in Bose creation and annihilation operators.

2. Notations

Let $\mathscr{K}$, $\mathscr{H}$ be complex Hilbert spaces with an involution $*, \mathscr{K} \ni f, g$. Let $\mathscr{U}_F(\mathscr{K})$ be a $*$ representation of the CAR algebra over $\mathscr{K}$ acting in $\mathscr{H} = \mathscr{H}_F$. The generating elements $b(f), b(g)^*$ fulfill:

$$[b(f), b(g)^*]_+ = (f^*, g)\, \mathbb{1}_F$$
$$[b(f), b(g)]_+ = 0 \tag{2.1}$$

where $(\cdot, \cdot)$ is a bilinear form in $\mathscr{K}$, $\mathbb{1}$ is the unit operator in $\mathscr{U}_F(\mathscr{K})$. Let further $\mathscr{U}_B(\mathscr{K})$ be a $*$ representation of the CCR algebra over $\mathscr{K}$ acting in some $\mathscr{D} \subset \mathscr{H} = \mathscr{H}_B$, $\mathbb{1}_B$ is a unit operator in $\mathscr{U}_B(\mathscr{K})$. For generating elements $a(f), a(g)^*$ we have

$$[a(f), a(g)^*]_- = (f^*, g)\, \mathbb{1}_B$$
$$[a(f), a(g)]_- = 0. \tag{2.2}$$

If there exist vacuum vectors $|0\rangle_F$ and $|0\rangle_B$ respectively:

$$a(f)|0\rangle_B = 0 = b(f)|0\rangle_F \tag{2.3}$$

we say about Fock representations.

Let E_n be a bounded operator acting on the n-th tensor product $\bigotimes_1^n \mathcal{K} = \mathcal{H}_n$ with properties:

$$E_n^3 = E_n, \qquad E_n^* = E_n, \qquad P_{ik}E_n = -E_nP_{ik} \tag{2.4}$$

where P_{ik} is an operator of permutation of i-th and k-th $\mathcal{K}$ in $\bigotimes_1^n \mathcal{K}$. (For a more detailed study of E_n as well as for examples, see [7, 8].)

Due to (2.4) E_n^2 is a projector and provides the following decomposition of $\mathcal{H}_n$:

$$\mathcal{H}_n = \overset{1}{\mathcal{H}}_n \oplus \overset{2}{\mathcal{H}}_n \tag{2.5}$$

with:

$$E_n^2 \mathcal{H}_n = \overset{1}{\mathcal{H}}_n, \qquad (1 - E_n^2)\mathcal{H}_n = \overset{2}{\mathcal{H}}_n. \tag{2.6}$$

Let us decompose additionally $\bigotimes_1^n \mathcal{K} = \mathcal{H}_n$ into the irreducible parts with respect to the symmetry group $\mathcal{S}_n$:

$$\mathcal{H}_n = \sum_y Y_n \mathcal{H}_n \tag{2.7}$$

where Y_n denotes the Young's operator in $\mathcal{H}_n$.

We are able to prove:

Lemma 1. E_n *is an automorphism of* $\overset{1}{\mathcal{H}}_n$ *(a homomorphism* $\mathcal{H}_n \to \overset{1}{\mathcal{H}}_n$ *with the kernel* $\ker E_n = \overset{2}{\mathcal{H}}_n$*) consisting of isomorphisms*

$$E_n : Y_n \overset{1}{\mathcal{H}}_n \leftrightarrow Y_n^* \overset{1}{\mathcal{H}}_n \tag{2.8}$$

where Y_n^* *is the operator corresponding to the dual Young's scheme.*

Proof. Is given in [7], Section 2.

In totally symmetric ($Y_n = S_n$) and antisymmetric ($Y_n = A_n$) cases we have:

$$E_n : \overset{1}{\mathcal{H}}_n^F = A_n \overset{1}{\mathcal{H}}_n \leftrightarrow \overset{1}{\mathcal{H}}_n^B = S_n \overset{1}{\mathcal{H}}_n. \tag{2.9}$$

In the present paper we assume (see also [7]):

$$A_n \overset{2}{\mathcal{H}}_n = 0. \tag{2.10}$$

Let $\mathcal{F}$ be given by:

$$\mathcal{F} = \{f = \{f_n\}_{n=0,1,\ldots}, f_n \in \mathcal{H}_n, \|f\| < \infty\} = \bigoplus_1^\infty \mathcal{H}_n. \tag{2.11}$$

Here $\mathcal{H}_0 = \mathbb{C}$.

