Commun. math. Phys. 43, 69—71 (1975)

On the Gel'fand-Kirillov Conjecture

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Received December 5, 1974

Abstract. In this paper we exhibit a simple counterexample to the Gel'fand-Kirillov conjecture on the structure of the quotient field of every algebraic Lie algebra over a commutative field of characteristic zero.

Acknowledgment. The authors are indebted to Prof. A. Galindo for some valuable discussions.

1. Introduction

Let G be a finite dimensional Lie algebra over a commutative field $\mathbb{K}$ of characteristic zero. Let $\mathfrak{A}(G)$ denote the envelopping algebra of G; $\mathcal{D}(G)$ the quotient field of $\mathfrak{A}(G)$, and $C(G)$ the center of $\mathcal{D}(G)$. Gel'fand and Kirillov [1] proposed a conjecture on the structure of $\mathcal{D}(G)$ based on the following model. Two nonnegative integers n, r, define a ring $R_{n,r}(\mathbb{R})$ generated over the polynomial ring on r indeterminates $\mathbb{K}[x_1, \ldots, x_r]$ by $2n$ elements $p_1, \ldots, p_n, q_1, \ldots, q_n$ satisfying:

$$p_i q_j - q_j p_i = \delta_{ij}, \quad q_i q_j - q_j q_i = p_i p_j - p_j p_i = 0 .$$

In the ring $R_{n,r}(\mathbb{K})$ we have a filtration:

$$(R_{n,r}(\mathbb{K}))_0 \subset (R_{n,r}(\mathbb{K}))_1 \subset \cdots ,$$

where $(R_{n,r}(\mathbb{K}))_i$ is the set of all elements in $R_{n,r}(\mathbb{K})$, which can be written as (noncommutative) polynomials of degree $\leqq i$ in $\{q_k, p_j\}_{k,j=1}^n$ with coefficients in $\mathbb{K}[x_1, \ldots, x_r]$. It is obvious that the associated graded ring $\operatorname{gr} R_{n,r}(\mathbb{K})$ is isomorphic to the polynomial ring:

$$\mathbb{K}[x_1, \ldots, x_r, p_1, \ldots, p_n, q_1, \ldots, q_n] .$$

Moreover $R_{n,r}(\mathbb{K})$ is a Ore ring. We shall denote $\mathcal{D}_{n,r}(\mathbb{K})$ its quotient field.

Conjecture. *If G is an algebraic Lie algebra over a commutative field $\mathbb{K}$ of characteristic zero, then $\mathcal{D}(G)$ is isomorphic to the field $\mathcal{D}_{n,r}(\mathbb{K})$ where:*

r = transcendence degree of $C(G)$ over $\mathbb{K}$

$$n = \tfrac{1}{2}(\dim_{\mathbb{K}} G - r) . \quad \square$$

The conjecture was verified by Gel'fand and Kirillov [1] for $\mathrm{GL}(n, \mathbb{K})$, $\mathrm{SL}(n, \mathbb{K})$ and every nilpotent G over $\mathbb{K}$ and in a modified form for G semisimple over $\mathbb{C}$ [2]. Recently it has been demonstrated for G solvable over $\mathbb{C}$ [3].

To make the discussion as self-contained as possible we review some of the basic concepts that are relevant to the conjecture.

Let π_i denote the canonical projection:

$$\pi_i : (R_{n,r}(\mathbb{K}))_i \to \mathrm{gr}^{(i)} R_{n,r}(\mathbb{K}) = (R_{n,r}(\mathbb{K}))_i / (R_{n,r}(\mathbb{K}))_{i-1} .$$

For any nonzero $a \in R_{n,r}(\mathbb{K})$ there exists a unique integer i such that $\pi_i(a)$ is well-defined and different from zero. The integer i is the degree of a, and it will be denoted by $d(a)$. Let $[a]$ denote $\pi_i(a)$. It is an homogeneous polynomial in the variables p_i, q_j with coefficients in $\mathbb{K}[x_1, \ldots, x_r]$.