Now:

$$\mathcal{F}_B = \overset{1}{\mathcal{F}}_B \oplus \overset{2}{\mathcal{F}}_B, \qquad \mathcal{F}_F = \overset{1}{\mathcal{F}}_F \oplus \overset{2}{\mathcal{F}}_F \tag{2.12}$$

with:

$$\overset{i}{\mathscr{F}}_B = \bigoplus_{n=0}^{\infty} \overset{i}{\mathscr{H}}_n^B, \qquad \overset{i}{\mathscr{F}}_F = \bigoplus_{n=0}^{\infty} \overset{i}{\mathscr{H}}_n^F \tag{2.13}$$

where:

$$\| f_B \|^2 = \sum_{n=0}^{\infty} \| f_n^B \|^2 \tag{2.14}$$

and:

$$\| f_F \|^2 = \sum_{n=0}^{\infty} \| \overset{1}{f}_n^F \|^2 = \sum_{n=0}^{\infty} \| \overset{1}{f}_n^B \|^2 = \| \overset{1}{f}_B \|^2 \tag{2.15}$$

if:

$$f_n^F = E_n f_n^B . \tag{2.16}$$

Here the assumption (2.10) is used.

3. Results

All the considerations of the previous section are applicable without any change to more complicated cases $(\mathscr{K} = \bigoplus_1^N k)$ admitting internal degrees of freedom in the theory. This fact was proved in [8]. Therefore we restrict ourselves to the scalar case.

Let $E_n(k_n, p_m)$ be an integral kernel of the E_n introduced above, $f_{nm}(k_n, p_m)$ be the $n+m$-point symmetric function, $k_n = (k_1, \ldots, k_n)$, $p_m = (p_1, \ldots, p_m)$; $k, p \in \mathbb{R}^3$.

Theorem. *Given a Fock representation $\mathscr{U}_B(\mathscr{K})$ of the CCR algebra acting in $\mathscr{D} \subset \mathscr{F}_B$, $(a^*, a) = \int dk a^*(k) a(k)$. Assume the properties (2.4) and (2.10) for E_n. Operators $b(f)$, $b(g)^*$ with:*

$$b(f) = :\exp\{-(a^*, a)\} \sum_{nm} \frac{1}{\sqrt{n!m!}} \int dk_n \int dp_m f_{nm}(k_n, p_m)$$

$$\cdot \overset{*}{a}(k_1) \ldots \overset{*}{a}(k_n) a(p_1) \ldots a(p_m):$$

$$= :\exp\{-(a^*, a)\} f[a^*, a]: \tag{3.1}$$

where:

$$f_{nm}(k_n, p_m) = \sqrt{n+1} \, \delta_{m, 1+n} \int dq_n \int dr E_n(k_n, q_n) f^*(r) E_{1+n}(r, q_n, p_{1+n}) \tag{3.2}$$

generate a Fock representation $\mathscr{U}_F(\mathscr{K})$ of the CAR algebra acting on the following subspace of $\mathscr{F}_B : \mathscr{F}_B = \bigoplus_{n=0}^{\infty} E_n^2 \mathscr{H}_n^B$ with the unit operator:

$$\mathbb{1}_F = :\exp\{-(a^*, a)\} E^2[a^*, a]:$$

$$= :\exp\{-(a^*, a)\} \sum_n \frac{1}{n!} \int dk_n \int dp_n \int dr_n$$

$$\cdot E_n(k_n, r_n) E_n(r_n, p_n) a^*(k_1) \ldots a^*(k_n) a(p_1) \ldots a(p_n):$$

$$dk_n = dk_1 \ldots dk_n . \tag{3.3}$$

Proof. For the proof one can perform an explicit calculation using the Wick rule and taking advantage of the properties of E_n. We will use a more elegant method, based on the notion of functional representations of the CCR [4–6] and CAR [8, 7]. The Theorem is formulated in a representation independent way. Therefore it is enough to prove it for any Fock representation. Let us suppose, the notion of functional derivative (Gateaux derivative) with respect to elements of $\mathcal{K} = \mathcal{L}^2(\mathbb{R}^3)$ is established, see [4, 5, 10]. For functional power series:

$$F[z^*, z] = \sum_{nm} \frac{1}{\sqrt{n!m!}} (f_{nm}, z^{*n}z^m) = \sum_{nm} \frac{1}{\sqrt{n!m!}} \int dk_n \int dp_m$$
$$\cdot f_{nm}(k_n, p_m) z^*(k_1) \dots z^*(k_n) z(p_1) \dots z(p_m) \tag{3.4}$$

where $z \in \mathcal{K}$, we assume the following action rule:

$$(F_1 F_2)[z^*, z] = F_1\left[z^*, \frac{d}{dy^*}\right] F_2[y^*, z]_{|y^* = 0} . \tag{3.5}$$