Every $b \in \mathscr{D}_{n,r}(\mathbb{K})$ can be decomposed in the following way:

$$b = a_1^{-1} a_2 , \qquad a_1, a_2 \in R_{n,r}(\mathbb{K}) .$$

Althought this decomposition is not unique, we have the following result [1]:

Lemma. *The rational function $[b] \equiv [a_1]^{-1} [a_2]$ depends only on the element b and not on the way b is decomposed. Moreover:*

$$[b_1 b_2] = [b_1][b_2] \qquad b_1, b_2 \in \mathscr{D}_{n,r}(\mathbb{K}) . \quad \square$$

2. A Counterexample to the Gel'fand-Kirillov Conjecture

Let G be the three-dimensional real Lie algebra, with basis $\{A_j\}_1^3$, such that:

$$[A_1, A_2] = A_3, \ [A_1, A_3] = -A_2, \ [A_2, A_3] = 0 .$$

Actually this is the Lie algebra of the Euclidean group of the plane (A_1 rotations, A_2, A_3 translations). Clearly $C(G)$ is generated over $\mathbb{R}$ by the element $A_2^2 + A_3^2$.

Hence the conjecture would require $\mathscr{D}(G)$ to be isomorphic to $\mathscr{D}_{1,1}(\mathbb{K})$, the field generated over the field $\mathbb{R}(x)$ by two elements p, q such that:

$$[p, q] = 1 .$$

Given such an isomorphism $\phi : \mathscr{D}(G) \to \mathscr{D}_{1,1}(\mathbb{R})$, we put $B_j = \phi(A_j)$ $j = 1, 2, 3$. Then, since $\phi|_{C(G)}$ maps $C(G)$ isomorphically onto the center $\mathbb{R}(x)$ of $\mathscr{D}_{1,1}(\mathbb{R})$ there is a choice of x such that:

$$\phi(A_2^2 + A_3^2) = B_2^2 + B_3^2 = x .$$

Let us descompose $B_3^2 = V^{-1} U$, where $V, U \in R_{1,1}(\mathbb{R})$. From the previous lemma we conclude:

$$[B_3^2] = [B_3]^2 = \frac{[U]}{[V]}$$

$$[B_2^2] = [B_2]^2 = [x - B_3^2] = [V^{-1}(xV - U)] = \frac{[xV - U]}{[V]} .$$

Let us investigate all possible situations

$$d(V) > d(U) \Rightarrow [xV - U] = [xV] = x[V] \Rightarrow [B_2]^2 = x \qquad (1)$$

which is impossible, $[B_2]$ being a real-valued rational function.

$$d(V) < d(U) \Rightarrow [xV - U] = -[U] \Rightarrow [B_2]^2 + [B_3]^2 = 0 \qquad (2)$$

which is also impossible (the trivial case $B_2 = B_3 = 0$ being excluded by the isomorphic character of ϕ).

$$d(V) = d(U) = g. \text{ Now two different situations arise.} \tag{3}$$

$$d(xV - U) = g \Rightarrow [xV - U] = x[V] - [U] \Rightarrow [B_2]^2 + [B_3]^2 = x, \tag{3a}$$

$$d(xV - U) < g \Rightarrow [U] = x[V] \Rightarrow [B_3]^2 = \frac{[U]}{[V]} = x. \tag{3b}$$

Both of these are clearly impossible.

The conclusion is that $\mathscr{D}(G)$ does not admits any isomorphism onto $\mathscr{D}_{1,1}(\mathbb{R})$.

3. Final Remark

The failure of the conjecture depends on the use of a field $\mathbb{K} = \mathbb{R}$ which is not algebraically closed. In fact, accordingly to [3], the complexified Lie algebra G admits an isomorphism $\phi : \mathscr{D}(G) \to \mathscr{D}_{1,1}(\mathbb{C})$, which can be realized in the following way:

$$\phi(A_1) = -ip \cdot q$$
$$\phi(A_2) = \tfrac{1}{2}(q + xq^{-1})$$
$$\phi(A_3) = \frac{-i}{2}(q - xq^{-1}).$$

Therefore, we think that the Gel'fand-Kirillov conjecture must be weakened to read:

Conjecture. *If G is an algebraic Lie algebra over an algebraically closed commutative field $\mathbb{K}$ of characteristic zero, then $\mathscr{D}(G)$ is isomorphic to $\mathscr{D}_{n,r}(\mathbb{K})$, where:*

$$r = \text{transcendence degree of } C(G) \text{ over } \mathbb{K}$$
$$n = \tfrac{1}{2}(\dim_{\mathbb{K}} G - r). \quad \square$$

References

1. Gel'fand, I. M., Kirillov, A. A.: I.H.E.S. Publications mathématiques **31**, 5—19 (1966)
2. Gel'fand, I. M., Kirillov, A. A.: Funkcional'nyj Analiz, 7—26 (1969)
3. Joseph, A.: Proc. Amer. Math. Soc. **45**, 1—10 (1974)

Communicated by H. Araki

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