So these double power series can be treated as operators acting in the vector space of single power series $V[z^*] = \sum_n \frac{1}{\sqrt{n!}} (v_n, \overset{*}{z}{}^n)$ due to:

$$(FV)[z^*] = V'[z^*] = F\left[z^*, \frac{d}{dy^*}\right] V[y^*]_{|y^* = 0} . \tag{3.6}$$

How to provide these series a nonformal meaning, see [6]. In this place we mention the following computational rules:

(i) $F_1 = F_2$ if and only if the following holds for all n, m

$$S_m S_n (f_1)_{mn}(k_m, p_n) = S_m S_n (f_2)_{mn}(k_m, p_n) \tag{3.7}$$

where S_m and S_n denote symmetrizations with respect to k_m and p_n respectively.

(ii) The product $F = F_1 F_2$ is given by

$$f_{nm}(k_m, p_n) = \sum_l \int dq_l (f_1)_{ml}(k_m, q_l) S_l (f_2)_{ln}(q_l, p_n) . \tag{3.8}$$

We have:

Lemma 2. *Let $f_0 \in \mathbb{C}$, $\mathcal{L}^2(\mathbb{R}^3) = \mathcal{K} \ni f, z$.*
The triple $\{a, a^*, f_0\}$ with

$$a(f)[z^*, z] = \exp(z^*, z) \cdot (z, f^*)$$
$$a(f)^*[z^*, z] = \exp(z^*, z) \cdot (z^*, f) \tag{3.9}$$

generates a functional Fock representation of the CCR algebra acting in some $\mathcal{D} \subset \mathcal{F}_B$.

Proof. By an explicit use of (3.5), (3.6) we are able to show:

$$[a(f), a(g)^*]_-[z^*, z] = (f^*, g) \exp(z^*, z)$$
$$[a(f), a(g)]_-[z^*, z] = 0$$
$$a(f)f_0 = 0 . \tag{3.10}$$

For details see [4, 6].

Lemma 3. *Let* $f_0 \in \mathbb{C}$, $\mathscr{L}^2(\mathbb{R}^3) = \mathscr{K} \ni z, f$.
The triple $\{b, b^*, f_0\}$ *with*

$$b(f)[z^*, z] = \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!m!}} (f_{nm}, z^{*n} z^m), \tag{3.11}$$

$f_{nm}(k_n, p_m)$ *given by the theorem and:*

$$b(f)^*[z^*, z] = b(f)[z^*, z]^* \tag{3.12}$$

generates a functional Fock representation of the CAR algebra in $\mathscr{F}_B$.

Proof. Is given in [7, 8], where functional representations of the CAR were first introduced.

$$[b(f), b(g)^*]_+[z^*, z] = (f^*, g) E^2[z^*, z]$$
$$[b(f), b(g)]_+[z^*, z] = 0$$
$$b(f)f_0 = 0. \tag{3.13}$$

In the above Lemmas $\exp(z^*, z)$ and $E^2[z^*, z]$ represent unit operators in $\mathscr{U}_B(\mathscr{K})$ and $\mathscr{U}_F(\mathscr{K})$ respectively.

Lemma 4. *Given:*

$$:f[a^*, a]: = \sum_{n,m} \frac{1}{\sqrt{n!m!}} (f_{nm}, a^{*n} a^m). \tag{3.14}$$

A functional representation of this operator is given by:

$$:f[a^*, a]:[z^*, z] = \exp(z^*, z)f[z^*, z]. \tag{3.15}$$

Proof. By an immediate application of Lemma 2 (see also [6]).
Collecting the Lemmas we are able to prove the theorem. We have by Lemma 4:

$$:\exp\{-(a^*, a)\} f[a^*, a]:[z^*, z] = \exp(z^*, z) \exp\{-(z^*, z)\} f[z^*, z]$$
$$= f[z^*, z] \tag{3.16}$$

and:

$$:\exp\{-(a^*, a)\} E^2[a^*, a]:[z^*, z] = E^2[z^*, z] \tag{3.17}$$

where the operator $:\exp\{-(a^*, a)\} E^2[a^*, a]:$ is the projection operator onto the subspace $\mathscr{F}_B$.
But:

$$f[z^*, z] = b(f)[z^*, z] \tag{3.18}$$

and therefore Lemma 3 implies the canonical anticommutation relations for $b(f)$. The Fock vacuum of $\mathscr{F}_B$ is obviously annihilated by $b(f)$. In the above, the domain was $\mathscr{D} \cap \mathscr{F}_B$. Since $b(f)$ is bounded (a general property of the CAR), its closure is defined on the whole space $\mathscr{F}_B$. This completes the proof of the Theorem.

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